Hochschild homology of finite dimensional algebras

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1 Introduction

Let A be an augmented algebra over a field k. By definition, the Hochschild homology of A with coefficients in A (called here Hochschild homology) is the homology of the Hochschild complex $(C_*(A), b)$ and is denoted $HH_*(A) = \underset{n\geq 0}{\oplus} HH_n(A)$, [Lo]. We denote $A = k \oplus \bar{A}$ where $\bar{A}$ is the augmentation ideal and we assume that $\bar{A}$ is a finite dimensional k-vector space.Then each homology group $HH_n(A)$is finite dimensional. Recall the following formulas :

$$C_n(A) = A \otimes \bar{A}^{\otimes n}$$

$$b(a_0 \otimes a_1 \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \ldots \otimes a_{n-1}$$

where $a_0 \in A$, $a_i \in \bar{A}$ if $i \geq 1$.

All the tensor products are over k.

We introduce the cyclic permutation $t_n : \bar{A}^{\otimes n} \to \bar{A}^{\otimes n}$ defined by $t_n(a_1 \otimes \ldots \otimes a_n) = (-1)^{n-1}$ $a_n \otimes a_1 \ldots \otimes a_{n-1}$.

¿From Loday, [Lo], Proposition 2.2.14, the reduced cyclic homology groups $H\tilde{C}_n(A) := HC_n(A)/HC_n(k)$ can be computed as the homology groups of the complex $\bar{A}^{\otimes(n+1)}/(Id - t_{n+1})$ endowed with the differential induced by b, when char$k = 0$, or char$k = p$ and $n < p-1$.

2 Characterization of the trivial algebra structure

Proposition 2.1 [Ro] *Let A be an augmented algebra, where the augmentation ideal $\bar{A}$ has finite dimension d and satisfies $\bar{A} \cdot \bar{A} = 0$, then*

1. $HH_n(A) = \mathrm{Coker}(\mathrm{Id} - t_{n+1}) \oplus \mathrm{Ker}(\mathrm{Id} - t_n)$ *as k-vector spaces for all $n > 0$*

2. *if* $\operatorname{char} k = 0$, $\tilde{HC}_n(A) = \operatorname{Coker}(\mathrm{Id} - t_{n+1})$ *for all* $n > 0$.

3. *if* $\operatorname{char} k = 0$, *for all* $n \geq 2$,

$$a_{n-1} = \dim HC_{n-1}(A) = (1/n)\sum_{i=1}^{n}(-1)^{(n-1)i}d^{q(i,n)}$$

where $q(i,n) = g.c.d.(i,n)$.

4. *if* $\operatorname{char} k = 0$, *then* $\quad \dim HH_n(A) = a_n + a_{n-1} \quad$ *for all* $n \geq 1 \quad$ *and* $\quad a_0 = d$.

Corollary 2.2 *Let A be an augmented algebra, where the augmentation ideal $\bar{A}$ has finite dimension d, $d \geq 1$ and satisfies $\bar{A} \cdot \bar{A} = 0$, then*

1. $HH_n(A) \neq 0$ *for all* $n > 0$

2. *if* $\operatorname{char} k = 0$, $\quad \lim_{n\to\infty} \sqrt[n]{\dim HH_n(A)} = d$.

Theorem 2.3 *Let A be an augmented algebra over a field k. Let $\bar{A}$ be its augmentation ideal, with* $\dim_k \bar{A}$ *finite. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$ and $\bar{A} = \bar{A}_t$ as k-vector space. We assume that* $\operatorname{char} k = 0$, *or* $\operatorname{char} k \neq 0$ *and there exists $N \geq 2$ such that $\bar{A}^N = 0$ in A ; then we have*

1. $\dim HH_n(A) \leq \dim HH_n(A_t)$ *for all* $n \geq 0$

2. $\dim HC_n(A) \leq \dim HC_n(A_t)$ *for all* $n \geq 0$

Proof: We define an increasing filtration A_k on A

$A_k = A \quad$ if $k \geq 0$, $\quad A_{-p} = \bar{A}^p \quad$ if $p \geq 1$

We filter $\bar{A}$ by

$\bar{A}_k = \bar{A} \quad$ if $\quad k \geq 0$, $\quad \bar{A}_{-p} = \bar{A}^p \quad$ if $\quad p \geq 1$

