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A NOTE ON THE FOURIER COEFFICIENTS AND PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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Abstract. The aim of this paper is to investigate Paley type and Hardy-Littlewood type inequalities and strong convergence theorem of partial sums of Vilenkin-Fourier series.

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive numbers, not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ, of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$

is the Haar measure on G_m, with $\mu(G_m) = 1$. If $\sup_n m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded then G_m is said to be an unbounded Vilenkin group. *In this paper we discuss bounded Vilenkin groups only.*

The elements of G_m represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m,\ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\bar{I}_n := G_m \setminus I_n$.

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If we define the so-called generalized number system, based on m in the following way :

$$M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differ from zero.

Let $|n| := \max \{j \in \mathbb{N} : n_j \neq 0\}$. Denote by $\mathbb{N}_{n_0}$ the subset of positive integers $\mathbb{N}_+$, for which $n_{|n|} = n_0 = 1$. Then every $n \in \mathbb{N}_{n_0}$, $M_k < n < M_{k+1}$ can be written as

$$n = M_0 + \sum_{j=1}^{k-1} n_j M_j + M_k = 1 + \sum_{j=1}^{k-1} n_j M_j + M_k,$$

where $n_j \in \{0, m_j - 1\}$, $(j \in \mathbb{N}_+)$.

By simple calculation we get

$$\sum_{\left\{n: M_k \leq n \leq M_{k+1}, \ n \in \mathbb{N}_{n_0}\right\}} 1 = \frac{M_{k-1}}{m_0} \geq c M_k, \tag{1}$$

where c is absolute constant.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space. Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi \iota x_k / m_k), \quad \left(\iota^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley one if $m \equiv 2$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 14].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums, the Dirichlet kernels, with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases} \tag{2}$$

and

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right). \tag{3}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu \right)^{1/p} \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\lambda>0} \lambda \mu (f > \lambda)^{1/p} < +\infty.$$

The σ-algebra, generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n $(n \in \mathbb{N})$. The conditional expectation operators relative to F_n $(n \in \mathbb{N})$ are denoted by E_n. Then

$$E_n f(x) = S_{M_n} f(x) = \sum_{k=0}^{M_n-1} \widehat{f}(k) w_k = \frac{1}{|I_n(x)|} \int_{I_n(x)} f(x) d\mu(x),$$

where $|I_n(x)| = M_n^{-1}$ denotes the length of $I_n(x)$.

A sequence $f = \left(f^{(n)},\ n \in \mathbb{N}\right)$ of functions $f_n \in L_1(G)$ is said to be a dyadic martingale if

(i) $f^{(n)}$ is F_n measurable, for all $n \in \mathbb{N}$,
(ii) $E_n f^{(m)} = f^{(n)}$, for all $n \leq m$

(for details see e.g. [15]).

The maximal function of a martingale f is denoted by

$$f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1$, then it is easy to show that the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is martingale, then the Vilenkin-Fourier coefficients

must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)}(x)\, \overline{\Psi}_i(x)\, d\mu(x). \tag{4}$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from f.

A bounded measurable function a is p-atom, if there exist a dyadic interval I, such that

(i) $\int_I a d\mu = 0$
(ii) $\|a\|_\infty \leq \mu(I)^{-1/p}$
(iii) $\operatorname{supp}(a) \subset I$.

The Hardy martingale spaces $H_p(G_m)$, for $0 < p \leq 1$ have an atomic characterization. Namely, the following theorem is true.

Theorem W (Weisz, [17]). *A martingale $f = \left(f^{(n)},\ n \in \mathbb{N}\right)$ is in H_p $(0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$:*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \tag{5}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of f of the form (5).

When $0 < p \leq 1$, the Hardy martingale space H_p is proper subspace of Lebesgue space L_p. It is well known that for $1 < p < \infty$ the space H_p is nothing but L_p.

The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,

$$\sum_{k=1}^{\infty} \frac{\left|\widehat{f}(k)\right|}{k} \leq c \|f\|_{H_1},$$

where the function f belongs to the Hardy space H_1 and c is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [6] (see also Coifman and Weiss [2]) and for Walsh system it can be found in [8].

Weisz [15, 18] generalized this result for Vilenkin system and proved:

Theorem A (Weisz). *Let $0 < p \leq 2$. Then there is an absolute constant c_p, depend only p, such that*

$$\sum_{k=1}^{\infty} \frac{\left|\widehat{f}(k)\right|^p}{k^{2-p}} \leq c_p \|f\|_{H_p}, \tag{6}$$

for all $f \in H_p$.

