Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
15 (1999), 27–34
www.bgytf.hu/~amapn

ON THE A.E. CONVERGENCE OF FOURIER SERIES ON UNBOUNDED VILENKIN GROUPS

G. Gát

Abstract. It is well known that the 2^nth partial sums of the Walsh-Fourier series of an integrable function converges a.e. to the function. This result has been proved [Sto] by techniques known in the martingale theory. The author gave "purely dyadic harmonic analysis" proof of this in the former volume of this journal [Gát]. The Vilenkin groups are generalizations of the Walsh group. We prove the a.e. convergence $S_{M_n} f \to f \, (n \to \infty)$, $f \in L^1(G_m)$ even in the case when G_m is an unbounded Vilenkin group. The nowelty of this proof is that we use techniques, which are elementary in dyadic harmonic analysis. We do not use any technique in martingale theory used in the former proof [Sto].

First we give a brief introduction to the Vilenkin systems. The Vilenkin systems were introduced in 1947 by N.Ja. Vilenkin (see e.g. [Vil]). Let $m := (m_k, k \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, \dots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} be the m_k -th discrete cyclic group, i.e. Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operation is the $\mod m_k$ addition and every subset is open. Haar measure on Z_{m_k} is given in the way that the measure of a singleton is $1/m_k \, (k \in \mathbf{N})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

The elements $x \in G_m$ can be represented by the sequence $x = (x_i, i \in \mathbf{N})$, where $x_i \in Z_{m_i} \, (i \in \mathbf{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ) and the topology is the product measure and topology , resp. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbf{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. G_m is a (bounded or not) Vilenkin group in this paper.

Give a base for the neighborhoods of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbf{P} := \mathbf{N} \setminus \{0\}$. Denote by $0 = (0, i \in \mathbf{N}) \in G_m$ the nullelement of $G_m, I_n := I_n(0) \, (n \in \mathbf{N})$. Denote by $L^p(G_m) \, (1 \le p \le \infty)$ the usual Lebesgue

1991 *Mathematics Subject Classification.* primary: 42C10, secondary: 43A75.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. F020334 and by the Hungarian "Művelődési és Közoktatási Minisztérium", grant no. FKFP 0710/1997

spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbf{N})$ $(E_{-1}f := 0\,(f \in L^1).)$ If m is bounded then $\mathcal{I} := \{I_n(x) : x \in G_m, n \in \mathbf{N}\}$ is called the set of intervalls on G_m. If the sequence m is not bounded, then we define the set of intervalls in a different way ([Sim]), that is we have "more" intervalls than in the bounded case.

A set $I \subset G_m$ is called an interval if for some $x \in G_m$ and $n \in \mathbf{N}, I$ is of the form $I = \bigcup_{k \in U} I_n(x,k)$ where U is obtained from

$$U^0_{n,0} = \Big\{0,\ldots,m_n-1\Big\}, U^1_{n,0} = \Big\{0,\ldots,\Big[\frac{m_n}{2}\Big]-1\Big\}, U^1_{n,1} = \Big\{\Big[\frac{m_n}{2}\Big],\ldots,m_n-1\Big\}$$

$$U^2_{n,0} = \Big\{0,\ldots,\Big[\frac{[m_n/2]-1}{2}\Big]-1\Big\}, U^2_{n,1} = \Big\{\Big[\frac{[m_n/2]-1}{2}\Big],\ldots,\Big[\frac{m_n}{2}\Big]-1\Big\},\ldots$$

etc, where $I_n(x,k) := \{y \in G_m : y_j = x_j (j < n), y_n = k\}$, $(x \in G_m, k \in Z_{m_n}, n \in \mathbf{N})$. The sequence of U's: (i.e. U is one of the following sets)

$$U^0_{n,0}, U^1_{n,0}, U^1_{n,1}, U^2_{n,0}, U^2_{n,1}, U^2_{n,2}, U^2_{n,3},\ldots, U^{v_n}_{n,0}, U^{v_n}_{n,1},\ldots, U^{v_n}_{n,2^{v_n}-1}.$$

The set of intervals is denoted by $\mathcal{I}$.

