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PYTHAGOREAN SIDE AND DIAGONAL NUMBERS

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Abstract. The Pythagorean discovery that the ratio of the diagonal (diameter) and side of a square cannot be expressed as the ratio of two positive integers means in modern language that is not a rational number. The above incommensurability had a great effect on Greek philosophy and mathematics. Many papers deal with it and its impact, but relatively few with the approximation of the ratio in question. Even not Euclid's Elements that was purely theoretical and neglected the calculation methods. The aim of our paper to analyse the sources and the literature on this problem and give a new (and hopefully a satisfactory) explanation of Pythagorean's approximation method, i.e. of their so-called side and diagonal (or diameter) numbers.

The Pythagorean discovery that the ratio of the diagonal (diameter) and side of a square cannot be expressed as the ratio of two positive integers means in modern language that $\sqrt{2}$ is not a rational number. The above incommensurability had a great effect on Greek philosophy and mathematics. Many papers deal with it and its impact, but relatively few with the approximation of the ratio in question. Even not Euclid's Elements that was purely theoretical and neglected the calculation methods. The aim of our paper to analyse the sources and the literature on this problem and give a new (and hopefully a satisfactory) explanation of Pythagorean's approximation method, i.e. of their so-called side and diagonal (or diameter) numbers. The word 'number' will be used in Greek sense: greater-than-one integer. The main problem is that no original sources survived from pre-Platonic times. We have only some fragments from later commentaries and a few allusions from later authors. Among these short and obscure allusions Plato's passage (Republic 546c) on nuptial (or side and diagonal) numbers is perhaps the most obscure one. It has different translations and a vast literature. Now we quote some lines from two English translations only:

> ... the other a rectangle, one of its sides being a hundred of the numbers from the rational diameters of five, each diminished by one (or a hundred of the numbers from the irrational diameters of five, each diminished by two), ...
>
> ([9], vol. I, p. 399)

> ... and the other a figure having one side equal to the former, but oblong, consisting of a hundred numbers squared upon rational diameters of a square (i.e. omitting fractions), the side of which is five ($7 \times 7 = 49 \times 100 = 4900$), each of them being less by one (than the perfect square which includes the fractions, sc. 50) or less by two perfect squares of irrational diameters (of square the side of which is five $= 50 + 50 = 100$); ...
>
> ([6], p. 403)

We can get a better picture on these rational and irrational diameters if we quote from Proclus' commentaries on this passage:

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> 1. The Pythagoreans proposed this elegant theorem about the diameters and sides, that when the diameter receives the side of which it is diameter, while the sides added to itself and receiving its diameter, becomes a diameter. And this is proved graphically in the second book of the Elements by him (sc. Euclid). If a straight line be bisected and a straight line be added to it, the square of the whole line including the added straight line the square on the latter by itself are together double of the square on the half and of the square on the straight line made up of the half and the added straight line.
> (Proclus: Commentary on Plato's Republic, ed. Kroll, ii. 27. 11-22; [9], vol. I, pp. 137-138.)
>
> 2. Thus in seeking the causes of certain results we turn to numbers, ... when we are content with approximations, as when, in geometry we have found a square double a given square but do not have it in numbers, we say that a square number is the double of another square number when it is short by one, like the square of seven, which is one less than the double of the square of five.
> [7, p. 49.]

The second quotation (that is not cited in literature) clearly shows that 7 is a rational approximation of the irrational diagonal (diameter) $\sqrt{50}$ of the square of side 5, and consequently, the ratio 7/5 is an approximation of $\sqrt{2}$. More approximations and details can be obtained from a book of Theon of Smyrna aiming to help the understanding of Plato's mathematical thoughts:

> Even as numbers are invested with power to make triangles, squares, pentagons and the other figures, so also we find side and diameter ratios appearing in numbers in accordance with the generative principles;for it is these which give harmony to the figures . Therefore since the unit, according to the supreme generative principle, is the starting-point of all figures, so also in the unit will be found the ratio of the diameter to the side. To make this clear, let two units be taken, of which we set one to be a diameter and the other a side , since the unit, as the beginning of all things, must have it in its capacity to be both side and diameter. Now let there be added to the side a diameter and to the diameter two sides, for as often as the square on the diameter is taken once, so often is the square on the side taken twice. The diameter will therefore become the greater and the side will become the less. Now in the case of the first side and diameter the square on the unit diameter will be less by a unit than twice the square on the unit side;for units are equal, and 1 is less by a unit than twice 1. Let us add to the side a diameter, that is, to the unit let us add a unit; therefore the [second] side will be two units. To the diameter let us now add two sides, that is, to the unit let us add two units;the [second] diameter will therefore be three units. Now the square on the side of two units will be 4, while the square on the diameter of three units will 9; and 9 is greater by a unit than twice the square on the side 2.
>
> Again, let us add to the side 2 the diameter 3; the [third] side will be 5. To the diameter 3 let us add two sides, that is, twice 2; the third diameter will be 7. Now the square from the side 5 will be 25, while that from the diameter 7 will be 49; and 49 is less by a unit than twice 25. Again, add to the side

> 5 the diameter 7; the result will be 12. And the diameter 7 add twice the side 5; the result will be 17. And the square of 17 is greater by a unit than twice the square of 12. Proceeding in this way in order, there will be the same alternating proportion; the square on the diameter will be now greater by a unit, now less by a unit, than twice the square on the side; and such sides and diameters are both rational.
> (Theon of Smyrna, ed. Hiller 42. 10-44. 17; [9], vol. I, pp. 133-137.)

The interpretations of these citations are quite similar and answer many questions, but leave some unanswered or not properly answered. Perhaps the most detailed study is that of Waerden's [10] tracing back to Heath' two works: [3], vol. I. pp. 91-93; and [4], pp. 32-33. Let us recall Waerden's presentation.

After stating the recursion formula for the side and diameter numbers (denoted by a_n and d_n, respectively)

$$a_{n+1} = a_n + d_n;\ d_{n+1} = 2a_n + d_n, \tag{1}$$

he continues to write (p. 126):

> The names side- and diagonal-numbers hint the fact that the ratio $a_n : d_n$ is an approximation for the ratio of the side of a square to its diagonal. This follows from the identity $d_n^2 = 2a_n^2 \pm 1$ which, according to Proclus, was proved by use of II. 10.

Waerden first conjecture was verified by Proclus, as we saw above. Next, Waerden asks how the Pythagoreans got the recursion formulas (1). He guesses the following method. In trying to find the common measure of the side and diagonal of a square they used the

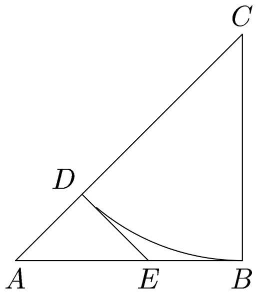

FIGURE 1. $AD = DE = EB = d - a$, $AE = 2a - d$.

so-called successive subtraction (antanairesis) method. (Fig. 1). Subtract from the greater d the smaller a. We get two new magnitudes: a and $d - a$. Again, subtract from the greater a the smaller $d - a$. The result is $2a - d$. In the next step $a_1 = d - a$ must be subtracted from $d_1 = 2a - d$. These magnitudes are again a side and diagonal of a smaller square. Their connection to the original side and diagonal is:

$$a = a_1 + d_1;\ d_1 = 2a_1 + d_1 \tag{2}$$

Waerden states that (2) 'present the same form' as (1). From this antanairesis method there follows two things:

i. The process continues ad infinitum (supposing that the indefinite divisibility of line accepted as axiom), a and d have no common measure, therefore their ratio cannot be given in numbers, i.e. as a ratio of two positive integers.
ii. During the process both a_n and d_n become smaller and smaller, and consequently so does their difference. Therefore we can take a_n approximately equal d_n for some n. By Waerden's words [10, p. 127]: 'If the process is continued until the difference between, say a''' and d''' has become too small to be observed and if one approximates by setting $a''' = b'''$, then, choosing a''' as the unit of length, a'' and b'', a' and b', and finally a and b are represented by means of (2), in the form of the successive side- and diagonal-numbers.' (We use lower indexes instead of dashes, and d for b.)

