Graded-commutativity of the Yoneda product of Hopf bimodules.

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Abstract

We prove that the Yoneda product for Hopf bimodule extensions is graded-commutative and we give a candidate for a graded Lie bracket on the cohomology of Hopf algebras defined by Gerstenhaber and Schack.

2000 Mathematics Subject Classification: 16E40, 16W30, 57T05.

Keywords: Hopf bimodules, cohomology, cup-product.

1 Introduction

The object of this paper is to give evidence towards proving that M. Gerstenhaber and S. Schack's conjecture on the cohomology of Hopf algebras is true.

Let H be a Hopf algebra. In [T1, T2] we identified various cohomologies for Hopf algebras and Hopf bimodules with the Ext^* functor in the category $\mathcal{H}$ of Hopf bimodules over H. We then defined a cup-product on some of these cohomologies, and proved that it corresponds to the Yoneda product of extensions *via* these identifications. We are in fact interested in the algebraic structure of one of these cohomologies, $\mathrm{H}_b^*(H,H)$: we would like to know whether this cohomology is a Gerstenhaber algebra. Gerstenhaber algebras are graded algebras, which are graded-commutative, and which are endowed with a graded Lie product (for a shifted grading) compatible with the cup-product. This graded Lie bracket appears in many cohomologies adapted to the study of deformations of algebraic structures, and can be used to describe the obstructions to their deformations. In the case of $\mathrm{H}_b^*(H,H)$, M. Gerstenhaber and S. Schack conjectured in [GS] that this cohomology is a Gerstenhaber algebra.

In this paper, we prove that the cup-product is graded-commutative, using techniques of S. Schwede (*cf.* [S]); this result provides evidence of the truth of the above conjecture. We also give a candidate for a graded Lie bracket, which is graded anticommutative.

We will not define the cohomology $\mathrm{H}_b^*(H,H)$, but we will consider instead the cohomology $\mathrm{Ext}_{\mathcal{H}}^*(H,H)$, which is isomorphic to it. $\otimes$ denotes the tensor product over the base field k.

An extended version of this paper can be found in [T3].

2 Graded-commutativity of the Yoneda product of Hopf bimodule extensions

In this section, we prove that the Yoneda product of Hopf bimodule extensions of H by H is graded-commutative, using paths of extensions, as did Schwede in [S] in the case of modules over an algebra. This implies that the cup-product defined in [T1] and [T2] on the cohomology defined by Gerstenhaber and Schack in [GS] is graded-commutative.

We refer to [S] Sectino 1 for the definitions and results on the homotopy of (the nerve of) a category which we shall need. Let us fix notations.

We shall denote by $\mathcal{E}xt^n_{\mathcal{H}}(H,H)$ the category of extensions of Hopf bimodules from H to H. Then $\pi_0\mathcal{E}xt^n_{\mathcal{H}}(H,H) \cong \mathrm{Ext}^n_{\mathcal{H}}(H,H)$ and $\pi_1\mathcal{E}xt^n_{\mathcal{H}}(H,H) \cong \mathrm{Ext}^{n-1}_{\mathcal{H}}(H,H)$, where paths and loops are defined as in [S] Section 1. Given two extensions $\mathbf{E}$ and $\mathbf{F}$, we denote by $\mathbf{E}\sharp\mathbf{F}$ their Yoneda product. If $\mathbf{E} = 0 \to H \overset{i_{\mathbf{E}}}{\to} E_{m-1} \to \cdots \to E_0 \overset{p_{\mathbf{E}}}{\to} H \to 0$ we denote by $(-1)\mathbf{E}$ the sequence obtained from $\mathbf{E}$ by replacing $p_{\mathbf{E}}$ by $-p_{\mathbf{E}}$; this represents the inverse of $\mathbf{E}$ in $\pi_0\mathcal{E}xt^m_{\mathcal{H}}(H,H)$ with respect to Baer sum.

To prove that this product is graded-commutative, we need to prove that the classes in $\pi_0\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$ of $\mathbf{F}\sharp\mathbf{E}$ and of $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ are equal, so we need to find a path from $\mathbf{F}\sharp\mathbf{E}$ to $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$. This will be done by relating both these extensions to a tensor product of extensions of Hopf bimodules.

Proposition 2.1 *Let E and F be Hopf bimodules over H. We define $E\underline{\otimes}_H F$ as the usual tensor product over H with the standard H-bimodule structure and with codiagonal bicomodule structure (*see *for instance [T1]).*

Let $\mathbf{E}$ and $\mathbf{F}$ be Hopf bimodule extensions of degrees m and n respectively. Then their tensor product over H, defined by $(\mathbf{E}\underline{\otimes}_H\mathbf{F})_r := \bigoplus_{\substack{s+t=r\\ s,t\geqslant 0}} E_s\underline{\otimes}_H F_t$ for $0 \leqslant r \leqslant m+n$ and $(\mathbf{E}\underline{\otimes}_H\mathbf{F})_{-1} := H$, is also a Hopf bimodule extension of H by itself.

