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THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF ONE-DIMENSIONAL VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate (H_p, L_p)-type inequalities for maximal operators of logarithmic means of one-dimensional bounded Vilenkin-Fourier series.

1. Introduction

In one-dimensional case the weak type inequality

$$\mu\left(\sigma^* f > \lambda\right) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)$$

can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujji [3] and Simon [12] verified that σ^* is bounded from H_1 to L_1. Weisz [17] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p > 1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava ([7], see also [2]).

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mention, for instance, the paper by Szász [14] and Yabuta [19]. This means with respect to the Walsh and Vilenkin systems was discussed by Simon[13] and Gát[4].

Móricz and Siddiqi[9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. The case when $q_k = 1/k$ is excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. Among there, they gave a negative answer to the question of Móricz and

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Siddiqi [9]. Gát and Goginava [6] proved that for each measurable function $\phi(u) = \circ\left(u\sqrt{\log u}\right)$ there exists an integrable function f, such that

$$\int_{G_m} \phi\left(|f(x)|\right) d\mu\ (x) < \infty$$

and there exist a set with positive measure, such that the Walsh-logarithmic means of the function diverge on this set.

The main aim of this paper is to investigate (H_p, L_p)-type inequalities for the maximal operators of Riesz and Nörlund logarithmic means of one-dimensional Vilenkin-Fourier series. We prove that the maximal operator R^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. We also shows that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$ for which

$$\|R^* f\|_{L_p} = +\infty.$$

For the Nörlund logarithmic means we prove that when $0 < p \leq 1$ there exists a martingale $f \in H_p$ for which

$$\|L^* f\|_{L_p} = +\infty.$$

Analogical theorems for Walsh-Paley system is proved in [8].

2. Definitions and notation

Let N_+ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the addition group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of the groups Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_{m_k} with $\mu(G_m) = 1$.

If the sequence m is bounded then G_m is called a bounded Vilenkin group, else it is called an unbounded one. *In this paper we discuss bounded Vilenkin groups only.* The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots)\ \left(x_i \in Z_{m_j}\right).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x) := G_m$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in N)$$

Denote $I_n := I_n(0)$ for $n \in N_+$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1,\ M_{k+1} := m_k M_k \quad (k \in N),$$

then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$ where $n_j \in Z_{m_j}$ $(j \in N_+)$ and only a finite number of n_js differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, define the complex valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad \left(i^2 = -1, x \in G_m, k \in N\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in N).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 15].

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \quad (k \in N),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad (n \in N_+, S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in N_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in N_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n. \end{cases}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \quad (0 < p < \infty).$$

The $\sigma-$algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ is denoted by $F_n (n \in N)$. Denote by $f = \left(f^{(n)}, n \in N\right)$ a martingale with respect to $F_n (n \in N)$ (for details see e.g. [16]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} \left| f^{(n)} \right|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingale for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in N)$ is a martingale.

If $f = \left(f^{(n)}, n \in N\right)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x)\, \overline{\Psi}_i(x)\, d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in N)$ obtained from f.

In the literature, there is the notion of Riesz's logarithmic means of the Fourier series. The n-th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{k},$$

where

$$l_n := \sum_{k=1}^{n} (1/k).$$

Let $\{q_k : k > 0\}$ be a sequence of nonnegative numbers. The n-th Nö rlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=1}^{n} q_k.$$

If $q_k = 1/k$, then we get Nörlund logarithmic means

$$L_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{n-k}$$

It is a kind of "reverse" Riesz's logarithmic mean. In this paper we call these means logarithmic means.

For the martingale f we consider the following maximal operators of

$$R^*f(x) := \sup_{n\in N} |R_n f(x)|,$$
$$L^*f(x) := \sup_{n\in N} |L_n f(x)|,$$
$$\sigma^*f(x) := \sup_{n\in N} |\sigma_n f(x)|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a d\mu = 0$,
b) $\|a\|_\infty \leq \mu(I)^{-1/p}$,
c) $\operatorname{supp}(a) \subset I$.