Let $M_0 := 1, M_{n+1} := m_n M_n\,(n \in \mathbf{N})$. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0,1,\ldots,m_i-1\},\ i \in \mathbf{N}),$$

where only a finite number of n_i 's differ from zero. Set

$$r_n(x) := \exp(2\pi\imath \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbf{N}, \imath := \sqrt{-1})$$

the generalized Rademacher functions ,

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbf{N})$$

the Vilenkin functions. The system $\psi := (\psi_n : n \in \mathbf{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m -adic addition:

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\operatorname{mod} m_j)) M_j \quad (k,n \in \mathbf{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n,\ \psi_n(x+y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1\ (k,n \in \mathbf{N}, x,y \in G_m)$.

Define the Fourier coefficients , the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means and the Fejér kernels with respect to the Vilenkin system ψ as follows.

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n,\ S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_n(y)\bar{\psi}_n(x),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f,\ K_n(y,x) = K_n(y-x) := \frac{1}{n}\sum_{k=1}^{n} D_n(y-x),$$

$$(n \in \mathbf{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f,\ f \in L^1(G_m)).$$

It is well-known that

$$(S_n f)(y) = \int_{G_m} f(x) D_n(y-x)d\mu(x),\ (\sigma_n f)(y) = \int_{G_m} f(x) K_n(y-x)d\mu(x)$$

$$(n \in \mathbf{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases},$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) (f \in L^1(G_m), n \in \mathbf{N}).$$

We say that an operator $T : L^1(G_m) \to L^0(G_m)$ ($L^0(G_m)$ is the space of measurable functions on G_m) is of type (p,p) (for $1 \le p \le \infty$) if $\|Tf\|_p \le c_p \|f\|_p$ for all $f \in L^p(G_m)$ and constant c_p depends only on p. We say that T is of weak type $(1,1)$ if $\mu(\{|Tf| > \lambda\}) \le c\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

In this paper c denotes an absolute constant which may not be the same at different occurences. For more on the Vilenkin system see [AVD, Tai, Vil].

Theorem 1. (The Calderon-Zygmund decomposition ([Sim]). Let $f \in L^1(G_m)$, $\lambda > \|f\|_1$. *Then there exists a decomposition*

$$f = \sum_{j=0}^{\infty} f_j,\ I^j := \cup_{l \in U^{a_j}_{k_j, b_j}} I_{k_j}(u^j, l) \in \mathcal{I}$$

disjoint intervals for which supp $f_j \subseteq I^j$, $\int_{I^j} f_j = 0$, $\mu(I^j)^{-1}\int_{I^j} |f_j| \le c\lambda, (u^j \in G_m,\ k_j, a_j, b_j \in \mathbf{N}, j \in \mathbf{P})$, $\|f_0\|_\infty \le c\lambda$, $\mu(F) \le c\|f\|_1/\lambda$, *where* $F = \cup_{j \in \mathbf{P}} I^j$.

The proof of Theorem 1 uses the fact that the M_nth partial sums of the Walsh-Fourier series of an integrable function converges a.e. to the function. This later statement was proved by techniques known in the martingale theory. We give a new proof for Theorem 1, which use techniques known in the theory of dyadic harmonic analysis, only. First we prove the following lemma which is similar to Theorem 1, but differs in the conditions to be proved for f_0.

Lemma 2. Let $f \in L^1(G_m)$, $\lambda > \|f\|_1$. *Then there exists a decomposition*

$$f = \sum_{j=0}^{\infty} f_j,\ I^j := \cup_{l \in U^{a_j}_{k_j, b_j}} I_{k_j}(u^j, l) \in \mathcal{I}$$

disjoint intervals for which supp $f_j \subseteq I^j$, $\int_{I^j} f_j = 0$, $\mu(I^j)^{-1}\int_{I^j}|f_j| \le c\lambda, (u^j \in G_m, k_j, a_j, b_j \in \mathbf{N}, j \in \mathbf{P})$, $\limsup_{n\to\infty} S_{M_n}|f_0| \le c\lambda$, $\mu(F) \le c\|f\|_1/\lambda$, *where* $F = \cup_{j\in\mathbf{P}} I^j$.