Proof: The proof uses the fact that Hopf bimodules are free as right H-modules (*see* for instance [Mo] Theorem 1.9.4), and therefore flat as right H-modules. ■

Given two extensions $\mathbf{E}$ and $\mathbf{F}$ as above, we now construct a path from $\mathbf{F}\sharp\mathbf{E}$ to $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$. In fact, as in [S] Section 2, we construct a loop in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$ going through these two extensions.

We can define, as in [S] p160, maps $\lambda_{\mathbf{E},\mathbf{F}} : \mathbf{E}\underline{\otimes}_H\mathbf{F} \to \mathbf{F}\sharp\mathbf{E}$ and $\rho_{\mathbf{E},\mathbf{F}} : \mathbf{E}\underline{\otimes}_H\mathbf{F} \to (-1)^{mn}\mathbf{E}\sharp\mathbf{F}$, that are morphisms of Hopf bimodule extensions. We may now consider the following loop of extensions in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$:

$$\begin{tikzcd}[column sep=small]
 & \mathbf{E}\underline{\otimes}_H\mathbf{F} \arrow[dl, "\lambda_{\mathbf{E},\mathbf{F}}"'] \arrow[dr, "\rho_{\mathbf{E},\mathbf{F}}"] & \\
\mathbf{F}\sharp\mathbf{E} & & (-1)^{mn}\mathbf{E}\sharp\mathbf{F} \\
 & (-1)^{mn}\mathbf{F}\underline{\otimes}_H\mathbf{E} \arrow[ul, "\rho_{\mathbf{F},\mathbf{E}}"] \arrow[ur, "\lambda_{\mathbf{F},\mathbf{E}}"'] &
\end{tikzcd}$$

We orient it counter-clockwise and call it $\Omega(\mathbf{F}, \mathbf{E})$.

There is therefore a path in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H, H)$ from $\mathbf{F}\sharp\mathbf{E}$ to $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ (either the upper part or the lower part of the diagram above), so $\mathbf{F}\sharp\mathbf{E}$ and $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ represent the same element in $\pi_0\mathcal{E}xt^{m+n}_{\mathcal{H}}(H, H) \cong \mathrm{Ext}^{m+n}_{\mathcal{H}}(H, H)$.

Furthermore, as in [S] p161, the construction of the loop $\Omega(\mathbf{F}, \mathbf{E})$ is functorial in $\mathbf{F}$ and $\mathbf{E}$, so this defines a map:

$$\Omega : \pi_0\mathcal{E}xt^m_{\mathcal{H}}(H, H) \times \pi_0\mathcal{E}xt^n_{\mathcal{H}}(H, H) \longrightarrow \pi_1\mathcal{E}xt^{m+n}_{\mathcal{H}}(H, H),$$

called the *loop bracket*.

Remark 2.2 *The graded-commutativity of the Yoneda product has also been proved by M. Farinati and A. Solotar in [FS] using different methods.*

3 The loop bracket

We shall now consider some properties of the loop bracket, which give some evidence that M. Gerstenhaber and S. Schack's conjecture might be

true in general.

Let us recall the definition of a G-algebra:

Definition 3.1 [*See* for instance [GS] Section 5] *A* G-algebra *is a graded k-module $\Lambda = \oplus_n\Lambda_n$ equipped with two multiplications, $(\lambda, \nu) \mapsto \lambda \smile \nu$ and $(\lambda, \nu) \mapsto [\lambda, \nu]$, satisfying the following properties:*

(1) $\smile$ *is an associative graded (by degree) commutative product;*

(2) $[-, -]$ *is a graded Lie bracket for which the grading is reduced degree, this being one less than the degree;*

(3) $[-, \eta^p]$ *is a graded derivation of degree $p-1$ of the associative algebra structure for all $\eta^p \in \Lambda_p$:*

$$[\lambda^m \smile \nu^n, \eta^p] = [\lambda, \eta] \smile \nu + (-1)^{m(p-1)}\lambda \smile [\nu, \eta] \quad \forall\lambda, \nu.$$

We shall now consider $\Lambda = \mathcal{E}xt^*_{\mathcal{H}}(H,H)$. We know from Section 2 that the first condition is satisfied. Regarding the bracket, in the previous section we described a map

$$\Omega : \pi_0\mathcal{E}xt^m_{\mathcal{H}}(H,H) \times \pi_0\mathcal{E}xt^n_{\mathcal{H}}(H,H) \longrightarrow \pi_1\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H).$$