3. Formulation of main results

Theorem 1. *Let $p > 1/2$. Then the maximal operator R^* is bounded from the Hardy space H_p to the space L_p.*

Theorem 2. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|R^*f\|_p = +\infty.$$

Corollary 1. *Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that*

$$\|\sigma^*f\|_p = +\infty.$$

Theorem 3. *Let $0 < p \leq 1$. Then there exists a martingale $f \in L_p$ such that*

$$\|L^*f\|_p = +\infty.$$

4. Auxiliary propositions

Lemma 1. [18] *A martingale $f = \left(f^{(n)}, n \in N\right)$ is in $H_p\,(0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in N)$ of p-atoms and a sequence $(\mu_k, k \in N)$ of a real numbers such that for every $n \in N$*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \tag{1}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{K=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (1).

5. Proof of the theorems

Proof of Theorem 1: Using Abel transformation we obtain

$$R_n f\left(x\right)=\frac{1}{l_n}\sum_{j=1}^{n-1}\frac{\sigma_j f\left(x\right)}{j+1}+\frac{\sigma_n f\left(x\right)}{l_n},$$

Consequently,

$$L^{*}f\leq c\sigma^{*}f. \tag{2}$$

On the other hand Weisz[17] proved that σ^{*} is bounded from the Hardy space H_p to the space L_p when $p>1/2$. Hence, from (2) we conclude that R^{*} is bounded from the martingale Hardy space H_p to the space L_p when $p>1/2$. □

Proof of Theorem 2: Let $\{\alpha_k : k\in N\}$ be an increasing sequence of the positive integers such that

$$\sum_{k=0}^{\infty}\alpha_k^{-p/2}<\infty, \tag{3}$$

$$\sum_{\eta=0}^{k-1}\frac{\left(M_{2\alpha_\eta}\right)^{1/p}}{\sqrt{\alpha_\eta}}<\frac{\left(M_{2\alpha_k}\right)^{1/p}}{\sqrt{\alpha_k}}, \tag{4}$$

$$\frac{\left(M_{2\alpha_{k-1}}\right)^{1/p}}{\sqrt{\alpha_{k-1}}}<\frac{M_{\alpha_k}}{\alpha_k^{3/2}}. \tag{5}$$

We note that such an increasing sequence $\{\alpha_k : k\in N\}$ which satisfies conditions (3)-(5) can be constructed.

Let

$$f^{(A)}\left(x\right)=\sum_{\{k;\ 2\alpha_k<A\}}\lambda_k a_k,$$

where

$$\lambda_k=\frac{m_{2\alpha_k}}{\sqrt{\alpha_k}}$$

and

$$a_k\left(x\right)=\frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}}\left(D_{M_{(2\alpha_k+1)}}\left(x\right)-D_{M_{2\alpha_k}}\left(x\right)\right).$$

It is easy to show that

$$\left\|a_k\right\|_{\infty}\leq\frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}}M_{2\alpha_k+1}\leq\left(M_{2\alpha_k}\right)^{1/p}=\left(\mu(\text{supp }a_k)\right)^{-1/p},$$

$$S_{M_A}a_k\left(x\right)=\begin{cases}a_k\left(x\right), & 2\alpha_k<A,\\ 0, & 2\alpha_k\geq A.\end{cases} \tag{6}$$

$$f^{(A)}(x)=\sum_{\{k;2\alpha_k<A\}}\lambda_k a_k=\sum_{k=0}^{\infty}\lambda_k S_{M_A}a_k(x),$$
$$\operatorname{supp}(a_k)=I_{2\alpha_k},$$
$$\int_{I_{2\alpha_k}}a_k d\mu=0.$$

From (3) and Lemma 1 we conclude that $f=\left(f^{(n)},n\in N\right)\in H_p$. Let

$$q_A^s=M_{2A}+M_{2s}-1,\quad A>S.$$

Then we can write

$$R_{q^s_{\alpha_k}}f(x)=\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=1}^{q^s_{\alpha_k}}\frac{S_jf(x)}{j}\tag{7}$$
$$=\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=1}^{M_{2\alpha_k}-1}\frac{S_jf(x)}{j}+\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}}\frac{S_jf(x)}{j}=I+II.$$

It is easy to show that

$$\widehat{f}(j)=\begin{cases}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}},\ \text{if } j\in\{M_{2\alpha_k},\dots,\ M_{2\alpha_k+1}-1\},k=0,1,2\dots,\\ 0,\qquad j\notin\bigcup\limits_{k=1}^{\infty}\{M_{2\alpha_k},\dots,\ M_{2\alpha_k+1}-1\}.\end{cases}\tag{8}$$