Proof of Lemma 2. Construct the following decomposition of the Vilenkin group G_m.

$$\Omega_0^0 := \{I_0(x) : M_0 \int_{I_0(x)} |f(y)|d\mu(y) > \lambda, x \in G_m\} = \emptyset,$$

$$\Omega_0^1 := \{\cup_{k\in U^1_{0,b}} I_0(x,k) =: I \,:\, \mu(I)^{-1}\int_I |f(y)|d\mu(y) > \lambda,\ \nexists J \in \Omega_0^0 : I \subset J,$$
$$b = 0,1, x \in G_m\}, \ldots$$

$$\Omega_0^{v_0} := \{\cup_{k\in U^{v_0}_{0,b}} I_0(x,k) =: I \,:\, \mu(I)^{-1}\int_I |f(y)|d\mu(y) > \lambda,\ \nexists J \in \cup_{j<v_0}\Omega_0^j : I \subset J,$$
$$b = 0,1,\ldots,2^{v_0}-1, x \in G_m\}, \ldots$$

$$\Omega_n^a := \{\cup_{k\in U^a_{n,b}} I_n(x,k) =: I \,:\, \mu(I)^{-1}\int_I |f(y)|d\mu(y) > \lambda,$$
$$\nexists J \in \left(\cup_{j<n}\cup_{i\le v_j}\Omega_j^i\right) \cup \left(\cup_{i<a}\Omega_n^i\right) : I \subset J, b = 0,1,\ldots,2^a-1, x \in G_m\}$$
$$(a = 0,1,\ldots,v_n)\ldots$$

$(n \in \mathbf{P})$. Then, the elements of $\Omega_n^k\, n,k \in \mathbf{N}$ are disjoint intervalls. Moreover, if $i \ne \tilde{i}$, then for all $J \in \Omega_i^a, K \in \Omega_{\tilde{i}}^{\tilde{a}}$ we have $J \cap K = \emptyset$ $(a, \tilde{a} \in \mathbf{N})$. If $x \in I \in \Omega_n^a$, then since there is no $J \in \cup_{j<n}\cup_{i\le v_j}\Omega_j^i \cup_{i<a}\Omega_n^i$ for which $I \subset J$, then we have $M_j\int_{I_j(x)}|f(y)|d\mu(y) \le \lambda$ for $j = 0,1,\ldots,n-1$. and for all $K \in \Omega_n^{a-1}$ for which $x \in K$ we have $\mu(K)^{-1}\int_K |f(y)|d\mu(y) \le \lambda$. This implies $\lambda < \mu(I)^{-1}\int_I |f(y)|d\mu(y) \le 3\lambda$. Since Ω_n^a has a finite number of elements, then set the notation:

$$\Omega_n^a = \{I^{n,a,i} : i = 1,\ldots,l_{n,a}\} \in \mathcal{I}, \quad F := \cup_{n=0}^\infty \cup_{a=0}^{v_n} \cup_{i=1}^{l_{n,a}} I^{n,a,i}.$$

Then,

$$f_{n,a,i} := f1_{I^{n,a,i}} - \mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f \quad (i \le l_{n,a},\ a \le v_n, n \in \mathbf{N}),$$

where $1_B(x) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B \end{cases}$ the characteristic function of set $B \subset G_m$ $(x \in G_m)$.

$$\begin{aligned}
\mu(F) &= \sum_{n=0}^\infty \sum_{a=0}^{v_n} \sum_{i=1}^{l_{n,a}} \mu(I^{n,a,i}) \\
&= \frac{1}{\lambda}\sum_{n=0}^\infty \sum_{a=0}^{v_n} \sum_{i=1}^{l_{n,a}} \lambda\mu(I^{n,a,i}) \\
&\le \frac{1}{\lambda}\sum_{n=0}^\infty \sum_{a=0}^{v_n} \sum_{i=1}^{l_{n,a}} \int_{I^{n,a,i}} |f| \le \frac{1}{\lambda}\int_{G_m} |f| = \|f\|_1/\lambda.
\end{aligned}$$

