Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
28 (2012), 197–206
www.emis.de/journals
ISSN 1786-0091

GENERALIZED WARPED PRODUCT MANIFOLDS AND CRITICAL RIEMANNIAN METRIC

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Abstract. In this paper, we present some properties of generalized warped product manifold. We establish the relationship between the scalar curvature of a generalized warped product $M \times_f N$ of Riemannian manifolds and those ones of M an N, and we obtain a necessary condition for a critical Riemannian metrics on $M \times_f N$.

1. Introduction

Let (M^m, g) be a compact oriented Riemannian manifold. A critical Riemannian metric is a critical point of the functional:

$$H(g) = \int_M |S_g|^2 v_g$$

where S_g is the scalar curvature of (M, g) and v_g is the volume element measured by g.

Locally, if $(x_i)_{i=1}^m$ denote a local coordinates on M, then g is a critical Riemannian metric, if and only if we have:

$$m\nabla_i\nabla_j S_g - mS_g \operatorname{Ric}_{ij}^g - (\Delta S_g)g_{ij} + S_g^2 g_{ij} = 0 \tag{1}$$

where $\operatorname{Ric}_{ij}^g$ denote the components of Ricci tensors with respect to g. For more details, we can refer to [1] and [7].

2. Results on generalized warped product

In this section, we give the definition and some geometric properties of generalized warped product manifolds.

2010 *Mathematics Subject Classification.* 53A30, 53B20, 53C21.
Key words and phrases. warped product manifolds, scalar curvature, critical metric.
Partially supported by the Algerian National Research Agency an Laboratory of Geometry, Analysis, Control and Applications.

Definition 1. Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $f\colon M \times N \to \mathbb{R}$ be a smooth positive function. The generalized warped metric on $M \times_f N$ is defined by

$$G_f = \pi^* g + (f)^2 \eta^* h \tag{2}$$

where $\pi\colon (x, y) \in M \times N \to x \in M$ and $\eta\colon (x, y) \in M \times N \to y \in N$ are the canonical projections.

For all $X, Y \in T(M \times N)$, we have

$$G_f(X, Y) = g(d\pi(X), d\pi(Y) + (f)^2 h(d\eta(X), d\eta(Y))$$

and we denote by $X \wedge_{G_{f^2}} Y$, the linear map:

$$Z \in \mathcal{H}(M) \times \mathcal{H}(N) \to \Big(X \wedge_{G_{f^2}} Y\Big)Z = G_{f^2}\Big(Z, Y\Big)X - G_{f^2}\Big(Z, X\Big)Y. \tag{3}$$

Theorem 1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\overline{\nabla}$ denote the Levi-Civita connection on $(M \times_f N, G_f)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$ we have:*

$$\begin{aligned}\overline{\nabla}_X Y = \nabla_X Y + X(\ln f)(0, Y_2) + Y(\ln f)(0, X_2) \\ - \frac{1}{2} h(X_2, Y_2)(\operatorname{grad}_M f^2, \frac{1}{f^2} \operatorname{grad}_N f^2)\end{aligned} \tag{4}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $\nabla_X Y = (\nabla^M_{X_1} Y^1, \nabla^N_{X_2} Y^2)$

The geometry of product manifolds is considered in [2], [5], [8].

Proof of Theorem 1 follows from the Kosul formula and the following formulas:

$$\begin{aligned} X(f^2).h(Y_2, Z_2) &= 2X(\ln f) G_f((0, Y_2), Z) \\ Y(f^2).h(X_2, Z_2) &= 2Y(\ln f) G_f((0, X_2), Z) \\ Z(f^2).h(X_2, Y_2) &= h(X_2, Y_2) G_f((\operatorname{grad}_M f^2, \frac{1}{f^2} \operatorname{grad}_N f^2), Z) \\ G_f(\nabla_X Y, Z) &= g(\nabla^M_{X_1} Y_1, Z_1) \circ \pi + f^2.h(\nabla^N_{X_2} Y_2, Z_2) \circ \eta \end{aligned}$$

where $Z = (Z_1, Z_2) \in \mathcal{H}(M) \times \mathcal{H}(N)$.

