On semisimple Hopf algebras of low dimension

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Abstract

We announce recent progress on the question about the semisolvability of semisimple Hopf algebras of dimension < 60.

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1 Introduction

Let H be a finite dimensional Hopf algebra over a field k.

Definition 1.1 *A Hopf subalgebra $A \subseteq H$ is called* normal *if $h_1 A \mathcal{S}(h_2) \subseteq A$, for all $h \in H$. If H does not contain proper normal Hopf subalgebras then it is called* simple*.*

If $A \subseteq H$ is a normal Hopf subalgebra then the structure of H can be reconstructed from A and the quotient Hopf algebra $\overline{H} = H/HA^+$; more precisely, it is known that in this case H is isomorphic to a bicrossed product $H \simeq A\#^{\rho,\tau}_{\rightharpoonup,\sigma}\overline{H}$, where $(\rightharpoonup, \sigma, \rho, \tau)$ is a *compatible datum*; see for instance [2, 5, 9]. This fact implies that, when trying to classify Hopf algebras of a given finite dimension, it is an important problem to decide whether the Hopf algebra is simple or not.

We shall assume from now on that the field k is algebraically closed of characteristic zero.

We say that a finite dimensional Hopf algebra H is *trivial* if it is isomorphic to a group algebra or to a dual group algebra. Then, H is trivial if and only if it is commutative or cocommutative.

The following definition appears in [11]. It generalizes the corresponding notion for finite groups.

Definition 1.2 *H is called* lower semisolvable *if there exists a chain of Hopf subalgebras $H_{n+1} = k \subseteq H_n \subseteq \dots \subseteq H_1 = H$ such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all* factors *$\overline{H}_i := H_{i+1}/H_{i+1}H_i^+$ are trivial. Dually, H is called* upper semisolvable *if there exists a chain of quotient Hopf algebras $H_{(0)} = H \to H_{(1)} \to \dots \to H_{(n)} = k$ such that each of the maps $H_{(i-1)} \to H_{(i)}$ is normal, and all* factors *$H_i := H_{(i-1)}^{\mathrm{co}\pi_i}$ are trivial.*

Remark 1.3 *We have that H is upper semisolvable if and only if H^* is lower semisolvable See [11]. If this is the case, then H can be obtained from group algebras and their duals by means of (a finite number of) extensions; in particular, H is semisimple.*

The following question was posed by S. Montgomery.

Question 1.4 *[10, Question, pp. 269]. Let H be a semisimple Hopf algebra of dimension less than 60. Must H be semisolvable?*

Let H be a semisimple Hopf algebra over k. If $\dim H = p^n$, where p is a prime number, then H has a non-trivial central group-like element [8]; inductively, one can see that H is both upper and lower semisolvable [11]. Also, if $\dim H = pq^2$, where $p \neq q$ are prime numbers, then it was shown in [12, 13, 14] that, under the assumption that H and H^* are both of Frobenius type, either H or H^* contains a non-trivial central group-like element. This implies that these Hopf algebras are also semisolvable, since semisimple Hopf algebras of dimension p, pq and q^2 are trivial. In [14] we showed that all semisimple Hopf algebras of dimension $pq^2 < 100$ are of Frobenius type; so that these are all semisolvable.

However, not every non-trivial semisimple Hopf algebra H is semisolvable. The smallest known example was constructed by D. Nikshych in [17]: in this case H is a cocycle twist of the group algebra of the simple group $\mathbf{A}_5$. Moreover, it was shown in [17] that if G is a finite simple group and $\phi \in kG \otimes kG$ is a non-trivial invertible pseudo 2-cocycle, then the twisted group algebra $(kG)_\phi$ is a non-trivial semisimple Hopf algebra, which is simple as a Hopf algebra.

In dimension less than 60, all known examples of semisimple Hopf algebras are semisolvable. The dimensions where the problem remains open are 24, 30, 36, 40, 42, 48, 54 and 56. We refer the reader to [1, 10] for an account of previous results on the problem of classification.

We also point out that, in the related context of Kac algebras, several classification results in low dimension were obtained by Izumi and Kosaki in their work [4]; in that paper, the authors classify all Kac algebras of dimensions 16, 24, $pq^2 < 60$ and $pqr < 60$.

2 Main results

The following is our main theorem.

Theorem 2.1 *Let H be a semisimple Hopf algebra of dimension 24, 30, 36, 40, 42, 54 or 56 over k. Then H is either upper or lower semisolvable.*

This theorem, combined with previous results, leaves open the question of semisolvability in dimension < 60 only in the case of dimension 48.

We also give the complete classification in dimensions 30 and 42.

Theorem 2.2 *Let H be a semisimple Hopf algebra of dimension* 30 *over k. Then H is trivial.*

The known non-trivial examples in dimension 42, $\mathcal{A}_7(2,3)$ and $\mathcal{A}_7(3,2) \simeq \mathcal{A}_7(2,3)^*$, were constructed in [3].