Let $j<M_{2\alpha_k}$. Then from (4) and (8) we have

$$|S_jf(x)|\le\sum_{\eta=0}^{k-1}\sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1}\left|\widehat{f}(v)\right|\tag{9}$$
$$\le\sum_{\eta=0}^{k-1}\sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1}\frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}}\le c\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}}\le\frac{cM_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

Consequently,

$$|I|\le\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=1}^{M_{2\alpha_k}-1}\frac{|S_jf(x)|}{j}\le\frac{c}{\alpha_k}\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}\sum_{j=1}^{M_{2\alpha_k}-1}\frac{1}{j}\le c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.\tag{10}$$

Let $M_{2\alpha_k} \leq j \leq q^s_{\alpha_k}$. Then we have the following

$$S_j f(x) = \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \widehat{f}(v)\psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \widehat{f}(v)\psi_v(x)$$
$$= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x)\right)$$
$$+ \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left(D_j(x) - D_{M_{2\alpha_k}}(x)\right). \tag{11}$$

This gives that

$$II = \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}} \frac{1}{j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x)\right)\right)$$
$$+ \frac{1}{l_{q^s_{\alpha_k}}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}} \frac{\left(D_j(x) - D_{M_{2\alpha_k}}(x)\right)}{j} = II_1 + II_2. \tag{12}$$

To discuss II_1, we use (4). Thus, we can write that

$$|II_1| \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq \frac{cM_{2\alpha_{k-1}}}{\sqrt{\alpha_{k-1}}}. \tag{13}$$

Since,

$$D_{j+M_{2\alpha_k}}(x) = D_{M_{2\alpha_k}}(x) + \psi_{M_{2\alpha_k}}(x) D_j(x), \qquad \text{when} \quad j < M_{2\alpha_k}, \tag{14}$$

for II_2 we have,

$$II_2 = \frac{1}{l_{q^s_{\alpha_k}}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=0}^{M_{2s}} \frac{D_{j+M_{2\alpha_k}}(x) - D_{M_{2\alpha_k}}(x)}{j+M_{2\alpha_k}}$$
$$= \frac{1}{l_{q^s_{\alpha_k}}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}}. \tag{15}$$

We write

$$R_{q^s_{\alpha_k}} f(x) = I + II_1 + II_2,$$

Then by (5), (7), (10) and (12)-(15) we have

$$\left|R_{q^s_{\alpha_k}} f(x)\right| \geq |II_2| - |I| - |II_1| \geq |II_2| - c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}$$
$$\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| - c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}$$

Let $0<p\leq 1/2$, $x\in I_{2s}\backslash I_{2s+1}$ for $s=[\alpha_k/2],\dots,\alpha_k$. Then it is evident

$$\left|\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{j+M_{2\alpha_k}}\right|\geq\frac{cM_{2s}^2}{M_{2\alpha_k}}$$

Hence, we can write

$$\begin{aligned}\left|R_{q_{\alpha_k}^s}f(x)\right|&\geq\frac{c}{\alpha_k}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\frac{cM_{2s}^2}{M_{2\alpha_k}}-c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}\\&\geq\frac{cM_{2\alpha_k}^{1/p-2}M_{2s}^2}{\alpha_k^{3/2}}-c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}\geq\frac{cM_{2\alpha_k}^{1/p-2}M_{2s}^2}{\alpha_k^{3/2}}.\end{aligned}$$

Then we have

$$\begin{aligned}\int_{G_m}|R^*f(x)|^p\,d\mu(x)&\geq\sum_{s=[\alpha_k/2]}^{\alpha_k}\int_{I_{2s}\backslash I_{2s+1}}\left|R_{q_{\alpha_k}^s}f(x)\right|^p d\mu(x)\\&\geq\sum_{s=[\alpha_k/2]}^{\alpha_k}\int_{I_{2s}\backslash I_{2s+1}}\left(\frac{cM_{2\alpha_k}^{1/p-2}M_{2s}^2}{\alpha_k^{3/2}}\right)^p d\mu(x)\\&\geq c\sum_{s=[\alpha_k/2]}^{\alpha_k}\frac{M_{2\alpha_k}^{1-2p}M_{2s}^{2p-1}}{\alpha_k^{3p/2}}\\&\geq\begin{cases}\frac{2^{\alpha_k(1-2p)}}{\alpha_k^{3p/2}}, & \text{when } 0<p<1/2\\ c\alpha_k^{1/4}, & \text{when } p=1/2\end{cases}\to\infty\text{ , when }k\to\infty.\end{aligned}$$

