N-Koszul Algebras, A summary

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Abstract

In this paper we study N-koszul algebras which were introduced by R. Berger. We show that when $n \geq 3$, these are classified by the Ext-algebra being generated in degrees 0, 1, and 2. We give a description of the Ext-algebra using the analogous of the Koszul complex and we also show that it is, is a Koszul algebra, after regrading. This notions can be generalized in various ways one was is to study categories of homogeneous algebras and this can be found in [2]

1 A general overview

In [1], Roland Berger introduced what he called "generalized" Koszul algebras. The generalized Koszul algebras are graded algebras $A = K \oplus A_1 \oplus A_2 \oplus \cdots$ which are generated in degrees 0 and 1 such that there is a graded projective resolution of K for which the k^{th} projective in the resolution is generated in degree $\delta(k)$ where

$$\delta(k) = \left\{ \begin{array}{l} \frac{k}{2}n \text{ if } k \text{ is even} \\ \frac{k-1}{2}n + 1 \text{ if } k \text{ is odd,} \end{array} \right. \quad \text{for some } n.$$

We generalize this definition to the nonlocal case, i.e., K is replaced by a semisimple K-algebra and we call this class of algebras *N-koszul algebras*.

We provide some general tools to study when the Yoneda product map

$$\mathrm{Ext}^k_A(A_0, A_0) \otimes \mathrm{Hom}_A(M, A_0) \to \mathrm{Ext}^k(M, A_0)$$

is surjective where $A = A_0 \oplus A_1 \oplus \cdots$ is a graded algebra and M is a graded left A-module.

We summarize the contents of the paper. In order to do that we need to introduce some notation:

2 Notation

Let K be a commutative ring and $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded K-algebra where the direct sum is as K-modules. Assume that A is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$. Let $\mathrm{Gr}(A)$ denote the category of graded A-modules and degree 0 homomorphisms and $\mathrm{Mod}(A)$ denote the category of left A-modules. We denote by $\mathrm{gr}(A)$ and $\mathrm{mod}(A)$ the full subcategories of $\mathrm{Gr}(A)$ and $\mathrm{Mod}(A)$ respectively, consisting of finitely generated modules. Let $F : \mathrm{Gr}(A) \to \mathrm{Mod}(A)$ denote the forgetful functor and $\mathrm{Gr}_0(A)$ (respectively $\mathrm{gr}_0(A)$) be the full subcategory of $\mathrm{Gr}(A)$ whose objects are the graded modules (respectively, finitely generated modules) generated in degree 0.

We assume A_0 is a semisimple Artin algebra. The graded Jacobson radical of A, which we denote by $\mathbf{r}_A$, or simply $\mathbf{r}$, when no confusion can arise, is $A_1 \oplus A_2 \oplus \cdots$. Since A is generated in degrees 0 and 1, it follows that $\mathbf{r}^i = A_i \oplus A_{i+1} \oplus \cdots$. We fix a minimal graded projective resolution of A_0,

$$\mathcal{P}^\bullet : \quad \cdots \to P^k \to \cdots \to P^1 \to P^0 \to A_0 \to 0,$$

where A_0 is viewed as a graded A-module generated in degree 0.

3 Summary

One of the fundamental results of the paper is the following:

Proposition 3.1 *Let $M \in Gr_0(A)$ be finitely generated with minimal graded projective resolution $(\mathcal{Q}^\bullet)$. Assume that P^n is generated in degree s and Q^n is finitely generated. Then Q^n is generated in degree s if and only if $Ext_A^k(A_0, A_0) \otimes_K Hom_A(M, A_0) \to Ext_A^k(M, A_0)$ is surjective; that is, $Ext_A^k(A_0, A_0) \cdot Hom_A(M, A_0) = Ext_A^k(M, A_0)$.*

One immediately consequence of this theorem is the following:

Corollary 3.2 *Let $M \in Gr_0(A)$ with minimal graded projective resolution $\mathcal{Q}^\bullet$. Assume that P^k is generated in degree s and Q^k is generated in degree t. Then $t \geq s$.* $\square$

If M_1, M_2, M_3 are A-modules, we let

$$\mathcal{Y}_{m,k} : \mathrm{Ext}_A^m(M_2, M_3) \otimes_K \mathrm{Ext}_A^k(M_1, M_2) \to Ext_A^{m+k}(M_1, M_3)$$

be the Yoneda product. We will usually write $\mathcal{Y}_{m,k}$ as $\mathcal{Y}$ when no confusion can arise. Furthermore, we will denote the image of $\mathcal{Y}$ in $Ext_A^{m+k}(M_1, M_3)$ by $\mathrm{Ext}_A^m(M_2, M_3) \cdot \mathrm{Ext}_A^k(M_1, M_2)$.