Remark 1. (1) If $f\colon (x, y) \in M \times N \mapsto f(x, y) = f(x)$, then

$$\begin{aligned}\overline{\nabla}_X Y = \nabla_X Y + X_1(\ln f)\Big(0, Y_2\Big) + Y_1(\ln f)\Big(0, X_2\Big) \\ - \frac{1}{2} h(X_2, Y_2)\Big(\operatorname{grad}_M(f^2), 0\Big)\end{aligned}$$

is the Levi-Civita connection of warped product manifolds.

(2) If $f\colon (x,y)\in M\times N\mapsto f(x,y)=f(y)$, then

$$\overline{\nabla}_X Y=\nabla_X Y+X_2(\ln f)\Big(0,Y_2\Big)+Y_2(\ln f)\Big(0,X_2\Big)$$
$$-\frac{1}{2}h(X_2,Y_2)\Big(0,\frac{1}{f^2}\operatorname{grad}_N(f^2)\Big)=\Big(\nabla^M_{X_1}Y_1,\widehat{\nabla}_{X_2}Y_2\Big)$$

is the Levi-Civita connection of product Riemannian manifolds (M,g) and (N,f^2h), where

$$\widehat{\nabla}_{X_2}Y_2=\nabla^N_{X_2}Y_2+X_2(\ln f)Y_2+Y_2(\ln f)X_2-h(X_2,Y_2)\operatorname{grad}_N\ln f.$$

From Theorem 1, we obtain

Corollary 1. *For all $X_1,Y_1\in\mathcal{H}(M)$ and $X_2,Y_2\in\mathcal{H}(N)$, we have:*

$$\overline{\nabla}_{(X_1,0)}(Y_1,0)=(\nabla^M_{X_1}Y_1,0)$$
$$\overline{\nabla}_{(X_1,0)}(0,Y_2)=X_1(\ln f)(0,Y_2)$$
$$\overline{\nabla}_{(0,X_2)}(Y_1,0)=Y_1(\ln f)(0,X_2)$$

and

$$\overline{\nabla}_{(0,X_2)}(0,Y_2)=(0,\nabla^N_{X_2}Y_2)+X_2(\ln f)(0,Y_2)+Y_2(\ln f)(0,X_2)$$
$$-\frac{1}{2}h(X_2,Y_2)(\operatorname{grad}_M f^2,\frac{1}{f^2}\operatorname{grad}_N f^2).$$

By Theorem 1, Corollary 1 and formula of curvature tensor, we have

Theorem 2. *Let (M^m,g) and (N^n,h) be two Riemannian manifolds and $f\colon M\times N\to\mathbb{R}$ be smooth positive function. If R and $\overline{R}$ denote the curvatures tensors of product manifold $(M\times N,G)$ and generalized warped product manifold $(M\times_f N,G_f)$ respectively, then*

$$(5)\,\overline{R}(X,Y)Z=R(X,Y)Z+\frac{1}{f}\Big((\nabla^M_{Y_1}\operatorname{grad}_M f,0)\wedge_{G_f}(0,X_2)\Big)Z$$
$$-\frac{1}{f}\Big((\nabla^M_{X_1}\operatorname{grad}_M f,0)\wedge_{G_f}(0,Y_2)\Big)Z$$
$$+\frac{1}{f^2}\Big[(0,\nabla^N_{Y_2}\operatorname{grad}_N\ln f-Y_2(\ln f)\operatorname{grad}_N\ln f)\wedge_{G_f}(0,X_2)$$
$$-(0,\nabla^N_{X_2}\operatorname{grad}_N\ln f-X_2(\ln f)\operatorname{grad}_N\ln f)\wedge_{G_f}(0,Y_2)$$
$$-(f^2\mid\operatorname{grad}_M\ln f\mid^2+\mid\operatorname{grad}_N\ln f\mid^2)(0,X_2)\wedge_{G_f}(0,Y_2)\Big]Z$$
$$+\Big[X_1(Z_2(\ln f))+X_2(Z_1(\ln f))\Big](0,Y_2)$$
$$-\Big[Y_1(Z_2(\ln f))+Y_2(Z_1(\ln f))\Big](0,X_2)$$

for all $X, Y, Z \in \mathcal{H}(M) \times \mathcal{H}(N)$, *where* $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2)$ *and* $R(X,Y)Z = (R^M(X_1, Y_1)Z_1, R^N(X_2, Y_2)Z_2)$.