which completes the proof of the Theorem 2. □

Proof of Theorem 3: We write

$$\begin{aligned}L_{q_{\alpha_k}^s}f(x)&=\frac{1}{l_{q_{\alpha_k},s}}\sum_{j=1}^{q_{\alpha_k}^s}\frac{S_jf(x)}{q_{\alpha_k}^s-j}\\&=\frac{1}{l_{q_{\alpha_k}^s}}\sum_{j=1}^{M_{2\alpha_k}-1}\frac{S_jf(x)}{q_{\alpha_k}^s-j}+\frac{1}{q_{\alpha_k}^s}\sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s}\frac{S_jf(x)}{q_{\alpha_k}^s-j}=III+IV.\end{aligned}\tag{16}$$

Since (see 9)

$$|S_jf(x)|\leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}},\quad j<M_{2\alpha_k}.$$

For III we can write

$$|III|\leq\frac{c}{\alpha_k}\sum_{j=0}^{M_{2\alpha_{k-1}}}\frac{1}{q_{\alpha_k}^s-j}\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}\leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.\tag{17}$$

Using (11) we have

$$IV=\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}}\frac{1}{q_{\alpha_k,s}-j}\left(\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}}\left(D_{M_{2\alpha_\eta+1}}(x)-D_{M_{2\alpha_\eta}}(x)\right)\right) \tag{18}$$
$$+\frac{1}{l_{q^s_{\alpha_k}}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}}\frac{\left(D_j(x)-D_{M_{2\alpha_k}}(x)\right)}{q^s_{\alpha_k}-j}=IV_1+IV_2.$$

Applying (4) in IV_1 we have

$$|IV_1|\leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}, \tag{19}$$

From (14) we obtain

$$IV_2=\frac{1}{l_{q_{\alpha_k,s}}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\psi_{M_{2\alpha_k}}\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{M_{2s}-j}. \tag{20}$$

Let $x\in I_{2s}\backslash I_{2s+1}$. Then $D_j(x)=j$, $j<M_{2s}$. Consequently,

$$\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{M_{2s}-j}=\sum_{j=0}^{M_{2s}-1}\frac{j}{M_{2s}-j}=\sum_{j=0}^{M_{2s}-1}\left(\frac{M_{2s}}{M_{2s}-j}-1\right)\geq csM_{2s}.$$

Then

$$|IV_2|\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s},\quad x\in I_{2s}\backslash I_{2s+1}. \tag{21}$$

Combining (5), (16)-(21) for $x\in I_{2s}\backslash I_{2s+1}$, $s=[\alpha_k/2],\dots,\alpha_k$, and $0<p\leq 1$ we have

$$\left|L_{q^s_{\alpha_k}}f(x)\right|\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s}-c\frac{M_{\alpha_k}}{\alpha_k}\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s}$$

Then

$$\begin{aligned}\int_{G_m}|L^*f(x)|^p\,d\mu(x) &\geq \sum_{s=[m_k/2]}^{m_k}\int_{I_{2s}\setminus I_{2s+1}}|L^*f(x)|^p\,d\mu(x)\\ &\geq \sum_{s=[m_k/2]}^{m_k}\int_{I_{2s}\setminus I_{2s+1}}\left|L_{q^s_{\alpha_k}}f(x)\right|^p d\mu(x)\\ &\geq c\sum_{s=[m_k/2]}^{m_k}\int_{I_{2s}\setminus I_{2s+1}}\left(\frac{M^{1/p-1}_{2\alpha_{k-1}}}{\alpha_k^{3/2}}sM_{2s}\right)^p d\mu(x)\\ &\geq c\sum_{s=[m_k/2]}^{m_k}\frac{M^{1-p}_{2\alpha_{k-1}}}{\alpha_k^{p/2}}M^{p-1}_{2s}\\ &\geq \begin{cases}\frac{2^{\alpha_k(1-p)}}{\alpha_k^{p/2}}, & \text{when } 0<p<1,\\ c\sqrt{\alpha_k}, & \text{when } p=1,\end{cases}\ \to\infty,\ \text{when } k\to\infty.\end{aligned}$$

Theorem 3 is proved. □

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