Lemma 3.3 *Let M be a finitely generated graded A-module. The map $\mathcal{Y} : Ext_A^k(A_0, A_0) \otimes_K Hom_A(M, A_0) \to Ext_A^k(M, A_0)$ factors through $Ext_A^k(M/\mathbf{r}M, A_0) \to Ext_A^k(M, A_0)$, the map induced from the canonical surjection $M \to M/\mathbf{r}M$. Furthermore, the induced map $\mathcal{Y}' : Ext_A^k(A_0, A_0) \otimes_K Hom_A(M, A_0) \to Ext_A^k(M/\mathbf{r}M, A_0)$ is surjective.*

The fundamental proposition, 3.1, is used to prove the following:

Proposition 3.4 *Suppose that P^i is finitely generated with generators in degree n_i, for $i = \alpha, \beta, \alpha + \beta$. Assume that*

$$n_{\alpha+\beta} = d_\alpha + d_\beta.$$

Then the Yoneda maps $Ext_A^\alpha(A_0, A_0) \otimes_K Ext_A^\beta(A_0, A_0) \to Ext_A^{\alpha+\beta}(A_0, A_0)$ and $Ext_A^\beta(A_0, A_0) \otimes_K Ext_A^\alpha(A_0, A_0) \to Ext_A^{\alpha+\beta}(A_0, A_0)$ are both surjective. Thus,

$$Ext_A^{\alpha+\beta}(A_0, A_0) = Ext_A^\alpha(A_0, A_0) \cdot Ext_A^\beta(A_0, A_0)$$

$$= Ext_A^\beta(A_0, A_0) \cdot Ext_A^\alpha(A_0, A_0).$$

In section 4 of the paper appears the definition of N-koszul algebras.

Let $A = A_0 + A_1 + A_2 + \cdots$ be a graded K-algebra generated in degrees 0 and 1, where A_0 is a product of the field K, A_1 is a finitely generated K-module and that $\mathcal{P}^\bullet$ is a minimal graded A-projective resolution of A_0. We say that A is a *N-koszul algebra* if, for each $n \geq 0$, P^n can be generated in exactly one degree, $\delta(n)$, and

$$\delta(k) = \begin{cases} \frac{k}{2}n & \text{if } k \text{ is even} \\ (\frac{k-1}{2}n) + 1 & \text{if } k \text{ is odd} \end{cases}.$$

We present the following characterization of N-koszul algebras.

Theorem 3.5 *Let $A = T_{A_0}(A_1)/I$ where I can be generated by elements of $\otimes^n_{A_0} A_1$ for some $n \geq 2$. Then A is a N-koszul algebra if and only if the Ext-algebra $E(A)$ can be generated in degrees 0, 1, and 2 in the ext-degree grading.*

From the proof of the above theorem, we get the following important result.

Corollary 3.6 *If A is a N-koszul algebra, with $n > 2$, then*

$$Ext_A^{2m+1}(A_0, A_0) \cdot Ext_A^{2k+1}(A_0, A_0) = (0),$$

for all $k, m \geq 0$. $\square$

In section 5 we introduce the notion of N-koszul modules which is a natural generalization of the analogous notion of Koszul modules.

We say a left graded A-module M is a *N-koszul module* if there is a graded A-projective resolution $\cdots \to Q^2 \to Q^1 \to Q^0 \to M \to 0$ such that Q^n is generated in degree $\delta(n)$ where

$$\delta(k) = \begin{cases} \frac{k}{2}n & \text{if } k \text{ is even} \\ (\frac{k-1}{2}n) + 1 & \text{if } k \text{ is odd} \end{cases}.$$

If M is a N-koszul module then $M \in \mathrm{Gr}_0(A)$ since Q^0 is generated in degree 0. Note that if $n = 2$, then a module is N-koszul if and only if it has a linear projective resolution. Thus, in this case, being a N-koszul coincides with being a Koszul module.

We state in this summary two results which are proved in section 5 of the paper:

Proposition 3.7 *Let A be a N-koszul algebra of type n and M a N-koszul module. Then $\Omega^2(M)[-n]$ and $\Omega^1(\mathbf{r}M)[-n]$ are both N-koszul A-modules.*

If M is a left A-module, let $\mathcal{E}(M)$ denote the left $E(A)$-module $\oplus_{n\geq 0}\mathrm{Ext}^k(M, A_0)$, where the module structure is given by the Yoneda product.