This allows us to filter the Hochschild complex as follows :

$$\begin{aligned} \mathcal{F}_k(C_n(A)) &= \sum_{k_0+k_1+\ldots+k_n \leq k} A_{k_0} \otimes \bar{A}_{k_1} \otimes \ldots \otimes \bar{A}_{k_n} && \text{for} \quad k < 0 \\ \mathcal{F}_k(C_n(A)) &= C_n(A) && \text{for} \quad k \geq 0 \end{aligned}$$

With the additional hypothesis that there exists $N \geq 2$ such that $\bar{A}^N = 0$, we get $\mathcal{F}_k(C_n(A)) = 0$ for $k < -(n+1)N$. This filtration gives rise to a spectral sequence (E^r_{**}, d^r) converging to $HH_*(A)$ with

$$\bigoplus_p E^0_{-p,n+p} = C_n(B)$$

$$B = k \oplus \frac{\bar{A}}{\bar{A}^2} \oplus \frac{\bar{A}^2}{\bar{A}^3} \oplus \dots$$

We check that $d^0((\lambda + \bar{a}_0) \otimes \bar{a}_1 \dots \otimes \bar{a}_n) = \lambda(Id - t_n)(\bar{a}_1 \dots \otimes \bar{a}_n)$ where $\bar{a}_i \in \oplus \bar{A}^p/\bar{A}^{p+1}$ for $i \geq 0$.

So we have $\underset{p}{\oplus} E^1_{-p,n+p} = HH_n(B)$ with B isomorphic to A_t as algebras.

A general fact about convergent spectral sequences implies that

$$\dim HH_n(A) \leq \dim HH_n(A_t).$$

To prove 2), we use the reduced bicomplex $\bar{B}(A)$ to compute cyclic homology ([Lo], page 58), and we define on it an increasing filtration as above. ■

Now, we are interested in algebras A for which inequality 1 or 2 of theorem 2.3 becomes an equality.

Theorem 2.4 *Let A be an augmented algebra over a characteristic zero field k. Let $\bar{A}$ be its augmentation ideal. We assume that $\bar{A}$ is a finite dimensional k-vector space. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$, and $\bar{A}_t = \bar{A}$ as k-vector space. Suppose that there exists $n \geq 1$ such that*

$$\dim HC_n(A) = \dim HC_n(A_t)$$

then A is commutative and is isomorphic to $A = S/I$ where S is the polynomial algebra $k[X_1, \dots, X_m]$ and I is generated by

$$f_i = X_i^2 - \lambda_i X_i,\ 1 \leq i \leq m, \qquad g_{ij} = X_i X_j - \frac{\lambda_j}{2} X_i - \frac{\lambda_i}{2} X_j,\ 1 \leq i < j \leq m, \qquad \text{and} \quad \lambda_j \in k.$$

Proof: It is a refinement of the proof of theorem 1.4 of [Vi]. ■

Corollary 2.5 *Let A be an augmented algebra over a characteristic zero field k. Let $\bar{A}$ be its augmentation ideal. We assume that $\bar{A}$ is a finite dimensional k-vector space and there exists $N \geq 2$ such that $\bar{A}^N = 0$, and $\bar{A} \neq 0$. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$, and $\bar{A}_t = \bar{A}$ as k-vector space. Suppose that there exists $n \geq 1$ such that*

$$\dim HC_n(A) = \dim HC_n(A_t)$$

then the multiplication is trivial in the augmented algebra A, (namely, A is isomorphic to A_t, as augmented algebras).

Proof:

The hypothesis $\bar{A}^N = 0$ implies $\lambda_i = 0, 1 \leq i \leq m$ so that $x^2 = 0$, for any $x \in \bar{A}$. ■

Remark Theorem 2.4 and Corollary 2.5 remain valid if char$k = p$, $p > 0$, and $n < p - 2$.