Theorem 2.3 *Let H be a non-trivial semisimple Hopf algebra of dimension* 42 *over k. Then H is isomorphic to one of the Hopf algebras $\mathcal{A}_7(2,3)$ or $\mathcal{A}_7(3,2)$.*

The classification of semisimple Hopf algebras of dimension pqr, where p, q and r are distinct prime numbers, was given in [12] under the assumption that H admits an extension with commutative 'kernel' and cocommutative 'cokernel' (a so called *abelian* extension). The proof of the above theorems consists of establishing the fact that semisimple Hopf algebras of dimension 30 and 42 admit abelian extensions.

3 About the proof of Theorems 2.1, 2.2 and 2.3

The lines of the proof of Theorem 2.1 are the following: for each fixed dimension, we first consider the possible algebra and coalgebra structures (which turn out to be of Frobenius type). Next, we discuss some properties of irreducible characters of low degree which allow, in most cases, to prove the existence of quotient Hopf algebras or Hopf subalgebras, for each fixed algebra or coalgebra structure, respectively. One of the main tools towards this end is the use of the Nichols-Richmond theorem on irreducible characters of degree 2 [16] and some of its consequences. Our final goal is to deduce the existence of proper normal Hopf subalgebras.

Another tool we repeatedely use is the reduction to the biproduct construction of Radford, often in combination with the main result in [20].

Proposition 3.1 *Let H is a non-trivial semisimple Hopf algebra of dimension < 60. Then the following are equivalent:*
(i) H is not simple;
(ii) H is either upper or lower semisolvable.

Proof. Suppose that the dimension of H is of the form pq^3 or p^2q^2, where $p \neq q$ are prime numbers. If H has a proper normal Hopf subalgebra A such that the quotient $\overline{H} := H/HA^+$ is commutative or cocommutative, then H is upper semisolvable; indeed, since $\dim A < \dim H$, A is upper semisolvable.

It is enough to prove that (i) $\Longrightarrow$ (ii); so let $1 \to A \to H \to \overline{H} \to 1$ be a short exact sequence of Hopf algebras, with $\dim A, \dim \overline{H} < \dim H$.

Suppose first that $\dim H = p^2q^2$ or pq^3, where $p \neq q$ are prime numbers. Then either A or $\overline{H}$ are trivial, and the implication follows.

It remains to consider the case $\dim H = 48$; so that, also in this case, either A or $\overline{H}$ are trivial. Assume that A is non-trivial. Then A admits a normal upper series of Hopf subalgebras: indeed, semisimple Hopf algebras of dimension 24 are both upper and lower semisolvable: this can be seen using the methods described in the next section. Hence H does too.

To prove Theorems 2.2 and 2.3 we use the methods outlined above to first show that these Hopf algebras are not simple, and discard several possibilities for the (co)algebra structures. Then we combine this with information on quotient coalgebras, as explained in the next section, to show that there exists a normal Hopf subalgebra $A \subseteq H$ such that the quotient Hopf algebra is cocommutative; that is, H fits into an abelian extension.

4 Quotient coalgebras

Let H be a finite dimensional semisimple Hopf algebra and let A be a Hopf subalgebra of H. Consider the quotient coalgebra $\overline{H} := H/HA^+$, which is a cosemisimple coalgebra. We discuss the corepresentation theory of $\overline{H}$ in relation with that of H and the corestriction functor ${}^H\mathcal{M} \to {}^{\overline{H}}\mathcal{M}$.

We show that if C is a simple subcoalgebra of H such that $Ca \subseteq C$, for all $a \in A$, then the dual of the quotient coalgebra C/CA^+ and the crossed product A_α, where $\alpha : A \otimes A \to k$ is a certain 2-cocycle, constitute a commuting pair in C^*. This is applied in combination with the results of Tambara and Yamagami [21] and Masuoka's main result in [7], in some instances of the proof of Theorem 2.1.

In particular, when $A = kG$ is the group algebra of a subgroup G of $G(H)$ and V is a simple H-comodule, we deduce that $\mathrm{End}^{\overline{H}}(V)$ is isomorphic as an algebra to a twisted group algebra $k_\alpha\Gamma$, where $\Gamma \subseteq G$ is the stabilizer of V, *i.e.* $\Gamma = \{g \in G : V \otimes g \simeq V\}$, and $\alpha : \Gamma \times \Gamma \to k^\times$ is a 2-cocycle. This result implies that the multiplicity of an irreducible $\overline{H}$-comodule in V is a divisor of the order of Γ. In particular, when the group Γ is abelian, all irreducible $\overline{H}$-comodules in the restriction of V to $\overline{H}$ appear with the same multiplicity d, where d divides the order of Γ. This allows us to recover the result in [7, Prop. 2.4].

Some of these results are applied to the case when H is a biproduct in the sense of Radford: $H \simeq R\#A$. Indeed, in this case R is isomorphic as a coalgebra to the quotient H/HA^+.

Detailed proofs will be given in [15].

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