Theorem 3.8 *Let $A = A_0 + A_1 + \cdots$ be a N-koszul algebra of type n with $n \geq 3$ and let M be a left N-koszul A-module. Then $\mathcal{E}(M)$ can be generated in degree 0. Moreover, $Ext^{2k+1}(A_0, A_0) \cdot Ext^{2m+1}(M, A_0) = (0)$ for all $k, m \geq 0$.*

Section 6 on the paper studies the sum of the even extensions. The main result in this section is the following:

Let M be a left N-koszul A-module. We let

$$E^{ev}(A) = \oplus_{k\geq 0}\mathrm{Ext}^{2k}(A_0, A_0)$$

and

$$\mathcal{E}^{ev}(M) = \oplus_{k\geq 0}\mathrm{Ext}^{2k}(M, A_0).$$

We grade $E^{ev}(A)$ by $E^{ev}(A)_k = \mathrm{Ext}^{2k}(A_0, A_0)$ and view $\mathcal{E}^{ev}(M)$ as a graded $E^{ev}(A)$-module where $\mathcal{E}^{ev}(M)_k = \mathrm{Ext}^{2k}_A(M, A_0)$. We call this the *even-grading.*

The following result is the main result of this section.

Theorem 3.9 *Let $A = A_0 + A_1 + \cdots$ be a N-koszul algebra of type n and M a N-koszul module. Then, in the even-grading, $E^{ev}(A)$ is a Koszul algebra and $\mathcal{E}^{ev}(M)$ is a Koszul $E^{ev}(A)$-module.*

In section 7 we study the Ext of a N-koszul algebra, the main result is the following: We denote by $\hat{E}(A)$ the Ext-algebra with a new grading where the homogeneous component of degree k is the sum of $Ext^{2k-1} + Ext^{2k}$. We show the following result.

Theorem 3.10 *Let $A = A_0 + A_1 + \cdots$ be a N-koszul algebra with $n \geq 3$. Let M be a N-koszul A-module. Then $\hat{E}(A)$ is a Koszul algebra and $\hat{\mathcal{E}}(M)$ is a Koszul $\hat{E}(A)$-module.*

In section 8 we give a new proof of one of Berger's result.

We introduced a N-koszul complex. This complex is defined for any algebra of the type $A = T_{A_0}(A_1)/I$ where I is generated by elements of degree n. It happens that this complex is a minimal resolution of A_0 if and only if the algebra is a N-koszul algebra. We end this summary by describing this complex. We point out that some of the results which we got in the earlier sections can also be obtained by using this complex, in particular one can get a good description of the Ext algebra of A, by using it.

Let $R = I \cap (\otimes^n_{A_0} A_1)$. Note that R is an A_0-A_0-submodule of $\otimes^n_{A_0} A_1$. We now assume K is a field and that A_0 is not only semisimple, but, as a ring, A_0 is $K \times K \times \cdots \times K$. Let $T = T_{A_0}(A_1)$ and if $x \in T$, let $\bar{x}$ denote $\pi(x)$ where $\pi : T \to A$ is the canonical surjection. In this case, T is isomorphic to a path algebra $K\Gamma$ for some quiver Γ. Let $\{v_1, \ldots, v_n\}$ be the arrows of Γ. Then the v_i's are a full set of orthogonal idempotents. We say a nonzero element $x \in T$ is *left uniform* if there exists a vertex v_i such that $x = v_i x$. If x is left uniform, we let $o(x) = v_i$ if $x = v_i x$.

We define the *generalized Koszul complex of* R as follows. Let $H_0 = A_0$, $H_1 = A_1$, and, for $k \geq n$,

$$H_k = \cap_{i+j+d=k} (\otimes^i_{A_0} A_1) \otimes_{A_0} R \otimes_{A_0} (\otimes^j_{A_0} A_1).$$

As usual, we let

$$\delta(k) = \left\{ \begin{array}{ll} \frac{k}{2}n & \text{if } k \text{ is even} \\ (\frac{k-1}{2}n) + 1 & \text{if } k \text{ is odd} \end{array} \right.$$

. We define $Q^k = A \otimes_{A_0} H_{\delta(k)}$ and note that Q^k is a projective left A-module for $k \geq 0$.