Example 2.6 Let $A = k[X]/(X^2 - X)$ and $A_t = k[X]/X^2$. We check,[B-V], that

$$\begin{aligned}
&\tilde{HC}_{2n}(A_t) = \tilde{HC}_{2n}(A) = k \\
&HC_{2n+1}(A_t) = HC_{2n+1}(A) = 0 \\
&HH_n(A_t) = k \quad \text{for all} \quad n > 0 \\
&HH_n(A) = 0 \quad \text{for all} \quad n > 0
\end{aligned}$$

This shows that the hypothesis $\bar{A}^N = 0$ cannot be omitted in corollary 2.5. On the other hand, the Hochschild homology groups of A_t and A are quite distinct.

This observation leads us to hope that the equality of the dimensions of one Hochschild homology group of A andA_t characterizes the trivial product.

Theorem 2.7 *Let A be an augmented algebra over a characteristic zero field k. Let $\bar{A}$ be its augmentation ideal and we assume that $\bar{A}$ has finite dimension. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$ and $\bar{A}_t = \bar{A}$ as k-vector space. Suppose that there exists $n \geq 1$ such that*

$$HH_n(A) = HH_n(A_t)$$

Then the multiplication is trivial in the augmented algebra A(namely A is isomorphic to A_t as augmented algebras).

Proof: It is analogous to the proof of theorem 1.6 in [Vi] but here we do not assume that A is commutative. ■

Remark Theorem 2.7 remains valid if char$k = p > 3$, and $1 \leq n < p - 1$.

3 Examples and remarks

Let A be an augmented algebra over a field k of characteristic zero. We assume that the augmentation ideal $\bar{A}$ has finite dimension d, $d \geq 2$.

We have seen, in §§2, that if $\bar{A} \cdot \bar{A} = 0$, then

$$\lim_{n \to \infty} \sqrt[n]{\dim HH_n(A)} = d.$$

Proposition 3.1 *Let A be the quotient of a polynomial algebra $k[X_1, \ldots, X_r]$ by an ideal generated by a regular sequence $(f_1, \ldots, f_r)$ where $f_i \in \mathfrak{m}^2$, for all i, and $\mathfrak{m} = (X_1, \ldots, X_r)$. Then there exist constants K_1 and K_2, such that*

$$K_2 \cdot n^r \leq \sum_{0 \leq p \leq n} \dim HH_p(A) \leq K_1 \cdot n^r$$

Proof: It relies on results proved in [B-V]. ■

Definition An algebra satisfying the hypothesis of proposition 3.1 is called a complete intersection.

Proposition 3.2 *Let A be a finite dimensional smooth commutative algebra, then we have*

$$\dim HH_n(A) = 0 \quad \text{for} \quad \text{all} \quad n > 0.$$

Proof: It is a direct consequence of the Hochschild-Kostant-Rosenberg, [H-K-R]. ■

Conjecture 3.3 *Let A be an augmented commutative algebra over a field, where the augmentation ideal has finite dimension d, $d \geq 2$. If A is neither smooth nor a complete intersection, then there exist real numbers C_1, C_2, $1 < C_2 \leq C_1 \leq d$ such that*

$$C_2^n \leq \sum_{0 \leq p \leq n} \dim HH_p(A) \leq C_1^n$$

Example 3.4 $A = k[x]/x^2 \times_k B$

A is the fiber product over k of $k[x]/x^2$ and B, where B is a finite dimensional augmented commutative algebra which is not smooth.

The fact that B is not smooth implies that there exists $y \in \bar{B}$ and $y \notin \bar{B}^2$. Consider $X = (x, 0) \in \bar{A}$ and $Y = (0, y) \in \bar{A}$; we have $X^2 = XY = 0$.

Proposition 9 of [La] implies that for $n = 4m, m \geq 1$, $\dim HH_n(A) \geq 2^{m-1}$. So we have:

$$C_2^n \leq \sum_{0 \leq p \leq n} \dim HH_p(A) \leq C_1^n$$

where $\quad C_2 = \sqrt[4]{2} \quad$ and $\quad C_1 \approx 1 + \dim \bar{B}$.

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