ON THE LATTICE OF COTILTING MODULES

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1. The lattice

Let Λ be an associative ring. In this note we show that the collection of (not necessarily finitely generated) cotilting modules over Λ carries the structure of a lattice. We work in the category $\operatorname{Mod}\Lambda$ of (right) Λ-modules and denote by $\operatorname{mod}\Lambda$ the full subcategory of finitely presented Λ-modules. Changing slightly[1] the definition in [1], we say that a Λ-module T is a *cotilting module* if

(T1) the injective dimension of T is finite;
(T2) $\operatorname{Ext}^i_\Lambda(T^\alpha, T) = 0$ for all $i > 0$ and every cardinal α;
(T3) there exists an injective cogenerator Q and a long exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to Q \to 0$ with T_i in $\operatorname{Prod} T$ for all $i = 0, 1, \ldots, n$;
(T4) T is pure-injective.

Here, $\operatorname{Prod} T$ denotes the closure under products and direct factors of T. Two cotilting modules T and T' are *equivalent* if $\operatorname{Prod} T = \operatorname{Prod} T'$. Our first result is a consequence of the fact that the equivalence class of a cotilting module T is determined by

$$^\perp T = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, T) = 0 \text{ for all } i \geq 1\}.$$

Theorem 1.1. *The equivalence classes of Λ-cotilting modules form a set of cardinality at most 2^κ where $\kappa = \max(\aleph_0, \operatorname{card}\Lambda)$.*

Proof. Recall that a class $\mathcal{X}$ of Λ-modules is *definable* if $\mathcal{X}$ is closed under taking products, filtered colimits, and pure submodules. In this case

$$\mathcal{X} = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Hom}_\Lambda(\phi, X) \text{ is surjective for all } \phi \in \Phi\}$$

where Φ is the set of all maps in $\operatorname{mod}\Lambda$ such that $\operatorname{Hom}_\Lambda(\phi, X)$ is surjective for all $X \in \mathcal{X}$; see [4, Section 2.3].

If T is a cotilting module, then $^\perp T$ is definable. This follows from Theorem 5.6 and Proposition 5.7 in [9]. The cardinality of the set of isomorphism classes of maps in $\operatorname{mod}\Lambda$ is bounded by κ, and therefore we have at most 2^κ equivalence classes of cotilting modules. $\square$

We denote by $\operatorname{Cotilt}\Lambda$ the set of equivalence classes of Λ-cotilting modules and we have a natural partial ordering via

$$T \leq T' \iff {}^\perp T \subseteq {}^\perp T'$$

for $T, T' \in \operatorname{Cotilt}\Lambda$. For finite dimensional algebras, the collection of finitely generated (co)tilting modules has some interesting combinatorial structure which is closely related to this partial ordering [10, 11, 3]. Our aim is to show that $\operatorname{Cotilt}\Lambda$

[1](T4) is added to avoid set-theoretic problems. For instance, the classification of modules satisfying (T1) – (T3) over a fixed Dedekind domain R seems to depend on set-theoretic properties of R.

is in fact a lattice. For this we need the concept of a cotorsion pair. We fix a pair $(\mathcal{X},\mathcal{Y})$ of full subcategories of $\operatorname{Mod}\Lambda$. Let

$$\mathcal{X}^{\perp} = \{Y \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,Y)=0 \text{ for all } i\geq 1 \text{ and } X\in\mathcal{X}\},$$

$$^{\perp}\mathcal{Y} = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,Y)=0 \text{ for all } i\geq 1 \text{ and } Y\in\mathcal{Y}\}.$$

The pair $(\mathcal{X},\mathcal{Y})$ is called a *cotorsion pair* for $\operatorname{Mod}\Lambda$ if the following conditions are satisfied:

(1) $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y}=\mathcal{X}^{\perp}$;
(2) every $A\in\operatorname{Mod}\Lambda$ fits into exact sequences $0\to Y_1\to X_1\to A\to 0$ and $0\to A\to Y_2\to X_2\to 0$ with $X_i\in\mathcal{X}$ and $Y_i\in\mathcal{Y}$.