Paley [7] proved that the Walsh–Fourier coefficients of a function $f \in L_p\,(1 < p < 2)$ satisfy the condition

$$\sum_{k=1}^{\infty} \left|\widehat{f}\left(2^k\right)\right|^2 < \infty.$$

This results fails to hold $p = 1$. However, it can be verified for functions $f \in L_1$, such that f^* belongs L_1, i.e. $f \in H_1$ (see e.g. Coifman and Weiss [2]).

For the Vilenkin system we have the following theorem.

Theorem B (Weisz [11]). *Let $0 < p \leq 1$. Then there is an absolute constant c_p, depend only p, such that*

$$\left(\sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left|\widehat{f}(jM_k)\right|^2\right)^{1/2} \leq c_p \|f\|_{H_p}, \tag{7}$$

for all $f \in H_p$.

It is well-known that Vilenkin system forms not basis in the space L_1. Moreover, there is a function in the dyadic Hardy space H_1, such that the partial sums of f are not bounded in L_1-norm. However, in Simon [9] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the k-th partial sum of the Walsh–Fourier series of f. (For the trigonometric analogue see Smith [12], for the Vilenkin system by Gát [3]). For the Vilenkin system Simon proved:

Theorem C (Simon [10]). *Let $0 < p < 1$. Then there is an absolute constant c_p, depends only p, such that*

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \tag{8}$$

for all $f \in H_p$.

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [16], Goginava [4], Gogoladze [5], Tephnadze [13].

The main aim of this paper is to prove the following theorem:

Theorem 1. *Let $\{\Phi_n\}_{n=1}^{\infty}$ is any nondecreasing sequence, satisfying the condition $\lim_{n\to\infty}\Phi_n=+\infty$. Then there exists a martingale $f\in H_p$, such that*

$$\sum_{k=1}^{\infty}\frac{\left|\widehat{f}\left(k\right)\right|^{p}\Phi_k}{k^{2-p}}=\infty,\ \textit{for } 0<p\leq 2, \tag{9}$$

$$\sum_{k=1}^{\infty}\frac{\Phi_{M_k}}{M_k^{2/p-2}}\sum_{j=1}^{m_k-1}\left|\widehat{f}\left(jM_k\right)\right|^{2}=\infty,\ \textit{for } 0<p\leq 1 \tag{10}$$

and

$$\sum_{k=1}^{\infty}\frac{\|S_kf\|_{L_{p,\infty}}^{p}\Phi_k}{k^{2-p}}=\infty,\ \textit{for } 0<p<1. \tag{11}$$

Proof. Let $0<p\leq 2$ and $\{\Phi_n\}_{n=1}^{\infty}$ is any nondecreasing, nonnegative sequence, satisfying condition $\lim_{n\to\infty}\Phi_n=\infty$.

For this function $\Phi\left(n\right)$, there exists an increasing sequence $\{\alpha_k\geq 2:k\in\mathbb{N}_+\}$ of the positive integers such that:

$$\sum_{k=1}^{\infty}\frac{1}{\Phi_{M_{\alpha_k}}^{p/4}}<\infty. \tag{12}$$

Let

$$f^{(A)}\left(x\right):=\sum_{\{k;\ \alpha_k<A\}}\lambda_k a_k\left(x\right),$$

where

$$\lambda_k=\frac{1}{\Phi_{M_{\alpha_k}}^{1/4}},\ a_k\left(x\right)=\frac{M_{\alpha_k}^{1/p-1}}{M}\left(D_{M_{\alpha_k+1}}\left(x\right)-D_{M_{\alpha_k}}\left(x\right)\right),$$

and $M=\sup_{n\in\mathbb{N}}m_n$.

It is easy to show that the martingale $f=\left(f^{(1)},f^{(2)},\ldots,f^{(A)},\ldots\right)\in H_p$. Indeed,

$$S_{M_A}\left(a_k\left(x\right)\right)=\begin{cases}a_k\left(x\right) & \alpha_k<A\\ 0, & \alpha_k\geq A,\end{cases} \tag{13}$$

$$\operatorname{supp}(a_k)=I_{\alpha_k},\quad \int_{I_{\alpha_k}}a_k d\mu=0,$$

and

$$\|a_k\|_{\infty}\leq\frac{M_{\alpha_k}^{1/p-1}}{M}M_{\alpha_k+1}\leq M_{\alpha_k}^{1/p}=\mu(\text{supp }a_k)^{-1/p}.$$

If we apply Theorem W and (12) we conclude that $f \in H_p$.

