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LATTICE OF DISTANCES BASED ON 3D-NEIGHBOURHOOD SEQUENCES

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Dedicated to Z. Muzsnay on the occasion of his 30th birthday.

Abstract. In this paper we give a natural ordering relation between distances based on 3D-neighbourhood sequences. We prove that this ordering induces a complete compact distributive lattice over the set of 3D-neighbourhood sequences.

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1. INTRODUCTION

Rosenfeld and Pfaltz identified in [3] two types of motions in two-dimensional digital geometry. The cityblock motion restricts movements only to the horizontal or vertical directions, while the second type – chessboard motion – allows diagonal movements only. These two types of motion in 2D determine two distances namely the cityblock distance and the chessboard distance. The alternate use of these motions results in the octagonal distance.

In case of cityblock movements there is a unit change in one coordinate at every step, while in case of chessboard motion there is a unit change in both coordinates. Recently Das and Chatterji [2] have extended the definition of ordinary octagonal distances to allow arbitrary long cycle sequences of cityblock and chessboard motions called neighbourhood sequences.

The neighbourhood sequence is a distance which is obtained by combining the cityblock and the chessboard motions.

P.P. Das in [1] shown that the distances, generated by the 2D-neighbourhood sequences, form a complete compact distributive lattice supplied with a naturally interpreted relation order.

In 3D digital geometry we can define three different distances. The 3D-neighbourhood sequences, obviously determine the combination of types of motions determined by these three distances.

In this paper we investigate distances generated by 3D-neighbourhood sequences and by using the results of P.P. Das [1] we prove that they form a complete compact distributive lattice supplied with the same ordering relation.

2. BASIC DEFINITIONS

In order to reach the aims formulated in the introduction we would like to define the basic definitions and notations in this chapter.

Definition 2.1. *Let p and q be any two points of the n-dimensional digital plane. The ith coordinate of the point p is indicated by $\mathrm{Pr}_i(p)$. The points p and q are m-neighbours $(0 \le m \le n)$, if the following two conditions hold:*

- $0 \le |\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \le 1$ $(1 \le i \le n)$,
- $\sum_{i=1}^{n} |\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \le m.$

Definition 2.2. *The finite sequence $B = \{b(i)\,|\,i = 1, 2, \ldots, l$ and $b(i) \in \{1, 2, \ldots, n\}\}$ with length l composed by the elements of the set $\{1, 2, \ldots, n\}$ $(n \in \mathbf{N})$ is called an nD-neighbourhood sequence with period l.*

Definition 2.3. *Let p and q be any two points of the n-dimensional digital plane and $B = \{b(1), b(2), \ldots, b(l)\}$ be an nD-neighbourhood sequence. The point sequence $\Pi(p, q; B)$ – which has the form $p = p_0, p_1, \ldots, p_m = q$, where p_i and p_{i-1} are r-neighbours $(0 < i \le m)$ and $r = b(((i-1) \bmod l) + 1)$ – is called the path from p to q determined by B. The length $|\Pi(p, q; B)|$ of the path $\Pi(p, q; B)$ is m.*

Definition 2.4. *Let p and q be any two points of the n-dimensional digital plane and B an n-dimensional neighbourhood sequence. The shortest path from p to q is denoted by $\Pi^*(p, q; B)$. The length of the minimal path is defined as the distance between p and q is written as*

$$d(p, q; B) = |\Pi^*(p, q; B)|.$$

Using the above distance we cannot obtain a metric on the n-dimensional digital plane for every n-dimensional neighbourhood sequence. In order to prove this, consider the following simple example. Let $B = \{2, 1\}$, $n = 2$, $p = (0, 0)$, $q = (1, 1)$ and $r = (2, 2)$. In this case $d(p, q; B) = 1$, $d(q, r; B) = 1$, but $d(p, r; B) = 3$.

The question is the following: knowing B, how can we decide whether the distance related to B is a metric on the n-dimensional digital plane, or not? The answer can be found in [2].

The following result of P.P. Das et al. (cf. [2]) provides an algorithm for the calculation of the above defined distance $d(p, q; B)$.