For $n\in\mathbb{N}$ we write $\mathcal{I}_n(\Lambda)=\{X\in\operatorname{Mod}\Lambda \mid \operatorname{id}X\leq n\}$ and let $\mathcal{I}(\Lambda)=\bigcup_{n\in\mathbb{N}}\mathcal{I}_n(\Lambda)$, where $\operatorname{id}X$ denotes the injective dimension of a module X. We need the following example:

Example 1.2. For all $n\in\mathbb{N}$ there exists a cotorsion pair $({}^{\perp}\mathcal{I}_n(\Lambda),\mathcal{I}_n(\Lambda))$. This follows from Theorem 10 in [6] since

$$\mathcal{I}_n(\Lambda)=\{Y\in\operatorname{Mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\Omega^n(\Lambda/\mathfrak{a}),Y)=0 \text{ for all right ideals } \mathfrak{a}\subseteq\Lambda\}$$

by Baer's criterion.

We have a description of cotilting modules in terms of cotorsion pairs which follows directly from work of Angeleri Hügel and Coelho [1, Theorem 4.2], in combination with [9, Proposition 5.7].

Proposition 1.3. *For a full subcategory $\mathcal{X}\subseteq\operatorname{Mod}\Lambda$ the following are equivalent:*

(1) *$\mathcal{X}={}^{\perp}T$ for some cotilting module T with $\operatorname{id}T\leq n$;*
(2) *$\mathcal{X}$ is definable and there is a cotorsion pair $(\mathcal{X},\mathcal{X}^{\perp})$ with $\mathcal{X}^{\perp}\subseteq\mathcal{I}_n(\Lambda)$.*

Moreover, in this case $\mathcal{X}\cap\mathcal{X}^{\perp}=\operatorname{Prod}T$.

Observe that Proposition 1.3 shows how to compute for a cotilting module T its injective dimension:

$$\operatorname{id}T=\inf\{n\in\mathbb{N}\mid {}^{\perp}\mathcal{I}_n(\Lambda)\subseteq{}^{\perp}T\}.$$

The next result describes the infimum of a collection of cotilting modules in $\operatorname{Cotilt}\Lambda$.

Proposition 1.4. *Let $(T_i)_{i\in I}$ be a family of cotilting modules and suppose that $\sup\{\operatorname{id}T_i\mid i\in I\}<\infty$. Then there exists a cotilting module T such that*

$${}^{\perp}T=\bigcap_{i\in I}{}^{\perp}T_i.$$

Moreover, $\operatorname{id}T=\sup\{\operatorname{id}T_i\mid i\in I\}$. The module T is unique up to equivalence and is denoted by $\bigwedge_{i\in I}T_i$.

Proof. We apply Proposition 1.3. There exists a cotorsion pair $(\mathcal{X},\mathcal{Y})$ with $\mathcal{X}={}^{\perp}(\prod_i T_i)$ since $\prod_i T_i$ is pure-injective; see [5, Corollary 10]. Each ${}^{\perp}T_i$ is definable and contains ${}^{\perp}\mathcal{I}_n(\Lambda)$ where $n=\sup\{\operatorname{id}T_i\mid i\in I\}$. Therefore $\mathcal{X}=\bigcap_{i\in I}{}^{\perp}T_i$ is definable and contains ${}^{\perp}\mathcal{I}_n(\Lambda)$. Thus $\mathcal{Y}\subseteq\mathcal{I}_n(\Lambda)$, and we obtain $\mathcal{X}={}^{\perp}T$ for some cotilting module T. $\square$

Example 1.5 (Happel)**.** Fix a field k and let Λ be the path algebra of the quiver $\begin{tikzcd}[cramped] \cdot \arrow[r] \arrow[rr, bend left] & \cdot \arrow[r] & \cdot \end{tikzcd}$ which is tame hereditary. Denote by $S_1=(1,0,1)$ and $S_2=(0,1,0)$ the quasi-simples from the unique exceptional tube of rank 2. Then there are cotilting modules

$$T_1 = (1,0,1) \oplus (2,1,1) \oplus (1,0,0) \quad \text{and} \quad T_2 = (0,1,0) \oplus (2,2,1) \oplus (1,1,0)$$

such that ${}^{\perp}T_1 \cap {}^{\perp}T_2 = {}^{\perp}T$ for $T = \widehat{S}_1 \amalg \widehat{S}_2 \amalg (\coprod_S S_\infty)$ where S runs through the isomorphism classes of quasi-simples different from S_1 and S_2. Here, S_∞ denotes the Prüfer and $\widehat{S}$ denotes the adic module corresponding to S. Moreover, no finite dimensional cotilting module is equivalent to T.