Some of Waerden's quoted arguments and conclusions are not correct, have some inconsistencies and misunderstanding of sources, and remain open some questions:

1. The formulas (2) in their general form

$$a_n = a_{n+1} + d_{n+1};\ \ d_n = 2a_{n+1} + d_{n+1} \tag{3}$$

   are not the same as (1).
2. If $a_n = d_n$ taken, how does $d_n : a_n$ approximate the ratio $d : a = \sqrt{2}$?
3. Figure 1 shows smaller and smaller sides and diagonals, while by Theon's construction we get greater and greater ones. Kroll also noticed this controversy in [5, p. 36]. 'Moreover, the step which seems so evident to us is never noted in the sources:namely, that the anthypairesis repeats ad infinitum to produce ever-decreasing line in the same ratio. The procedure is rather employed to yield a formula for ever-increasing integral values which approximates the ratio.'
4. From the quoted paragraphs of Proclus (and Theon) it can be concluded that Euclid's Prop. II. 10 proves not (only) the identity $d_n^2 = 2a_n^2 \pm 1$, but the correctness of the construction of the greater side and diagonal.

Now we try to eliminate these contradictions and establish a theory of side and diagonal numbers mainly by deductive reconstruction that corresponds better to sources and is of inner coherent.

When the Pythagoreans discovered the incommensurability of the side and diagonal of a square, i.e. they found that there does not exist a smallest length of which they are integer multiple, their belief in numbers suffered a big shock. Earlier they were convinced that every ratio can be expressed as the ratio of numbers. But the diagonal of a square proved to be 'irrational' to the side. By the irrational diagonal they meant that its ratio to the side is inexpressible by numbers, or as Proclus said: 'do not have it in numbers.' To restore at least partly their number-centered philosophy they tried to find the rational diagonal of the square as closest as possible to the irrational one. The 'closest' meant for them:their squares can differ only by one. (Fractions did not exist for them in pure mathematics!) This requirement in modern notation can be formulated as follows

$$d_n^2 = 2a_n^2 \pm 1 \tag{4}$$

where a_n, d_n denotes the side and diagonal, respectively, gained after the n^{th} step of approximation. They had two equivalent possibilities to find rational diagonals by which the $d : a$ ratio can be approximated 'in numbers:'

1. *Decreasing or subtraction method.* By successive subtraction to get smaller and smaller squares (as we saw in Fig. 1) making the difference between the side and diagonal as small as we please. Taking them equal they can serve as an (approximate) common measure, or a unit length. Then the length of both the side and diameter (consequently their ratio, too) can be given in numbers.
2. *Increasing or addition method.* By successive addition to get bigger and bigger squares. Regarding the original side and diagonal both of unit length, we have their length in numbers. Because the unit lengths are really different, the length of the rational diameter will differ from the 'exact' irrational one. But this difference will be decreasing indefinitely during the process.

Studying the sources it becomes clear that the Pythagoreans chose the second possibility. One reason for this is probably their disappointment in antanairesis: it led to the demolishing discovery of incommensurability. The discrepancies of Waerden's interpretation are caused by mixing of these two methods.

The Pythagoreans starting from a and d, used $a+d$ and $2a+d$ as first step of approximation. They proved by Euclid's Prop. II. 10 geometrically that these are also the side and diagonal of a 'bigger' square, and that the square of the rational diagonal $2a+d(=3)$ differs only by 1 from that of the irrational one belonging to the side $a+d(=2)$.

The heuristic idea behind forming the new side as $a+d$, and the new diagonal as $2a+d$ is included Theon's description quotend above: '... for as often as the square on the diameter is taken once, so often is the square on the side taken twice.' By Prop. II. 10 it is easy to verify that this construction results again a square (Fig. 2).

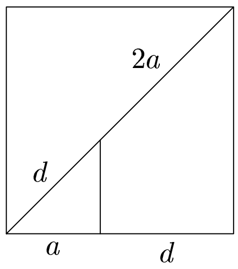

FIGURE 2.