Theorem 3. *Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds and* $f\colon M \times N \to \mathbb{R}$ *be smooth positive function. The Ricci curvature from generalized warped product manifolds* $(M \times_f N, G_f)$ *is given by the following formulas:*

$$\begin{aligned}
\operatorname{Ric}((X_1,0),(Y_1,0)) &= \operatorname{Ric}^M(X_1,Y_1)\\
&\quad -ng(\nabla^M_{X_1} \operatorname{grad}_M \ln f + X_1(\ln f) \operatorname{grad}_M \ln f, Y_1)\\
&= \operatorname{Ric}^M(X_1,Y_1) - \frac{n}{f} g(\nabla^M_{X_1} \operatorname{grad}_M f, Y_1)\\
\operatorname{Ric}((X_1,0),(0,Y_2)) &= -nX_1(Y_2(\ln f))\\
\operatorname{Ric}((0,X_2),(Y_1,0)) &= h(X_2, \operatorname{grad}_N(Y_1(\ln f))) - nX_2(Y_1(\ln f))\\
&= (1-n)X_2(Y_1(\ln f))\\
\operatorname{Ric}((0,X_2),(0,Y_2)) &= \operatorname{Ric}^N(X_2,Y_2) + (2-n)h(\nabla^N_{X_2} \operatorname{grad}_N \ln f, Y_2)\\
&\quad +(2-n)h(X_2,Y_2) \mid \operatorname{grad}_N \ln f \mid^2\\
&\quad -(2-n)X_2(\ln f)h(\operatorname{grad}_N \ln f, Y_2)\\
&\quad -nf^2 h(X_2,Y_2) \mid \operatorname{grad}_M \ln f \mid^2\\
&\quad -h(X_2,Y_2)\big[\Delta_N(\ln f) + f^2 \Delta_M(\ln f)\big]
\end{aligned}$$

for all $X_1, Y_1 \in \mathcal{H}(M)$ *and* $X_2, Y_2 \in \mathcal{H}(N)$.

Lemma 1. *Any local orthonormal frame* $\{e_i, i = 1, \ldots, m\}$ *and* $\{\overline{e}_j, j = 1, \ldots, n\}$ *with respect to* (M^m, g) *and* (N^n, h) *, such* $\nabla^M_{e_s} e_i = 0$ *and* $\nabla^N_{\overline{e}_k} \overline{e}_j = 0$, *induces a local orthonormal frame on* $(M \times_f N, G_f)$ *by:*

$$\{(e_i, 0), \quad (0, \frac{1}{f}\overline{e}_j) : \quad i = \overline{1, \ldots, m}, \quad j = \overline{1, \ldots, n}\}$$

and followings formulae:

$$\sum_j h(\nabla^N_{\overline{e}_j} \operatorname{grad}_N \ln f), \overline{e}_j) = \Delta_N(\ln f)$$

$$\sum_j h(\overline{e}_j, Y_2) h((\nabla^N_{X_2} \operatorname{grad}_N \ln f), \overline{e}_j) = h(\nabla^N_{X_2} \operatorname{grad}_N \ln f, Y_2)$$

$$\sum_j h(\overline{e}_j, Y_2) h(X_2, \overline{e}_j) = h(X_2, Y_2)$$

Proof of Theorem 3. Using Theorem 2 and the formula of Ricci curvature, by summing over the indexes i and j, we obtain:

$$\begin{aligned}
\operatorname{Ric}((X_1,0),(Y_1,0)) = {} & G_f(\overline{R}((e_i,0),(X_1,0))((Y_1,0),(e_i,0)) \\
& + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_i),(X_1,0))((Y_1,0),(0,\overline{e}_i)) \\
= {} & g(R_M(e_i,X_1)Y_1,e_i) \\
& - g(\nabla^M_{X_1}\operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, Y_1)h(\overline{e}_j,\overline{e}_j) \\
= {} & \operatorname{Ric}^M(X_1,Y_1) \\
& - ng(\nabla^M_{X_1}\operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, Y_1),
\end{aligned}$$

