Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 9–12
www.emis.de/journals

ON THE LOCAL LIMIT THEOREM FOR GENERAL LATTICE DISTRIBUTION

V. O. PETEN'KO

Abstract. The scheme of series of lattice casual vectors analogical to the probable is considered in the paper. A local limit theorem have been received, and the indicated directions of its use in differential methods theory of mathematical physics problems solving were received.

A local limit theorem for lattice distributions, and their quasiprobable analogues are successfully used in the approximate solution of some kinds of mathematical physics problems [1-3].

Quasiprobable distribution on the plain (according to Y. P. Studnyev [4]) on the whole is the complex number set $\{p(k,l)\}$, for which

$$\sum_{k,l} p(k,l) = 1; \qquad \sum_{k,l} |p(k,l)| < +\infty.$$

Let $\{(\xi_{nn}, \eta_{nn})\}$ be sequence of a mutually independent series of probable vectors (first index - series number, the second is changed from 1 to n), which are equally distributed on $\{kh_1, lh_2\}$ lattice, and let $\{p(n,k,l)\}$ be a quasiprobable vector distribution in series frames in the meaning of correspondence $p(n,k,l) = P\{(\xi_{nj}, \eta_{nj}) = (kh_1, lh_2)\}$, where $k, l \in Z; h_1, h_2 > 0; j = \overline{1,n}$. Then

$$w(t,s) = \sum_{k,l} e^{i(tkh_1 + slh_2)} p(n,k,l)$$

is a Fourier-Stieltjes distribution transformation $\{p(n,k,l)\}$.

As in the theory of probabilities, the formula of reverse is being proved:

$$p(n,k,l) = \frac{h_1 h_2}{4\pi^2} \iint_D e^{-i(tkh_1 + slh_2)} w(t,s)\, dt\, ds,$$

where D rectangle is in the form of $\left[-\frac{\pi}{h_1}, \frac{\pi}{h_1}, -\frac{\pi}{h_2}, \frac{\pi}{h_2}\right]$.

If $\{p_n(k,l)\}$ is $\left(\sum_{j=1}^{n} \xi_{nj}, \sum_{j=1}^{n} \eta_{nj}\right)$ sum vector distribution, then the reverse formula is in

$$P_n(n,k,l) = \frac{h_1 h_2}{4\pi^2} \iint_D e^{-i(tkh_1 + slh_2)} w(t,s)\, dt\, ds, \tag{1}$$

2000 *Mathematics Subject Classification.* 60B12, 60F05.
Key words and phrases. Lattice distribution, limit theorem.

where $\{P_n(k,l)\}$ distribution is $n-$divisible fold of the set $\{p_n(k,l)\}$ with itself. Let, further

$$\alpha_{r,q} = \sum_{k,l} (kh_1)^r (lh_2)^q p(n,k,l)$$

be an initial moment with (r,q) order of $\{p(n,k,l)\}$ distribution.

Let us consider two auxiliary lemmas.

Lemma 1. *Let for the casual vector with distribution conditions exist :*

1) $\alpha_{1,0} = \alpha_{0,1} = \alpha_{3,0} = \alpha_{2,1} = \alpha_{1,2} + \alpha_{0,3} = \alpha_{3,1} = \alpha_{2,2} = \alpha_{1,3} = 0$;

2) $\alpha_{2,0} = -\frac{2a}{\sqrt{n}}, \alpha_{1,1} = -\frac{2b}{\sqrt{n}}, \alpha_{0,2} = -\frac{2c}{\sqrt{n}}, at^2+bts+cs^2$ *is a positively determined square form;*

3) $\alpha_{4,0} = \alpha_{0,4} = 4!A, A > 0$;

4) $a_{r,q}$ *exist, when* $r = q = 5$.

Then

$$\left| w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) - e^{at^2+bts+cs^2-A\left(t^4+s^4\right)} dtds \right| = O\left(\frac{|t|^5+|s|^5}{\sqrt[4]{n}}\right), n \to \infty \tag{2}$$

correlation comes true.