The text of this theorem can be found in Proclus' cited passage. With modern notation we can express it by the formula

$$(2a+d)^2 + d^2 = 2a^2 + 2\,(a+d)^2 \tag{5}$$

Here a and d can denote any line segment, but if they are the side and diagonal of a square, that is if $d^2 = 2a^2$, than (5) reduces to $(2a+d)^2 = 2\,(a+d)^2$ demonstrating by the converse of the Theorem of Pythagoras that $a+d$ and $2a+d$ are again the side and diameter of a square. The recursion formulas for a_n and d_n in this method are those of (1):

$$a_{n+1} = a_n + d_n;\ d_{n+1} = 2a_n + d_n\ (n \geq 0,\ a_0 = a,\ b_0 = b). \tag{6}$$

Apply (5) to a_n and d_n:

$$(2a_n + d_n)^2 + d_n^2 = 2a_n^2 + 2\,(a_n + d_n)^2\,. \tag{7}$$

From here by using (6) we have:

$$d_{n+1}^2 + d_n^2 = 2a_n^2 + 2a_{n+1}^2. \tag{8}$$

Substitute here $n = 1$ and $a = d = 1$: $d_1^2 = 2a_1^2 + 1$. Further, substitute this in (8) for $n = 1$:

$$d_2^2 + d_1^2 = 2a_1^2 + 2a_2^2 \implies d_2^2 + 2a_1^2 + 1 = 2a_1^2 + 2a_2^2 \implies d_2^2 = 2a_2^2 - 1.$$

The Pythagoreans probably stopped here, or after some more steps, and concluded the rule: $d_n^2 = 2a_n^2 - 1$, if n is even; $d_n^2 = 2a_n^2 + 1$, if n is odd, in other words they got the formula (4).

I do not believe that they proved this antanairesis-like formula by induction argument as Waerden states it. By either complete or incomplete induction they must have gained (4) in this way. Waerden (and others) did not say anything about the genesis of this formula. They prove it simply by substitution of (7).

Now, the side and diagonal numbers including the famous 5 and 7 can easily be calculated from (6) starting with $a = d = 1$. Then d_n/a_n gives better and better approximations of $\sqrt{2}$.

Using the decreasing method we can get the same results as the Pythagoreans got by the above manner, but in a slightly different forms and in a more complicated way.

Here the recursion formulas (2) have to be used. We rewrite them in the following more convenient form:

$$a_n = a_{n+1} + d_{n+1};\ d_n = 2a_{n+1} + d_{n+1}\ (n \geq 0,\ a_0 = a,\ d_0 = d). \tag{9}$$

Substituting (9) in (7) we have:

$$d_{n-1}^2 + d_n^2 = 2a_n^2 + 2a_{n-1}^2,$$

from where taking $a_n = d_n = 1$ for some n, there follows

$$d_{n-1}^2 = 2a_{n-1}^2 + 1.$$

Now, write (7) for a_{n-1} and d_{n-1} and substitute it here:

$$d_{n-2}^2 + d_{n-1}^2 = 2d_{n-1}^2 + 2a_{n-2}^2 \implies d_{n-2}^2 = 2a_{n-2}^2 - 1.$$

What we have here is a similar rule to (4) but in an 'opposite' recursion form starting not from 1 but from some n.

The side and diagonal numbers can be obtained from (9) as a and d taking $a_n = d_n = 1$ for some n. Let, say, $a_3 = d_3 = 1$. Then $a_2 = 2$, $d_2 = 3$; $a_1 = 5$, $d_1 = 7$; $a = 12$, $d = 17$. The resulted approximation of $\sqrt{2}$ is 17/12. The square of the rational diagonal 17 (289) exceeds by one the square of the irrational diagonal ($2 \cdot 12^2 = 288$).

The calculations can be made easier by the successive application of (9) as follows:

$$a = a_1 + d_1 = 3a_2 + 2d_2 = 7a_3 + 5d_3 = \ldots$$
$$d = 2a_1 + d_1 = 4a_2 + 3d_2 = 10a_3 + 7d_3 = \ldots$$

From here the side and diagonal numbers belonging to some $a_n = d_n = 1$ can be read immediately.

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