Theorem 2.5. (see [2]). *Let p and q be any two points of the n-dimensional digital plane, and $B = \{b(1), b(2), \ldots, b(l)\}$ be an nD-neighbourhood sequence, and let*

$$x = (x(1), x(2), \ldots, x(n)),$$

where x is the nonascending ordering of $|\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)|$ that is $x(i) \ge x(j)$, if $i < j$. Put

$$\begin{aligned}
a_i &= \sum_{j=1}^{n-i+1} x(j),\\
b_i(j) &= \begin{cases} b(j) & , \text{ if } b(j) < n - i + 2,\\ n - i + 1 & , \text{ otherwise,} \end{cases}\\
f_i(j) &= \begin{cases} \sum_{k=1}^{j} b_i(k) & , \text{ if } 1 \le j \le l,\\ 0 & , \text{ if } j = 0, \end{cases}\\
g_i(j) &= f_i(l) - f_i(j-1) - 1, \qquad 1 \le j \le l.
\end{aligned}$$

The length of the minimal path from p to q determined by B, denoted by $d(p,q;B) = |\Pi^*(p,q;B)|$, is given by the following formula:

$$d(p,q;B) = \max_{i=1}^{n} d_i(p,q), \text{ where}$$
$$d_i(p,q) = \sum_{j=1}^{l} \left\lfloor \frac{a_i + g_i(j)}{f_i(l)} \right\rfloor .$$

3. NEIGHBOURHOOD SEQUENCES IN 3D

It is a natural question that what kind of relation exists between those distance functions generated by B_1 and B_2 in case of two given, B_1 and B_2 neighbourhood sequences. The complexity of the problem can be characterized by the following 2D example known from [1]. Let $B_1 = \{1,1,2\}$, $B_2 = \{1,1,1,2,2,2\}$. Choose the points $p = (3,1)$ and $q = (6,3)$. In this case we obtain that $d(0,p;B_1) = 3 < 4 = d(0,p;B_2)$, but $d(0,q;B_1) = 7 > 6 = d(0,q;B_2)$. So the distance generated by B_1 and B_2 cannot be compared.

In [1] the author has shown that in case of 2D the distance functions generated by the neighbourhood sequences form a distributive lattice. In this chapter we show that that the same feature is valid in 3D as well.

Definition 3.1. *The rth ($r \geq 1$) power B^r of the neighbourhood sequence B can be defined as follows:*

$$B^r = \{b'(i) \mid 1 \leq i \leq rl\}$$
$$b'(i) = b(((i-1) \bmod l) + 1),\ 1 \leq i \leq rl.$$

It follows from the above Definition and from the definition of the neighbourhood sequence that in case of any points p and q, $d(p,q;B^r) = d(p,q;B)$ ($r \geq 1$) holds.

Theorem 3.2. *Using the notation of Theorem 2.5, for any 3-dimensional neighbourhood sequences B_1 and B_2*

$$d(p,q;B_1) \leq d(p,q;B_2), \qquad \text{for all } p,q \in \mathbf{Z}^3$$

if and only if

$$f_k^{(1)*}(i) \geq f_k^{(2)*}(i), \qquad \text{for all } 1 \leq i \leq l,\ 1 \leq k \leq 3,$$

where $B_1^* = B_1^r$, $B_2^* = B_2^s$, $r = \frac{l}{|B_1|}$, $s = \frac{l}{|B_2|}$ and l is the least common multiple of $|B_1|$, $|B_2|$.

Proof. We start with the special case when $|B_1| = |B_2| = l$. Clearly $B_1^* = B_1$ and $B_2^* = B_2$.

First we prove that if $d(p,q;B_1) \leq d(p,q;B_2)$ for any p,q, then $f_k^{(1)}(i) \geq f_k^{(2)}(i)$ for every $1 \leq i \leq l$, $1 \leq k \leq 3$. The proof is indirect. Assume that there are such $1 \leq i \leq l$ and $1 \leq k \leq 2$, for which $f_k^{(1)}(i) < f_k^{(2)}(i)$ is true. If $k = 3$, then $f_3^{(1)}(j) = f_3^{(2)}(j)$ trivially holds for every $1 \leq j \leq l$.

Let u_j ($1 \leq j \leq l$) be the numbers of those $b^{(2)}(t)$, $1 \leq t \leq i$, which equal to j.