Then,

$$\begin{aligned}
f &= \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} f1_{I^{n,a,i}} + f1_{G_m\setminus F} \\
&= \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} \left(f - \mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f\right) 1_{I^{n,a,i}} \\
&+ \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} \left(\mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f\right) 1_{I^{n,a,i}} + f1_{G_m\setminus F} \\
&=: \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} f_{n,a,i} + f_0.
\end{aligned}$$

Discuss the functions $f_{n,a,i}$.
supp $f_{n,a,i} \subset I^{n,a,i}$,

$$\int_{I^{n,a,i}} f_{n,a,i} = \int_{I^{n,a,i}} (f(t) - \mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f(y)d\mu(y))d\mu(t) = 0,$$

$$\begin{aligned}
&\mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} |f_{n,a,i}| \\
&\leq \mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} |f| + |\mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f| \leq c\cdot\mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} |f| \\
&\leq c\lambda.
\end{aligned}$$

The only relation rest to prove is $\limsup_n S_{M_n}|f_0| \leq c\lambda$.

$$f_0 = \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} \left(\mu(I^{n,a,i})^{-1}\int_{I^{n,a,i}} f\right) 1_{I^{n,a,i}} + f1_{G_m\setminus F} =: f_0^1 + f_0^2.$$

First, discuss function f_0^1.

$$|f_0^1| \leq \sum_{n=0}^{\infty}\sum_{a=0}^{v_n}\sum_{i=1}^{l_{n,a}} c\lambda 1_{I^{n,a,i})} = c\lambda 1_F \leq c\lambda.$$

Thus,

$$S_{M_n}|f_0^1(x)| = M_n\int_{I_n(x)} |f_0^1(t)|d\mu(t) \leq c\lambda$$

for all $x \in G_m, n \in \mathbf{N}$. Consequently, $\limsup_n S_{M_n}|f_0^1| \leq c\lambda$ everywhere.

Secondly, discuss function f_0^2. If $x \in F$, then since set F is open (the union of intervalls (intervalls are both open and closed)), then there exists a $n \in \mathbf{N}$ such as $I_n(x) \subset F$. Since $f_0^2 = f1_{G_m\setminus F}$, then f_0^2 is zero on the intervall $I_n(x)$. Thus, for each $l \geq n$ we have $M_l\int_{I_l(x)} |f_0^2(t)|d\mu(t) = 0$. This implies, $\limsup_n S_{M_n}|f_0^2(x)| = 0$

for $x \in F$. Finally, let $x \notin F$. Then $M_j \int_{I_j(x)} |f(y)| d\mu(y) \leq \lambda$ for $j = 0, 1, \ldots$. This gives

$$S_{M_j} |f_0^2(x)| = M_j \int_{I_j(x)} |f(y) 1_{G_m \setminus F}(y)| d\mu(y) \leq M_j \int_{I_j(x)} |f(y)| d\mu(y) \leq \lambda$$

for $j = 0, 1, \ldots$. This follows that $\limsup_n S_{M_n} |f_0^2(x)| \leq \lambda$ in the case of $x \notin F$. Consequently, $\limsup_n S_{M_n} |f(x)| \leq \limsup_n S_{M_n} |f_0^1(x)| + \limsup_n S_{M_n} |f_0^2(x)| \leq c\lambda$. This completes the proof of Lemma 2. $\square$

Set the following maximal operators

$$S^\circ f := \limsup_n |S_{M_n} f(x)|, \quad Sf := \sup_n |S_{M_n} f(x)|.$$

for $f \in L^1(G_m)$.