Corollary 1.6. *The partially ordered set* $\operatorname{Cotilt}\Lambda$ *is a lattice. More precisely, for a family* $(T_i)_{i\in I}$ *in* $\operatorname{Cotilt}\Lambda$ *the following holds:*

(1) *The infimum* $\inf\{T_i \mid i \in I\}$ *of all* T_i *exists if and only if* $\sup\{\operatorname{id} T_i \mid i \in I\} < \infty$. *In this case* $\inf\{T_i \mid i \in I\} = \bigwedge_{i\in I} T_i$.
(2) *The supremum* $\sup\{T_i \mid i \in I\}$ *of all* T_i *equals the infimum* $\inf\{T \in \operatorname{Cotilt}\Lambda \mid T_i \leq T$ *for all* $i \in I\}$.

Corollary 1.7. *The map* $(\operatorname{Cotilt}\Lambda, \leq) \longrightarrow (\mathbb{N}, \leq)$ *sending* T *to* $\operatorname{id} T$ *has the following properties:*

(1) $T \leq T'$ *implies* $\operatorname{id} T' \leq \operatorname{id} T$.
(2) $\operatorname{id}(\inf\{T_i \mid i \in I\}) = \sup\{\operatorname{id} T_i \mid i \in I\}$ *for every family* $(T_i)_{i\in I}$, *provided that* $\sup\{\operatorname{id} T_i \mid i \in I\} < \infty$.
(3) $\operatorname{id}(\sup\{T_i \mid i \in I\}) \leq \inf\{\operatorname{id} T_i \mid i \in I\}$ *for every family* $(T_i)_{i\in I}$.

2. Finitistic dimension

In this section we relate the finitistic dimension of Λ to the structure of $\operatorname{Cotilt}\Lambda^{\mathrm{op}}$. Recall that the *finitistic dimension* $\operatorname{Fin.dim}\Lambda$ is the supremum of all projective dimensions of Λ-modules having finite projective dimension. Restriction to finitely presented Λ-modules gives $\operatorname{fin.dim}\Lambda$. The *finitistic injective dimension* of Λ is by definition

$$\operatorname{Fin.inj.dim}\Lambda = \sup\{\operatorname{id} X \mid X \in \operatorname{Mod}\Lambda \text{ and } \operatorname{id} X < \infty\}.$$

Observe that $\operatorname{Fin.dim}\Lambda = \operatorname{Fin.inj.dim}\Lambda^{\mathrm{op}}$ provided that Λ is artinian.

Proposition 2.1. *Let* Λ *be an artin algebra and let* $\mathcal{C}$ *be a class of finitely presented* Λ*-modules. If* $\operatorname{id}\mathcal{C} = \sup\{\operatorname{id} X \mid X \in \mathcal{C}\} < \infty$, *then there exists a cotilting module* T *such that* ${}^{\perp}T = {}^{\perp}\mathcal{C}$ *and* $\operatorname{id} T = \operatorname{id}\mathcal{C}$.