Proof. We should note, that the role of moments in apportion is the same, as for characteristic functions in the analogical situation. That is why

$$w(t,s) = 1 + \frac{1}{\sqrt{n}}\left(at^2+bts+cs^2\right) - A\left(t^4+s^4\right) + O\left(|t|^5+|s|^5\right), t,s \to 0.$$

But then

$$w\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) = 1 + \frac{at^2+bts+cs^2}{n} - \frac{A\left(t^4+s^4\right)}{n} + O\left(\frac{|t|^5+|s|^5}{n\sqrt[4]{n}}\right), \quad n \to \infty;$$

or

$$w\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) = e^{\frac{1}{n}\left(at^2+bts+cs^2\right) - \frac{A\left(t^4+s^4\right)}{n}} + O\left(\frac{|t|^5+|s|^5}{\sqrt[4]{n}}\right), \quad n \to \infty$$

for any fix couple $(t,s) \in R_2$. Hence, it follows, after the $n-$th power production of both parts of the last equality we receive (2). Then Lemma 1 is proved.

Let's present$w(t,s)$ into

$$w(t,s) = w_0(t,s) + \frac{1}{\sqrt{n}}\left(at^2+bts+cs^2\right)$$

form. □

Lemma 2. *Let in* $\{\xi_{nn}, \eta_{nn}\}$ *series sequence, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series Lemma 1 conditions and condition :* $|w_0(t,s)| < 1$, *when* $(t,s) \in D \backslash \{0,0\}$, *are executed. Then* $A_0 > 0$ *exists that in* $\sqrt[4]{n}D = \left[-\frac{\sqrt[4]{n}\pi}{h_1}, \frac{\sqrt[4]{n}\pi}{h_1}, -\frac{\sqrt[4]{n}\pi}{h_2}, \frac{\sqrt[4]{n}\pi}{h_2}\right]$ *rectangle execute*

$$\left| w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) \right| \le e^{at^2+bts+cs^2-A_0\left(t^4+s^4\right)}. \tag{3}$$

Proof. In the work [3] it has been shown that by Lemma 2 conditions $A_0 > 0$ would exist that $|w_0(t,s)| \le e^{-A_0\left(t^4+s^4\right)}, (t,s) \in D$.

Then in D executes

$$|w(t,s)| \le |w_0(t,s)| + \frac{at^2+bts+cs^2}{\sqrt{n}} \le e^{\frac{at^2+bts+cs^2}{\sqrt{n}} - A_0\left(t^4+s^4\right)},$$

whence directly (3) comes out. □

These lemmas permit to prove such a theorem.

Theorem. *Let in $\{(\xi_{nn}, \eta_{nn})\}$ series sequences, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series conditions :*

1) $\alpha_{1,0} = \alpha_{0,1} = \alpha_{3,0} = \alpha_{2,1} = \alpha_{1,2} = \alpha_{0,3} = \alpha_{3,1} = \alpha_{2,2} = \alpha_{1,3} = 0$;

2) $\alpha_{2,0} = -\frac{2a}{\sqrt{n}}, \alpha_{1,1} = -\frac{2b}{\sqrt{n}}, at^2 + bts + cs^2$ *is a positively determined square form;*

3) $\alpha_{4,0} = \alpha_{0,4} = 4!A, A > 0$ *;*

4) $\alpha_{r,q}$ *exist, when* $r + q = 5$ *;*

5) $|w_0(t,s)| < 1$*, when* $(t,s) \in D \backslash \{0,0\}$.

Then evenly on $k \in Z$ *by* $n \to \infty$ *:*

$$(4) \qquad \sqrt{n}\left(\frac{p_n(k,l)}{h_1 h_2} - \frac{1}{4\pi^2}\iint e^{-i(tkh_1+slh_2)+n\left(\frac{at^2+bts+cs^2}{\sqrt{n}} - A\left(t^4+s^4\right)\right)} dtds\right) \to 0.$$

Proof. Use the reverse formula (1) and in integrals of the left part of (4) correlation defined through R_n, execute the substitute: $t \to \frac{t}{\sqrt[4]{n}}, s \to \frac{s}{\sqrt[4]{n}}$.

We receive

$$4\pi^2 R_n = \iint\limits_{\sqrt[4]{n}D} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)} w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) dt\, ds -$$

$$- \iint e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right) + at^2+bts+cs^2 - A\left(t^4+s^4\right)} dt\, ds == I_1 + I_2 - I_3,$$

where

$I_1 = \iint\limits_{\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)} \left(w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) - e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}\right) dt\, ds,$

$I_2 = \iint\limits_{\sqrt[4]{n}D\backslash\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)} w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) dt\, ds,$

$I_3 = \iint\limits_{R_2\backslash\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right) + at^2+bts+cs^2-A\left(t^4+s^4\right)};$