Lemma 3. *Operators S° and S are of type (∞, ∞).*

Proof.

$$\begin{aligned} \|S^\circ f\|_\infty \leq \|Sf\|_\infty &= \| \sup_{n \in \mathbf{N}} |M_n \int_{I_n(x)} f(t) d\mu(t)| \|_\infty \\ &\leq \| \|f\|_\infty \sup_{n \in \mathbf{N}} M_n \int_{I_n(x)} 1 d\mu(t) \|_\infty = \|f\|_\infty. \end{aligned}$$

$\square$

Lemma 4. *Operator S° is of weak type $(1, 1)$.*

Proof. $\lambda > \|f\|_1$ can be supposed. Apply Lemma 2.

$$\mu(S^\circ f > 2c\lambda) \leq \mu(S^\circ f_0 > c\lambda) + \mu(S^\circ (\sum_{n,a,i} f_{n,a,i}) > c\lambda) =: l_1 + l_2.$$

Since $|S^\circ f_0| \leq c\lambda$ a.e., then $l_1 = 0$. On the other hand, by the σ-sublinearity of operator S

$$\begin{aligned} l_2 &\leq \mu(F) + \mu(x \in G_m \setminus F : S(\sum_{n,a,i} f_{n,a,i}) > c\lambda) \\ &\leq \mu(F) + \frac{c}{\lambda} \int_{G_m \setminus F} S(\sum_{n,a,i} f_{n,a,i}) \\ &\leq c\|f\|_1/\lambda + \frac{c}{\lambda} \int_{G_m \setminus F} \sum_{n,a,i} S(f_{n,a,i}) \\ &\leq c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{n,a,i} \int_{G_m \setminus F} S(f_{n,a,i}) \\ &\leq c\|f\|_1/\lambda + \frac{c}{\lambda} \sum_{n,a,i} \int_{G_m \setminus I^{n,a,i}} S(f_{n,a,i}). \end{aligned}$$

We prove that $\int_{G_m\setminus I^{n,a,i}} S(f_{n,a,i}) = 0$ for all $i \le l_{n,a}$, $a \le v_n$ $n \in \mathbf{N}$.

If $y \in G_m \setminus I^{n,a,i} = G_m \setminus \cup_{k\in U^a_{n,b}} I_n(x^{n,a,i},k)$, then we have two cases. If $y \in I_a(x^{n,a,i}) \setminus I_{a+1}(x^{n,a,i})$ for some $a = 0,\dots,n-1$, then If $N \ge a$ implies $S_{M_N} f_{n,a,i}(y) = M_N \int_{I_N(y)} f_{n,a,i} = 0$, because $I_N(y) \cap I_n(x^{n,a,i}) = \emptyset$.

If $N < a$, then $D_{M_N}(y-x) = M_N$ for $x \in \cup_{k\in U^a_{n,b}} I_n(x^{n,a,i},k) \subset I_n(x^{n,a,i})$. Consequently, $S_{M_N} f_{n,a,i}(y) = M_N \int_{I^{n,a,i}} f_{n,a,i} = 0$.

The second case: $y \in I_n(x^{n,a,i})$ but $y_n \notin U^a_{n,b}$. In this case $N \ge n+1$ implies $I_{n+1}(y) \cap I^{n,a,i} = \emptyset$, that is, $S_{M_N} f_{n,a,i}(y) = 0$. If $N \le n$, then for each $x \in I^{n,a,i} = \cup_{k\in U^a_{n,b}} I_n(x^{n,a,i},k)$, we have $D_{M_N}(y-x) = M_N$ which gives $S_{M_N} f_{n,a,i}(y) = M_N \int_{I^{n,a,i}} f_{n,a,i} = 0$. That is, in all cases for all $N \in \mathbf{N}$ we have $S_{M_N} f_{n,a,i}(y) = 0$, thus $Sf_{n,a,i}(y) = 0$ for all $y \in G_m \setminus I^{n,a,i}$. Consequently, $l_2 \le c\|f\|_1/\lambda$. The proof of Lemma 4 is complete. □

The proof of the following theorem known till now is based on the martingale theory (see e.g. [Sto]). We give a "pure dyadic analysis" proof for it.