We wish to define maps $d^k : Q^k \to Q^{k-1}$ for $k \geq 1$. For this we need the following lemma which relates to the condition (ec) in Berger's work. To simplify notation, we will denote $\otimes^i_{A_0} A_1$ as simply A_1^i and write $\otimes_{A_0}$ as simply $\otimes$.

Lemma 3.11 *notation, if A is N-koszul then, for $2 \leq i < n$,*

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subseteq A_1^{i-1} \otimes R \otimes A_1.$$

We define $d^m : Q^m \to Q^{m-1}$. Recall that $Q^m = A \otimes H_{\delta(m)}$. From the definition and that $R \subset A_1^n$, we note that $H_{\delta(m)} \subset A_1^{\delta(m)}$. We write elements of $H_{\delta(m)}$ as $x_1 \otimes \cdots \otimes x_{\delta(m)}$ where the x_i are in A_1. If $m = 2k$, define

$$d^m(a \otimes x_1 \otimes \cdots \otimes x_{kn}) = a x_1 x_2 \cdots x_{n-1} \otimes x_n \otimes \cdots \otimes x_{kn}.$$

If $m = 2k + 1$, define

$$d^m(a \otimes x_1 \otimes \cdots \otimes x_{kn+1}) = ax_1 \otimes x_2 \otimes \otimes \cdots \otimes x_{kn+1}.$$

Using the description of $H_{\delta(k)}$ in the form of the corollary one shows that the maps are well-defined. We now can state one of Berger's main results.

Theorem 3.12 *[1, Thm 2.1] Let $A = K\Gamma/I$ where I is an ideal generated in degree n. The following statements are equivalent.*

(i) A is a N-koszul algebra.

(ii) $\{Q^k, d^k\}$ is a minimal A-projective resolution of A_0.

In section 9 we provide a description of the Ext-algebra $E(A)$ when A is a N-koszul algebra with $n > 2$ We need to introduce some notation in order to be able to state this description.

Recall that since $A_0 = \prod_{i=1}^n K$, then the indecomposable A_0-A_0-bimodules are 1-dimensional over K and of the form $e_iA_0e_j$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 occurring in the i^{th} component. Furthermore, $A_0^{op} = A_0$ since A_0 is a commutative ring. Since $A_0 \otimes_K A_0$ is a semisimple ring, it follows that every A_0-A_0-bimodule is a direct sum of copies of the 1-dimensional simple modules $e_iA_0e_j$, $1 \leq i, j \leq n$.

Let V be a finitely generated A_0-A_0-bimodule. If W is an A_0-A_0-submodule of V, let $W^* = \mathrm{Hom}_{A_0}(W, A_0)$ where the Hom is as left A_0-modules. The right A_0-module structure on W gives W^* a left A_0-module structure. The right A_0-module structure on A_0 gives W^* an A_0-A_0-bimodule structure. Note that * is a duality on A_0-A_0-bimodules and that if V is a finitely generated bimodule, then V^{**} is naturally isomorphic to V as bimodules. Let $W^\perp = \{f \in V^* \mid f(W) = 0\}$. We see that $W^\perp$ is an A_0-A_0-bimodule if W is.

We identify A_0 and A_0^*. There is a natural isomorphism between $(A_1^i)^* = (\overset{i}{\otimes} A_1)^*$ and $\overset{i}{\otimes} A_1^* = (A_1^*)^i$, which we view as an identification. Let $R^\perp = \{f \in (A_1^*)^n \mid f(x) = 0 \text{ for all } x \in R\}$. Let T^* be the tensor algebra $T_{A_0}(A_1^*) = A_0 \oplus A_1^* \oplus (A_1^*)^2 \oplus \cdots$. The *dual algebra* of A is defined to be $A^! = T^*/ < R^\perp >$.

We see that $A^!$ is a graded algebra since $R^\perp$ is contained in $(A_1^*)^n$. Thus $A^! = A_0^! \oplus A_1^! \oplus A_2^! \oplus \cdots$. Let $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$ where $B_k = A^!_{\delta(k)}$ as vector spaces.

We main result in this section is the following:

Theorem 3.13 *If A is a N-koszul algebra and $n \geq 2$ then $E(A)$ is isomorphic to B as graded algebras. In particular, $Ext_A^k(A_0, A_0)$ is isomorphic to $A^!_{\delta(k)}$.*

In the final section we give some examples of N-koszul algebras with $n > 2$, in particular we characterize the monomial algebras which are N-koszul, this was done in [1], we show that his conclusions carry over quotient of quiver algebras.

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[AMA - Algebra Montpellier Announcements - 01-2003] [September 2003]