In case of $k = 1$ let $p = (0,0,0)$ and $q = (u_1+u_2+u_3, u_2+u_3, u_3)$. Using the definition of $d(p,q;B)$, it is clear that $d(p,q;B_2)$ is equal to i. On the other hand, by the assumption $f_1^{(2)}(i) > f_1^{(1)}(i)$, and by the definition of p and q, we have $d(p,q;B_1) > i$, which is a contradiction.

In case of $k = 2$ let $p = (0,0,0)$ and $q = (u_1 + u_2 + u_3, u_2 + u_3, 0)$. Similary as above, we obtain $d(p,q;B_2) = i$. However, using again the definition of p and q and the assumption $f_2^{(2)}(i) > f_2^{(1)}(i)$, we get $d(p,q;B_1) > i$, which is a contradiction, too.

Conversely, suppose that $f_k^{(1)}(i) \geq f_k^{(2)}(i)$ for every $1 \leq i \leq l$, $1 \leq k \leq 3$. To derive $d(p,q;B_1) \leq d(p,q;B_2)$, by Theorem 2.5 it is sufficient to show that

$$d_k^{(1)}(p,q) = \sum_{j=1}^{l} \left\lfloor \frac{a_k + g_k^{(1)}(j)}{f_k^{(1)}(l)} \right\rfloor \leq \sum_{j=1}^{l} \left\lfloor \frac{a_k + g_k^{(2)}(j)}{f_k^{(2)}(l)} \right\rfloor = d_k^{(2)}(p,q)$$

holds. For this we prove that for any fixed k with $1 \leq k \leq 3$

$$\left\lfloor \frac{a_k + g_k^{(1)}(j)}{f_k^{(1)}(l)} \right\rfloor \leq \left\lfloor \frac{a_k + g_k^{(2)}(j)}{f_k^{(2)}(l)} \right\rfloor \qquad \text{for } 1 \leq j \leq l.$$

Using the definition of $g_k(j)$, the above inequalities are equivalent to the following ones:

$$\left\lfloor \frac{a_k + f_k^{(1)}(l) - f_k^{(1)}(j-1) - 1}{f_k^{(1)}(l)} \right\rfloor \leq \left\lfloor \frac{a_k + f_k^{(2)}(l) - f_k^{(2)}(j-1) - 1}{f_k^{(2)}(l)} \right\rfloor, \qquad 1 \leq j \leq l,$$

from which

$$1 + \left\lfloor \frac{(a_k - 1) - f_k^{(1)}(j-1)}{f_k^{(1)}(l)} \right\rfloor \leq 1 + \left\lfloor \frac{(a_k - 1) - f_k^{(2)}(j-1)}{f_k^{(2)}(l)} \right\rfloor, \qquad 1 \leq j \leq l.$$

If $(a_k - 1) - f_k^{(2)}(j-1) \geq 0$, then we even have

$$\frac{(a_k - 1) - f_k^{(1)}(j-1)}{f_k^{(1)}(l)} \leq \frac{(a_k - 1) - f_k^{(2)}(j-1)}{f_k^{(2)}(l)}.$$

Indeed, this inequality is equivalent to

$$f_k^{(2)}(l)(a_k - 1 - f_k^{(1)}(j-1)) \leq f_k^{(1)}(l)(a_k - 1 - f_k^{(2)}(j-1)),$$

which clearly holds because of our assumption $f_k^{(2)}(i) \leq f_k^{(1)}(i)$, $1 \leq i \leq l$.

In case of $(a_k - 1) - f_k^{(2)}(j-1) < 0$, by the definitions of f_k and a_k, we obviously have

$$\left\lfloor \frac{(a_k - 1) - f_k^{(2)}(j-1)}{f_k^{(2)}(l)} \right\rfloor = -1.$$

However, using again $f_k^{(2)}(i) \leq f_k^{(1)}(i)$, $1 \leq i \leq l$, now the inequality

$$\left\lfloor \frac{(a_k - 1) - f_k^{(1)}(j-1)}{f_k^{(1)}(l)} \right\rfloor = -1$$

is also true, which completes the proof of the Theorem in the special case $|B_1| = |B_2|$.