Proof. We apply Proposition 1.3 to obtain the cotilting module T satisfying ${}^{\perp}T = {}^{\perp}\mathcal{C}$. It follows from Theorem 2 in [9] that every definable and resolving subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ induces a cotorsion pair $(\mathcal{X}, \mathcal{X}^{\perp})$. Recall that $\mathcal{X}$ is *resolving* if $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and contains all projectives. Clearly, ${}^{\perp}\mathcal{C}$ is resolving. Using the fact that the modules in $\mathcal{C}$ are finitely presented, it is not difficult to check that ${}^{\perp}\mathcal{C}$ is definable; see for example the proof of [9, Corollary 6.4]. Finally, we have ${}^{\perp}\mathcal{I}_n(\Lambda) \subseteq {}^{\perp}\mathcal{C}$ if and only if $\mathcal{C} \subseteq \mathcal{I}_n(\Lambda)$, because $({}^{\perp}\mathcal{I}_n(\Lambda))^{\perp} = \mathcal{I}_n(\Lambda)$. Therefore $\operatorname{id} T = \operatorname{id}\mathcal{C}$. □

Corollary 2.2. *Let* Λ *be an artin algebra. Then*

$$\operatorname{Fin.dim}\Lambda \geq \sup\{\operatorname{id} T \mid T \in \operatorname{Cotilt}\Lambda^{\mathrm{op}}\} \geq \operatorname{fin.dim}\Lambda.$$

3. Minimal cotilting modules

If $\operatorname{Fin.inj.dim} \Lambda < \infty$, then we define

$$T_{\min} = \bigwedge_{T \in \operatorname{Cotilt} \Lambda} T$$

to be the (unique) minimal element in $\operatorname{Cotilt} \Lambda$. We have always ${}^{\perp}\mathcal{I}(\Lambda) \subseteq {}^{\perp}T_{\min}$ and in this section we ask when both subcategories are equal. To this end we introduce another module which is of potential interest.

Lemma 3.1. *Let Λ be right noetherian and suppose that* $\operatorname{Fin.inj.dim} \Lambda < \infty$. *Then there exists a Λ-module T such that*

$${}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda) = \operatorname{Add} T.$$

Proof. We have a cotorsion pair $({}^{\perp}\mathcal{I}(\Lambda), \mathcal{I}(\Lambda))$ since $\operatorname{Fin.inj.dim} \Lambda < \infty$. Observe that $\mathcal{I}(\Lambda)$ and ${}^{\perp}\mathcal{I}(\Lambda)$ both are closed under taking kernels of epimorphisms. Therefore every epimorphism in ${}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda)$ splits. Now fix an exact sequence $0 \to \Lambda \to T \to X \to 0$ with $T \in \mathcal{I}(\Lambda)$ and $X \in {}^{\perp}\mathcal{I}(\Lambda)$. Clearly, $T \in {}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda)$. Taking coproducts we get for each cardinal α an exact sequence $0 \to \Lambda^{(\alpha)} \to T^{(\alpha)} \to X^{(\alpha)} \to 0$ with $T^{(\alpha)} \in {}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda)$ and $X^{(\alpha)} \in {}^{\perp}\mathcal{I}(\Lambda)$, since $\mathcal{I}(\Lambda)$ is closed under coproducts. Thus every map $\phi \colon \Lambda^{(\alpha)} \to Y$ with $Y \in \mathcal{I}(\Lambda)$ factors through $\Lambda^{(\alpha)} \to T^{(\alpha)}$ via some map $\phi' \colon T^{(\alpha)} \to Y$. If $Y \in {}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda)$ and ϕ is an epi, then ϕ' splits. Thus ${}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda) = \operatorname{Add} T$. □

By abuse of notation we denote by T_{inj} a module satisfying ${}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda) = \operatorname{Add} T_{\mathrm{inj}}$.

Lemma 3.2. *Let Λ be right noetherian and suppose that* $\operatorname{Fin.inj.dim} \Lambda = n < \infty$. *Then a Λ-module C has finite injective dimension if and only if there is an exact sequence*

$$(*) \qquad 0 \longrightarrow T_{n+1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow C \longrightarrow 0$$

with $T_i \in \operatorname{Add} T_{\mathrm{inj}}$ for all i.