$$\begin{aligned}
\operatorname{Ric}((X_1,0),(0,Y_2)) = {} & G_f(\overline{R}((e_i,0),(X_1,0))(0,Y_2),(e_i,0)) \\
& + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(X_1,0))(0,Y_2),(0,\overline{e}_j)) \\
= {} & \frac{1}{f^2}h(Y_2,\overline{e}_j)G_f((\nabla^M_{X_1}\operatorname{grad}_M \ln f,(0,\overline{e}_j)) \\
& + \frac{1}{f^2}h(Y_2,\overline{e}_j)G_f((X_1(\ln f)\operatorname{grad}_M \ln f,0),(0,\overline{e}_j)) \\
& - \frac{1}{f^2}X_1(Y_2(\ln f))G_f((0,\overline{e}_j),(0,\overline{e}_j)) \\
= {} & - nX_1(Y_2(\ln f)),
\end{aligned}$$

$$\begin{aligned}
\operatorname{Ric}((0,X_2),(Y_1,0)) = {} & G_f(\overline{R}((e_i,0),(0,X_2))(Y_1,0),(e_i,0)) \\
& + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(Y_1,0),(0,\overline{e}_j)) \\
= {} & g(\nabla^M_{e_i}\operatorname{grad}_M \ln f, Y_1)G_f((0,X_2),(e_i,0)) \\
& + g(e_i(\ln f)\operatorname{grad}_M \ln f, Y_1)G_f((0,X_2),(e_i,0)) \\
& + \frac{1}{f^2}\overline{e}_j(Y_1(\ln f))G_f((0,X_2),(0,\overline{e}_j)) \\
& - \frac{1}{f^2}X_2(Y_1(\ln f))G_f((0,\overline{e}_j),(0,\overline{e}_j)) \\
= {} & h(X_2,\operatorname{grad}_N(Y_1(\ln f))) - nX_2(Y_1(\ln f)),
\end{aligned}$$

and

$$\begin{aligned}
(6)\quad \operatorname{Ric}((0,X_2),(0,Y_2)) = {} & G_f(\overline{R}((e_i,0),(0,X_2))(0,Y_2),(e_i,0)) \\
& + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)).
\end{aligned}$$