If $|B_1| \neq |B_2|$ then by the above argument and using the definition B^r, we obtain the statement of the Theorem.

Notation 3.3. *Let $S(l)$ be the set of the all neighbourhood sequences of length l. We define the relation $\sim$ in the following way:*

$$B_1 \sim B_2 \qquad \Leftrightarrow \qquad f_k^{(1)}(i) \leq f_k^{(2)}(i)$$

for all $1 \leq i \leq l$ and $1 \leq k \leq 3$.

By the previous Theorem it is evident that $\sim$ is an ordering relation. Now we show that $\sim$ induces a distributive lattice over $S(l)$.

Theorem 3.4. *For all $l \geq 1$ $(S(l), \sim)$ is a distributive lattice with minimal element $S_{\min} = \{1\}^l$ and maximal element $S_{\max} = \{3\}^l$.*

Proof. From the definition of $\sim$ it follows that this relation is reflexive, antisymmetric and transitive on $S(l)$. Thus $(S(l), \sim)$ is a partially ordered set.
In case of any $B_1, B_2 \in S(l)$ we define the operations $\wedge$ and $\vee$ in the following way:

$$B = B_1 \wedge B_2, \text{ where } f(i) = \min(f_1(i), f_2(i)),$$
$$B = B_1 \vee B_2, \text{ where } f(i) = \max(f_1(i), f_2(i)).$$

It is clear that $B_1 \wedge B_2$ and $B_1 \vee B_2$ are elements of $S(l)$. Furthermore, $B_1 \wedge B_2 \sim B_1$, $B_1 \wedge B_2 \sim B_2$ and $B_1 \sim B_1 \vee B_2, B_2 \sim B_1 \vee B_2$. Thus $(S(l), \sim)$ is a lattice with operations $\wedge$ and $\vee$.

Now we show that this lattice is a distributive one. Let $B_1, B_2, B_3 \in S(l)$ and $B = B_1 \wedge (B_2 \vee B_3)$. It is evident that for any $1 \leq i \leq l$ the following equalities hold:

$$f(i) = \min(f_1(i), \max(f_2(i), f_3(i))) = \max(\min(f_1(i), f_2(i)), \min(f_1(i), f_3(i))).$$

This means that $B = (B_1 \wedge B_2) \vee (B_1 \wedge B_3)$, that is $\wedge$ is distributive over $\vee$. Similarly, we can prove that $\vee$ is distributive over $\wedge$.
It is trivial that $S_{\min}$ is the minimum and $S_{\max}$ is the maximum of $S(l)$.
After this, we define the set $S^*(l) = \bigcup\limits_{l'=1}^{l} S(l')$ and the relation $\sim^*$ in the following way:

$$B_1 \sim^* B_2 \qquad \Leftrightarrow \qquad B_1^* \sim B_2^*,$$

where B^* is defined in Theorem 3.2.

Theorem 3.5. *$(S^*(l), \sim^*)$ is a distributive lattice for all $l \geq 1$, with minimal element $T_{\min} = \{1\}$ and maximal element $T_{\max} = \{3\}$.*

Proof. This Theorem can be proved similarly to Theorem 3.4.
However, there is an important difference between the lattices $(S(l), \sim)$ and $(S^*(l), \sim^*)$. Namely, because of the different lengths of the neighbourhood sequences, there is an unpleasant feature in the second case, which can be illustrated by the following example.
Let $B_1 = \{1, 2\} \in S^*(4)$ and $B_2 = \{1, 2, 1, 2\} \in S^*(4)$. It is clear that $B_1 \sim^* B_2$ and $B_2 \sim^* B_1$, but $B_1 \neq B_2$. To exclude such cases we can use the following construction found in [1]:

- $S^*(1) \leftarrow S(1)$,
- $S^*(l) \leftarrow S^*(l-1) \cup \{B \,|\, B \in S(l)$ and $\neg\exists l', 1 \leq l' < l$ such that $B_1^{l'} = B$ for $B_1 \in S^*(l-1)\}$.

4. CONCLUSION

In this paper we have shown that the set of neighbourhood sequences and consequently the set of $d(p,q;B)$'s forms a complete distributive lattice in 3D under the natural comparison relation. This lattice has an important role in the approximation of the Euclidean distance by digital distances.

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