$\Delta = \left\{(t,s) : |t| \le n^{\lambda}, 0 < \lambda < \frac{1}{28}\right\}.$

Integral I_1 estimation comes out of Lemma 1. Really, according to (2) $B > 0$ exists, that

$$\left|w^k\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) - e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}\right| \le B\left(\frac{|t|^5 + |s|^5}{\sqrt[4]{n}}\right), (t,s) \in \Delta.$$

Then

$$|I_1| \le \frac{B}{\sqrt[4]{n}} \iint\limits_{\Delta} \left(|t|^5 + |s|^5\right) dtds = \frac{4B}{3} n^{7\lambda - \frac{1}{4}} = \frac{4B}{3} n^{-7\left(\frac{1}{28} - \lambda\right)}.$$

So as $\lambda < \frac{1}{28}$, then $I_1 \to 0$ by $n \to \infty$.

Out of the Lemma 2 the I_2 estimation comes out. Using (3), we receive

$$|I_2| \le \iint\limits_{\sqrt[4]{n}D\backslash\Delta} e^{at^2+bts+cs^2-A\left(t^4+s^4\right)} dtds.$$

The positively determined square form permits an upper estimation:

$$at^2 + bts + cs^2 \le a_0\left(t^2 + s^2\right), \quad a_0 > 0.$$

Then

$$|I_2| \le \iint\limits_{\sqrt[4]{n}D\backslash\Delta} e^{a_0\left(t^2+s^2\right)-A\left(t^4+s^4\right)}\,dt\,ds \le 4\left(\int\limits_{n^\lambda}^{\infty} e^{a_0t^2-A_0t^4}dt\right)^2.$$

We should take into consideration that by sufficiently large n for $t \ge n^\lambda$

$$4A_0t^3 - 2a_0t > 4A_0n^{3\lambda} - 2a_0n^\lambda,$$

I_2 integral permits the further estimation :

$|I_2| \le \frac{1}{(2A_0n^{3\lambda}-a_0n^\lambda)^2}\int\limits_{n_\lambda}^{\infty}\left(\left(4A_0t^3-2a_0t\right)e^{a_0t^2-A_0t^4}dt\right)^2 = \left(\frac{e^{a_0n^{2\lambda}-A_0n^{4\lambda}}}{2A_0n^{3\lambda}-a_0n^\lambda}\right)^2,$

therefore, by $n \to \infty$ $I_2 \to 0$.

I_3 integral estimation is analogical to the second stage of I_2 integral estimation. The Theorem is proved. □

The received theorem could easy be generalized in case of quasiprobable lattice distributions with Fourier-Stieltjes transformation in

$$w\left(t,s\right) = e^{\Psi_2(t,s)+\Psi_4(t,s)+\ldots+\Psi_{2q-2}(t,s)-\Psi_{2q}(t,s)}$$

form,where $\Psi_2\left(t,s\right), \Psi_4\left(t,s\right), \Psi_{2q}\left(t,s\right)$ are positively-determined forms with orders indicated by indexes.

Such generalization could be used to the approximate problems solving linked with the evolutional equation in the form of:

$$\frac{\partial u(x,y,\tau)}{\partial\tau} = \left(\left(-1\right)^{q+1}\Psi_{2q}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right) - \left(-1\right)^q\Psi_{2q-2}\left(i\frac{\partial}{dx}, i\frac{\partial}{\partial y}\right) - \right.$$
$$\left. - \ldots - \Psi_2\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)\right)u\left(x,y,\tau\right)$$

according to the scheme, for example, in [3].

References

[1] Peten'ko V. O., On the Some Local Qualities of Stable in Wide Sense Distributions on the Plane. *Ukrainian Mathematical Journal*, **9** (1979), 447–454.

[2] Peten'ko V. O., On the One Differential Solution of the Heat Conduction Problem in the Segment and Half-axis. *Uzhgorod State University Scientific Herald, Mathematical Series*, **3** (1994), 165–168.

[3] Peten'ko V. O., On the One Differential Solution of the Cauchy Problem for Some Class of the Parabolic According to G. E. Shilov Systems, *Mathematical Methods and Physical-Mechanical Fields*, **9** 1979, 20–25.

[4] Studnyev Y. P., On Some Theory of Probabilities Limit Theorem Generalization, *Theory of Probabilities and its Application*, **4** 1967, 729-734.

Received October 3, 2000.

Uzhhorod National University
88000 Uzhhorod, Ukraine