By Theorem 2, formula (3) and Lemma 1, we obtain

$$
\begin{aligned}
(7)\qquad & G_f(\overline{R}((e_i,0),(0,X_2))(0,Y_2),(e_i,0)) \\
&\quad = -f^2 h(X_2,Y_2) G_f((\nabla^M_{e_i}\operatorname{grad}_M \ln f,0),(e_i,0)) \\
&\qquad -f^2 h(X_2,Y_2) G_f((e_i(\ln f)\operatorname{grad}_M \ln f,0),(e_i,0)) \\
&\qquad +e_i(Y_2(\ln f)) G_f((0,X_2),(e_i,0)) \\
&\quad = -f^2 h(X_2,Y_2)\Big[\Delta_M(\ln f) + \mid \operatorname{grad}_M \ln f \mid^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) &= f^2 \operatorname{Ric}^N(X_2,Y_2) \\
&\quad + \frac{1}{f^2} G_f((0,\nabla^N_{X_2}\operatorname{grad}_N \ln f)) \wedge_{G_f} (0,\overline{e}_j))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2} G_f((0,X_2(\ln f)\operatorname{grad}_N \ln f)) \wedge_{G_f} (0,\overline{e}_j))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2} G_f((0,\nabla^N_{\overline{e}_j}\operatorname{grad}_N \ln f)) \wedge_{G_f} (0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad + \frac{1}{f^2} G_f((0,\overline{e_j}(\ln f)\operatorname{grad}_N \ln f)) \wedge_{G_f} (0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}(f^2 \mid \operatorname{grad}_M \ln f \mid^2) G_f(((0,\overline{e}_j) \wedge_{G_f} (0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}(\mid \operatorname{grad}_N \ln f \mid^2) G_f(((0,\overline{e}_j) \wedge_{G_f} (0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&= f^2 \operatorname{Ric}^N(X_2,Y_2) \\
&\quad + f^2 h(Y_2,\overline{e_j}) h(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f),\overline{e}_j) \\
&\quad - f^2 h(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f) h(\overline{e_j},\overline{e}_j) \\
&\quad - f^2 h(Y_2,X_2) h(\nabla^N_{\overline{e}_j}\operatorname{grad}_N \ln f - \overline{e_j}(\ln f)\operatorname{grad}_N \ln f,\overline{e_j}) \\
&\quad + f^2 h(Y_2,\nabla^N_{\overline{e}_j}\operatorname{grad}_N \ln f - \overline{e_j}(\ln f)\operatorname{grad}_N \ln f) h(X_2,\overline{e}_j) \\
&\quad - f^2(f^2 \mid \operatorname{grad}_M \ln f \mid^2)(h(X_2,Y_2)h(\overline{e}_j,\overline{e}_j) - h(\overline{e}_j,Y_2)h(X_2,\overline{e}_j)) \\
&\quad - f^2(\mid \operatorname{grad}_N \ln f \mid^2)(h(X_2,Y_2)h(\overline{e}_j,\overline{e}_j) - h(\overline{e}_j,Y_2)h(X_2,\overline{e}_j))
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{f^2} G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) = \\
&\quad = \operatorname{Ric}^N(X_2,Y_2) + h(Y_2,(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f)) \\
&\qquad - nh(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f) \\
&\qquad - h(Y_2,X_2)(\Delta_N(\ln f) - |\operatorname{grad}_N \ln f|^2) \\
&\qquad + h(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f) - h(Y_2,\operatorname{grad}_N \ln f) h(X_2,\operatorname{grad}_N \ln f) \\
&\qquad - (f^2 \mid \operatorname{grad}_M \ln f \mid^2 + \mid \operatorname{grad}_N \ln f \mid^2)(nh(X_2,Y_2) - h(Y_2,X_2))
\end{aligned}
$$

then

$$\begin{aligned}(8)\qquad \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) &= \mathrm{Ric}^N(X_2,Y_2)\\ &+(2-n)h(Y_2,(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f))\\ &-h(Y_2,X_2)(\Delta_N(\ln f) - |\operatorname{grad}_N \ln f|^2)\\ &-(n-1)(f^2 \mid \operatorname{grad}_M \ln f \mid^2 + \mid \operatorname{grad}_N \ln f \mid^2)h(X_2,Y_2).\end{aligned}$$

Substituting (7) and (8) in (6), we deduce:

$$\begin{aligned}\mathrm{Ric}((0,X_2),(0,Y_2)) &= \mathrm{Ric}^N(X_2,Y_2) + (2-n)h(X_2,Y_2)|\operatorname{grad}_N \ln f|^2\\ &+(2-n)h(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f, Y_2)\\ &-h(X_2,Y_2)\Big[nf^2 \mid \operatorname{grad}_M \ln f \mid^2 + \Delta_N(\ln f) + f^2\Delta_M(\ln f)\Big]. \quad \square\end{aligned}$$

Corollary 2. *Locally, the component of Ricci tensors from generalized warped product manifolds $(M \times_f N, G_f)$ are given by the following formulas:*

$$(9)\qquad \mathrm{Ric}_{ij} = \mathrm{Ric}^M_{ij} - \frac{n}{f}\nabla^M_i\nabla^M_j f$$

$$(10)\qquad \mathrm{Ric}_{ib} = -n\nabla^M_i\nabla^N_b(\ln f))$$

$$(11)\qquad \mathrm{Ric}_{aj} = (1-n)\nabla^N_a\nabla^M_j(\ln f))$$

$$\begin{aligned}(12)\qquad \mathrm{Ric}_{ab} &= \mathrm{Ric}^N_{ab} - h_{ab}\big[\Delta_N(\ln f) + f^2\Delta_M(\ln f)\big]\\ &\quad -h_{ab}\big[nf^2 \mid \operatorname{grad}_M \ln f \mid^2 + (n-2) \mid \operatorname{grad}_N \ln f \mid^2\big]\\ &\quad +(2-n)\big[\nabla^N_a\nabla^N_b \ln f - \nabla^N_a(\ln f)\nabla^N_b(\ln f))\big]\end{aligned}$$

$i,j = 1,\ldots,m$ and $a,b = 1,\ldots,n$.