We mentioned at the beginning of Section 2 that there are isomorphisms $\varphi : \mathrm{H}^*_b(H,H) \xrightarrow{\sim} \pi_0\mathcal{E}xt^*_{\mathcal{H}}(H,H)$ and $\psi : \mathrm{H}^{*-1}_b(H,H) \xrightarrow{\sim} \pi_1\mathcal{E}xt^*_{\mathcal{H}}(H,H)$. Transporting Ω *via* these isomorphisms, we obtain a map $\mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \to \mathrm{H}^{m+n-1}_b(H,H)$. We therefore have a candidate for the product $[-,-]$. More precisely, we consider the map $[-,-] : \mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \to \mathrm{H}^{m+n-1}_b(H,H)$ such that the diagram

$$\begin{tikzcd}
\mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \arrow[r, "{[-,-]}"] \arrow[d, "\varphi\times\varphi"'] & \mathrm{H}^{m+n-1}_b(H,H) \arrow[d, "\psi"] \\
\pi_0\mathcal{E}xt^m_{\mathcal{H}}(H,H) \times \pi_0\mathcal{E}xt^n_{\mathcal{H}}(H,H) \arrow[r, "\Omega"'] & \pi_1\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)
\end{tikzcd}$$

commutes *up to the sign* $(-1)^n$. However, we do not know any of the isomorphisms explicitly, so we do not have an expression for $[-,-]$, and we cannot prove directly conditions **(2)** and **(3)**.

Therefore, we now need to find out how the conditions **(2)** and **(3)** translate for Ω. These conditions involve in particular the sum of elements of $\mathrm{H}^*_b(H,H)$. Using a result of Retakh's (*see* [R], or [NR] Section 6), we can prove that this corresponds to concatenation of loops:

Proposition 3.2 *The isomorphism ψ is an isomorphism of groups, where the group law on $\mathrm{H}^*_b(H,H)$ is induced by Baer sum and that on $\pi_1\mathcal{E}xt^{*+1}_{\mathcal{H}}(H,H)$ is induced by concatenation of loops.*

Properties **(2)** and **(3)** (adapted to Ω) are satisfied in the case of an associative algebra when considering bimodule extensions, since S. Schwede proved in [S] Theorem 3.1 that Ω corresponds to Gerstenhaber's Lie bracket in that case. However, he doesn't give a direct proof of these properties for Ω, but uses universal properties to prove that Ω does correspond to the already existing bracket $[-,-]$ in the way described above.

In the case of finite dimensional Hopf algebras, these properties hold too: M. Farinati and A. Solotar defined a bracket on $\mathrm{H}^*_b(H,H)$ (*see* [FS]) by viewing this cohomology as a subalgebra of the Hochschild cohomology of the Drinfel'd double of H with coefficients in k. We can prove that in this case, the loop bracket again corresponds to their Lie bracket.

These results give strong indications that $\mathrm{H}^*_b(H,H)$ is a Gerstenhaber algebra in general. However, in the general case we would need to prove properties **(2)** and **(3)** directly. We do not yet know how to do this entirely, but we do have the equivalent of the graded anti-commutativity of the bracket:

Proposition 3.3 [*See* [T3] Proposition 5.4] *The graded anti-commutativity of the Lie bracket translates as*

$$\Omega(\mathbf{E},\mathbf{F}) = \Omega(\mathbf{F},\mathbf{E})^{(-1)^{mn}}$$

where $\mathbf{E}$ *is of degree* m *and* $\mathbf{F}$ *is of degree* n*. The loop bracket* Ω *satisfies this relation.*

Proof: The first part is straightforward. For the second part, only the case where both m and n are odd needs to be considered. For this, we first show that, given three extensions $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$, and changing base points so that the identities make sense, we have:

$$\begin{aligned} \Omega(\mathbf{G},\mathbf{E}+\mathbf{F}) &= \Omega(\mathbf{G},\mathbf{E}).\Omega(\mathbf{G},\mathbf{F}) \text{ and} \\ \Omega(\mathbf{F}+\mathbf{G},\mathbf{E}) &= \Omega(\mathbf{F},\mathbf{E}).\Omega(\mathbf{G},\mathbf{E}). \end{aligned}$$

We then prove that the loop $\Omega(\mathbf{F},\mathbf{0})$ is trivial, and the result follows. ∎

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12/12/2002.
[AMA - Algebra Montpellier Announcements - 01-2003] [September 2003]