Theorem 5. *Let $f \in L^1(G_m)$. Then $S_{M_n} f \to f$ a.e.*

Proof. Let $\epsilon > 0$. Then let P be a Vilenkin polynomial , that means $P = \sum_{i=0}^{k-1} d_i\psi_i$ for some $d_0,\dots,d_{k-1} \in \mathbf{C}$, $k \in \mathbf{P}$. Since $S_{M_n}P(x) \to P$ everywhere (moreover, $S_{M_n}P = P$ for $M_n \ge k$), then by lemmas 3 and 4 we have

$$\begin{aligned}
&\mu(\{x \in G_m : \limsup_n |S_{M_n} f(x) - f(x)| > \epsilon\})\\
&\le \mu(\{x \in G_m : \limsup_n |S_{M_n} f(x) - S_{M_n}P(x)| > \epsilon/3\})\\
&+ \mu(\{x \in G_m : \limsup_n |S_{M_n} P(x) - P(x)| > \epsilon/3\})\\
&+ \mu(\{x \in G_m : \limsup_n |P(x) - f(x)| > \epsilon/3\})\\
&\le \mu(\{x \in G_m : \limsup_n |S_{M_n}(f(x) - P(x))| > \epsilon/3\}) + 0 + \|P-f\|_1 \frac{3}{\epsilon}\\
&\le c\|P-f\|_1/\epsilon =: \delta.
\end{aligned}$$

Since the set of Vilenkin polynomial is dense in $L^1(G_m)$ (see e.g. [AVD]), then δ can be less than an arbitrary small positive real number. This follows $\mu(\{x \in G_m : \limsup_n |S_{M_n} f(x) - f(x)| > \epsilon\}) = 0$ for all $\epsilon > 0$. This gives the relation $S_{M_n} f \to f$ almost everywhere. □

The proof of Theorem 1. We apply Lemma 2 and Theorem 5. The proof follows the proof of Lemma 2. The only difference is that we have to prove $\|f_0\|_\infty \le c\lambda$ instead of $\limsup_{n\to\infty} S_{M_n}|f_0| \le c\lambda$. By Theorem 5 we have $S_{M_n} f_0 \to f_0$ a.e. Thus, we have the a.e. inequality

$$|f_0| = \limsup_n |S_{M_n} f_0| \le \limsup_n S_{M_n}|f_0| \le c\lambda.$$

That is, the proof is complete. □

Corollary 6. The operator S is of type (p,p) for each $1 < p$.

Proof. Since we have proved that operator S is of type (∞,∞) and of weak type $(1,1)$, then by the interpolation theorem of Marczinkiewicz (see e.g. [SWS]) the proof of Corollary 6 is complete. □

References

[AVD] Agaev, G.H., Vilenkin, N.Ja., Dzhafarli, G.M. Rubinstein, A.I., *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups (in Russaian)*, Izd.("ELM"), Baku, 1981.

[Gát] Gát, G., *On the Calderon-Zygmund decomposition lemma on the Walsh-Paley group*, Acta Math. Acad. Paed. Nyíregyhásiensis **14** (1998), 25-30, www.bgytf.hu/˜amapn.

[GES] Golubov, B.I, Efimov, A.V., Skvorcov, V.A., *Walsh series and transforms*, Nauka, Moskva, 1987.

[Sim] Simon, P., *Investigations with respect to the Vilenkin system*, Annales Univ. Sci. Budapestiensis, Sectio Math. **27** (1985), 87-101.

[SWS] Schipp, F., Wade, W.R., Simon, P., Pál, J., *Walsh series: an introduction to dyadic harmonic analysis*, Adam Hilger, Bristol and New York, 1990.

[Sto] Stout, W.F., *Almost sure convergence*, Academic Press, New York, 1974.

[Tai] Taibleson, M.H., *Fourier Analysis on Local Fields*, Princeton Univ. Press., Princeton, N.J., 1975.

[Vil] Vilenkin, N.Ja., *On a class of complete orthonormal systems (in Russian)*, Izv. Akad. Nauk. SSSR, Ser. Math. **11** (1947), 363-400.

(Received October 20, 1998)

Bessenyei College, Department of Mathematics,
Nyíregyháza, P.O.Box 166., H–4400, Hungary
E-mail address: `gatgy@agy.bgytf.hu`