2.1. **Scalar curvature.**

Theorem 4. *Let S^M, S^N and $\overline{S}$ denote the scalar curvature on (M^m,g), (N^n,h) and $(M \times_{G_f} N, G_f)$ respectively. Then the following equality holds:*

$$\begin{aligned}(13)\quad \overline{S} &= S^M + \frac{1}{f^2}S^N - 2n\Delta_M(\ln f) + \frac{2(1-n)}{f^2}\Delta_N(\ln f)\\ &\quad - n(n-1)|\operatorname{grad}_M \ln f|^2 - \frac{(n-1)(n-2)}{f^2}|\operatorname{grad}_N \ln f|^2.\end{aligned}$$

Proof. Let $\{e_i, i = 1,\ldots,m\}$ and $\{\overline{e}_a, a = 1,\ldots,n\}$ with respect to (M^m,g) and (N^n,h) , such $\nabla^M_{e_i}e_j = 0$ and $\nabla^N_{\overline{e}_a}\overline{e}_b = 0$, we have:

$$(14)\qquad \overline{S} = \sum_{i=1}^m \mathrm{Ric}((e_i,0),(e_i,0)) + \frac{1}{f^2}\sum_{a=1}^n \mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)).$$

From Theorem 3, we obtain

$$\sum_{i=1}^{m}\mathrm{Ric}((e_i,0),(e_i,0)) = S^M - n\sum_{i=1}^{m} g(\nabla^M_{e_i}\operatorname{grad}_M \ln f, e_i) - n\sum_{i=1}^{m} g(e_i(\ln f)\operatorname{grad}_M \ln f, e_i) = S^M - n|\operatorname{grad}_M \ln f|^2 - n\Delta_M(\ln f) \tag{15}$$

$$\sum_{a=1}^{n}\mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)) = S^N + (2-n)\sum_{a=1}^{n} h(\nabla^N_{\overline{e}_a}\operatorname{grad}_N \ln f, \overline{e}_a) + (2-n)\sum_{a=1}^{n}\big[h(\overline{e}_a,\overline{e}_a) \mid \operatorname{grad}_N \ln f \mid^2 - \overline{e}_a(\ln f)h(\operatorname{grad}_N \ln f, \overline{e}_a)\big] - \sum_{a=1}^{n} h(\overline{e}_a,\overline{e}_a)\big[nf^2 \mid \operatorname{grad}_M \ln f \mid^2 + \Delta_N(\ln f) + f^2\Delta_M(\ln f)\big]$$

$$\sum_{a=1}^{n}\mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)) = S^N + 2(1-n)\Delta_N(\ln f) + (2-n)(n-1) \mid \operatorname{grad}_N \ln f \mid^2 - n^2 f^2 \mid \operatorname{grad}_M \ln f \mid^2 - nf^2\Delta_M(\ln f) \tag{16}$$

Substituting (22) and (16) in (14), we deduce the equality (13). □

Corollary 3. *Let $U = f^{\frac{n+1}{2}}$ and $V = f^{\frac{n-2}{2}}$, $(n \geq 3)$, then*

$$\overline{S} = S^M + \frac{1}{f^2}S^N - \frac{4n}{n+1}U^{-1}\Delta_M U - \frac{4(n-1)}{n-2}V^{-\frac{n+2}{n-2}}\Delta_N V. \tag{17}$$

Particular cases:

(1) If $f(x,y) = f(y)$, $n \geq 3$ and $g = 0$. From the formula (17), we obtain the Yamabe equation associated to conformal metrics

$$\overline{S}.V^{\frac{n+2}{n-2}} = S^N.V - \frac{4(n-1)}{n-2}\Delta_N V.$$

(2) If $f(x,y) = f(x)$. From the formula (17), we obtain the scalar curvature equation of warped product manifolds:

$$\overline{S}.U = S^M.U + S^N.U^{\frac{n-3}{n+1}} - \frac{4n}{n+1}\Delta_M U.$$

the result is obtained in [3]