It is easy to show that

$$\widehat{f}(j) = \begin{cases} \frac{1}{M}\frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, & \text{if } j \in \{M_{\alpha_k}, \dots, M_{\alpha_k+1}-1\},\ k = 1, 2 \dots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, \dots, M_{\alpha_k+1}-1\}. \end{cases} \tag{14}$$

First we prove equality (9). Using (14) we can

$$\begin{aligned} \sum_{l=1}^{M_{\alpha_k+1}-1} \frac{\left|\widehat{f}(l)\right|^p \Phi_l}{l^{2-p}} &= \sum_{n=1}^{k} \sum_{l=M_{\alpha_n}}^{M_{\alpha_n+1}-1} \frac{\left|\widehat{f}(l)\right|^p \Phi_l}{l^{2-p}} \\ &\geq \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{\left|\widehat{f}(l)\right|^p \Phi_l}{l^{2-p}} \geq c\Phi_{M_{\alpha_k}} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{\left|\widehat{f}(l)\right|^p}{l^{2-p}} \\ &\geq c\Phi_{M_{\alpha_k}} \frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{1}{l^{2-p}} \geq c\Phi_{M_{\alpha_k}}^{1/2} M_{\alpha_k}^{1-p} \sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{1}{M_{\alpha_k+1}^{2-p}} \\ &\geq c\Phi_{M_{\alpha_k}}^{1/2} M_{\alpha_k}^{1-p} \frac{1}{M_{\alpha_k+1}^{1-p}} \geq c\Phi_{M_{\alpha_k}}^{1/2} \to \infty, \text{ when } k \to \infty. \end{aligned}$$

Next we prove equality (10). Let $0 < p \leq 1$. Using (14) we get

$$\begin{aligned} \sum_{l=1}^{k} M_{\alpha_l}^{2-2/p} \Phi_{M_{\alpha_l}} \sum_{j=1}^{m_{\alpha_l}-1} \left|\widehat{f}(jM_{\alpha_l})\right|^2 &\geq M_{\alpha_k}^{2-2/p} \Phi_{M_{\alpha_k}} \sum_{j=1}^{m_{\alpha_k}-1} \left|\widehat{f}(jM_{\alpha_k})\right|^2 \\ &\geq cM_{\alpha_k}^{2-2/p} \Phi_{M_{\alpha_k}} \sum_{j=1}^{m_{\alpha_k}-1} \frac{M_{\alpha_k}^{2/p-2}}{\Phi_{M_{\alpha_k}}^{1/2}} \\ &\geq c\Phi_{M_{\alpha_k}}^{1/2} \to \infty, \text{ when } k \to \infty. \end{aligned}$$

Finally we prove equality (11). Let $0 < p < 1$ and $M_{\alpha_k} \leq j < M_{\alpha_k+1}$. From (14) we have

$$\begin{aligned} S_j f(x) &= \sum_{l=0}^{M_{\alpha_{k-1}+1}-1} \widehat{f}(l)\psi_l(x) + \sum_{l=M_{\alpha_k}}^{j-1} \widehat{f}(l)\psi_l(x) \\ &= \sum_{\eta=0}^{k-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \widehat{f}(v)\psi_v(x) + \sum_{v=M_{\alpha_k}}^{j-1} \widehat{f}(v)\psi_v(x) \\ &= \sum_{\eta=0}^{k-1} \sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1} \frac{1}{M}\frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}}\psi_v(x) + \sum_{v=M_{\alpha_k}}^{j-1} \frac{1}{M}\frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}\psi_v(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\eta=0}^{k-1} \frac{1}{M} \frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}} \left(D_{M_{\alpha_{\eta+1}}}(x) - D_{M_{\alpha_\eta}}(x) \right) \\
&\quad + \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left(D_j(x) - D_{M_{\alpha_k}}(x) \right) \\
&= I + II.
\end{aligned}
$$

Let $j \in \mathbb{N}_{n_0}$ and $x \in G_m \backslash I_1$. Since $j - M_{\alpha_k} \in \mathbb{N}_{n_0}$ and

$$
D_{j+M_{\alpha_k}}(x) = D_{M_{\alpha_k}}(x) + \psi_{M_{\alpha_k}}(x) D_j(x),
$$

when $j < M_{\alpha_k}$. Combining (2) and (3) we can write

$$
\begin{aligned}
|II| &= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}}(x) \right| \\
&= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}}(x) \psi_{j-M_{\alpha_k}}(x) r_0^{m_0-1}(x) D_1(x) \right| \\
&= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.
\end{aligned} \tag{15}
$$

Applying (2) and condition $\alpha_n \geq 2$ $(n \in \mathbb{N})$ for I we have

$$
I = 0, \text{ for } x \in G_m \backslash I_1. \tag{16}
$$

It follows that

$$
|S_j f(x)| = |II| = \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, \text{ for } x \in G_m \backslash I_1.
$$

Hence

$$
\begin{aligned}
\|S_j(f(x))\|_{L_{p,\infty}} &\geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left(x \in G_m : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p} \\
&\geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left(x \in G_m \backslash I_1 : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p} \qquad (17) \\
&= \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} |G_m \backslash I_1| \\
&\geq \frac{c M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.
\end{aligned}
$$