(3) If $n = 1$ then $f = U$ and

$$\overline{S}.U = S^M.U + S^N.U^{-1} - 2\Delta_M U.$$

(4) If $n = 2$ and $\gamma = \ln f$ then

$$\overline{S} = S^M + e^{-2\gamma}S^N - 4\Delta_M(\gamma) - 2e^{-2\gamma}\Delta_N(\gamma) - 2|\operatorname{grad}_M \gamma|^2.$$

From Theorem 4 and Corollary 2 we deduce:

Corollary 4. *If* $\ln f(x,y) = f_1(x) + f_2(y)$, *then the following formulas holds*

$$\text{Ric}_{ij} = \text{Ric}^M_{ij} - \frac{n}{f}\nabla^M_i\nabla^M_j f \tag{18}$$

$$\text{Ric}_{ib} = 0 \tag{19}$$

$$\text{Ric}_{aj} = 0 \tag{20}$$

$$\begin{aligned}\text{Ric}_{ab} = \text{Ric}^N_{ab} &- h_{ab}\big[nf^2 \mid \text{grad}_M f_1 \mid^2 + \Delta_N(f_2) + f^2\Delta_M(f_1)\big] \\ &+ (2-n)\big[\nabla^N_a\nabla^N_b f_2 + h_{ab} \mid \text{grad}_N f_2 \mid^2 - \nabla^N_a(f_2)\nabla^N_b(f_2))\big]\end{aligned} \tag{21}$$

and

$$\begin{aligned}\overline{S} = S^M + e^{-2(f_1+f_2)}S^N - 2n\Delta_M(f_1) + 2(1-n)e^{-2(f_1+f_2)}\Delta_N(f_2) \\ -n(n-1)|\,\text{grad}_M f_1|^2 - (n-1)(n-2)e^{-2(f_1+f_2)}|\,\text{grad}_N f_2|^2.\end{aligned} \tag{22}$$

Theorem 5. *Let* (M^m, g) *and* (N^n, h) *be compact manifolds with scalar curvatures* S^M *and* S^N *respectively and* $\ln f(x,y) = f_1(x) + f_2(y)$. *If* G_f *is critical Riemannian metric on* $M \times N$, *then the warped product space* $(M \times_f N, G_f)$ *is Riemannian product space or*

$$S^N = e^{2.f_2} + 2(n-1)\Delta_N(f_2) + (n-1)(n-2)|\,\text{grad}_n f_2|^2. \tag{23}$$

Proof. Let

$$H = S^N + 2(1-n)\Delta_N(f_2) - (n-1)(n-2)|\,\text{grad}_N f_2|^2$$

For $i = 1, \ldots, n$ and $a = 1, \ldots, n$ and using formula (22), we obtain

$$\begin{aligned}\nabla_i\nabla_a\overline{S} &= \partial_i(\partial_a(e^{-2(f_1+f_2)}H)) \\ &= \partial_i(e^{-2(f_1+f_2)}(\partial_a(H) - 2H\partial_a(f_2)) \\ \nabla_i\nabla_a\overline{S} &= -2e^{-2(f_1+f_2)}\partial_i(f_1)[\partial_a(H) - 2H\partial_a(f_2)]\end{aligned} \tag{24}$$

Other hand, from formula (1) and Corollary 4, we have

$$\nabla_i\nabla_a\overline{S} = 0. \tag{25}$$

then

$$\partial_i(f_1)[\partial_a(H) - 2H\partial_a(f_2)] = 0. \tag{26}$$

Hence, the solution of equation 2 is given by

$$f_1 = \text{constant or } H = e^{2.f_2}.$$

If $f_1 = constant$, then the warped product space $(M \times_f N, G_f)$ is the product space $(M \times N, g \oplus f^2h)$ of Riemannian manifolds (M^m, g) and $(N^n, f^2.h)$. □

Remark 2. In the case where $f_2 = constant$, we recover the result obtained in [6].

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Received May 03, 2012.

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