Combining (1) and (17) we have

$$\begin{aligned}
\sum_{j=1}^{M_{\alpha_k+1}-1}\frac{\|S_j(f(x))\|_{L_{p,\infty}}^p\Phi_j}{j^{2-p}} &\geq \sum_{j=M_{\alpha_k}}^{M_{\alpha_k+1}-1}\frac{\|S_j(f(x))\|_{L_{p,\infty}}^p\Phi_j}{j^{2-p}}\\
&\geq \Phi_{M_{\alpha_k}}\sum_{\left\{j:M_k\leq j\leq M_{k+1},\ j\in\mathbb{N}_{n_0}\right\}}\frac{\|S_j(f(x))\|_{L_{p,\infty}}^p}{j^{2-p}}\\
&\geq c\Phi_{M_{\alpha_k}}\frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}}\sum_{\left\{j:M_k\leq j\leq M_{k+1},\ j\in\mathbb{N}_{n_0}\right\}}\frac{1}{j^{2-p}}\\
&\geq c\Phi_{M_{\alpha_k}}^{3/4}M_{\alpha_k}^{1-p}\sum_{\left\{j:M_k\leq j\leq M_{k+1},\ j\in\mathbb{N}_{n_0}\right\}}\frac{1}{M_{\alpha_k+1}^{2-p}}\\
&\geq c\frac{\Phi_{M_{\alpha_k}}^{3/4}}{M_{\alpha_k+1}}\sum_{\left\{j:M_k\leq j\leq M_{k+1},\ j\in\mathbb{N}_{n_0}\right\}}1\\
&\geq c\Phi_{M_{\alpha_k}}^{3/4}\to\infty,\ \text{when } k\to\infty. \quad \square
\end{aligned}$$

References

[1] G. N. Agaev, N. Y. Vilenkin, G. M. Dzhafarli, and A. I. Rubinshteĭn. *Multiplikativnye sistemy funktsii i garmonicheskii analiz na nulmernykh gruppakh*. "Èlm", Baku, 1981.

[2] R. R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83(4):569–645, 1977.

[3] G. Gát. Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hungar.*, 61(1-2):131–149, 1993.

[4] U. Goginava and G. L. D. Strong convergence of cubic partial sums of two-dimensional Walsh–Fourier series. In *Constructive Theory of Functions*, pages 108–117. Prof. Marin Drinov Academic Publishing House, Sofia, 2012.

[5] L. D. Gogoladze. The strong summation of Fourier series. *Sakharth. SSR Mecn. Akad. Moambe*, 52:287–292, 1968.

[6] G. H. Hardy and J. E. Littlewood. Notes on the Theory of Series (XIII): Some New Properties of Fourier Constants. *J. London Math. Soc.*, S1-6(1):3–9, 1931.

[7] R. E. A. C. Paley. A Remarkable Series of Orthogonal Functions (II). *Proc. London Math. Soc.*, S2-34(1):265–279.

[8] F. Schipp, W. R. Wade, and P. Simon. *Walsh series*. Adam Hilger Ltd., Bristol, 1990. An introduction to dyadic harmonic analysis, With the collaboration of J. Pál.

[9] P. Simon. Strong convergence of certain means with respect to the Walsh-Fourier series. *Acta Math. Hungar.*, 49(3-4):425–431, 1987.

[10] P. Simon. Strong convergence theorem for Vilenkin-Fourier series. *J. Math. Anal. Appl.*, 245(1):52–68, 2000.

[11] P. Simon and F. Weisz. Paley type inequalities for Vilenkin-Fourier coefficients. *Acta Sci. Math. (Szeged)*, 63(1-2):107–124, 1997.

[12] B. Smith. A strong convergence theorem for $H^1(\mathbf{T})$. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, volume 995 of *Lecture Notes in Math.*, pages 169–173. Springer, Berlin, 1983.

[13] G. Tephnadze. Strong convergence of two-dimensional walsh-fourier series. to appear.
[14] N. Vilenkin. On a class of complete orthonormal systems. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]*, 11:363–400, 1947.
[15] F. Weisz. *Martingale Hardy spaces and their applications in Fourier analysis*, volume 1568 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.
[16] F. Weisz. Strong convergence theorems for two-parameter Walsh-Fourier and trigonometric-Fourier series. *Studia Math.*, 117(2):173–194, 1996.
[17] F. Weisz. *Summability of multi-dimensional Fourier series and Hardy spaces*, volume 541 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2002.
[18] F. Weisz. Hardy-Littlewood inequalities for Ciesielski-Fourier series. *Anal. Math.*, 31(3):217–233, 2005.

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