Proof. We have a cotorsion pair $({}^{\perp}\mathcal{I}(\Lambda), \mathcal{I}(\Lambda))$. Starting with $Y_0 = C \in \mathcal{I}(\Lambda)$, we obtain exact sequences $\varepsilon_i \colon 0 \to Y_{i+1} \to T_i \to Y_i \to 0$ for each $i \geq 0$, with $Y_i \in \mathcal{I}(\Lambda)$ and $T_i \in \operatorname{Add} T_{\mathrm{inj}}$ for all i. Using dimension shift, one sees that ε_n splits. Thus $Y_{n+1} \in \operatorname{Add} T_{\mathrm{inj}}$, and splicing together the ε_i produces a sequence of the form $(*)$. Conversely, if C fits into a sequence $(*)$, then $C \in \mathcal{I}(\Lambda)$ since $\mathcal{I}(\Lambda)$ is closed under taking cokernels of monomorphisms. □

Recall that a module C is Σ-pure-injective if every coproduct $C^{(\alpha)}$ is pure-injective.

Theorem 3.3. *Let Λ be right noetherian and suppose that* $\operatorname{Fin.inj.dim} \Lambda < \infty$. *Then the following are equivalent:*

(1) ${}^{\perp}\mathcal{I}(\Lambda) = {}^{\perp}T_{\min}$*;*
(2) ${}^{\perp}\mathcal{I}(\Lambda)$ *is closed under taking products;*
(3) T_{inj} *is product complete, that is,* $\operatorname{Add} T_{\mathrm{inj}} = \operatorname{Prod} T_{\mathrm{inj}}$*;*
(4) T_{inj} *is a Σ-pure-injective cotilting module.*

Moreover, in this case $T_{\min}$ and T_{inj} are equivalent cotilting modules.

Proof. (1) $\Rightarrow$ (2): Clear, since ${}^{\perp}T_{\min}$ is closed under products.

(2) $\Rightarrow$ (3): If ${}^{\perp}\mathcal{I}(\Lambda)$ is closed under products, then ${}^{\perp}\mathcal{I}(\Lambda) \cap \mathcal{I}(\Lambda)$ is closed under products. Thus every product of copies of T_{inj} belongs to $\operatorname{Add} T_{\mathrm{inj}}$. It follows that

T_{inj} is Σ-pure-injective and therefore $\operatorname{Add} T_{\mathrm{inj}} \subseteq \operatorname{Prod} T_{\mathrm{inj}}$; see [8]. Thus T_{inj} is product complete.

(3) $\Rightarrow$ (4): A product complete module is Σ-pure-injective. For T_{inj}, the defining conditions of a cotilting module are obviously satisfied, except (T3) which follows from Lemma 3.2.

(4) $\Rightarrow$ (1): First observe that $\operatorname{Add} T_{\mathrm{inj}} \subseteq \operatorname{Prod} T_{\mathrm{inj}}$ since T_{inj} is Σ-pure-injective. The cotilting module T_{inj} induces a cotorsion pair $({}^{\perp}T_{\mathrm{inj}}, ({}^{\perp}T_{\mathrm{inj}})^{\perp})$ by Proposition 1.3. We claim that $\mathcal{I}(\Lambda) = ({}^{\perp}T_{\mathrm{inj}})^{\perp}$. We need to check $\mathcal{I}(\Lambda) \subseteq ({}^{\perp}T_{\mathrm{inj}})^{\perp}$ and this follows from Lemma 3.2 since $\operatorname{Add} T_{\mathrm{inj}} \subseteq \operatorname{Prod} T_{\mathrm{inj}}$. Thus ${}^{\perp}\mathcal{I}(\Lambda) = {}^{\perp}T_{\mathrm{inj}}$ and therefore T_{inj} is equivalent to the minimal cotilting module $T_{\min}$. □

Remark 3.4. A cotilting module T is Σ-pure-injective if and only if $({}^{\perp}T)^{\perp}$ is closed under coproducts. In this case let T' be the coproduct of a representative set of indecomposable modules in $\operatorname{Prod} T$. Then T' is a product complete cotilting module which is equivalent to T.

It seems to be an interesting project to describe the minimal cotilting module for a given algebra. For example, $T_{\min} = \Lambda$ if Λ is a Gorenstein artin algebra.

In fact, there is a more general result which discribes when $T_{\min}$ is finitely presented. This is inspired by a result about modules of finite projective dimension by Huisgen-Zimmermann and Smalø [7].

Proposition 3.5. *Let Λ be an artin algebra. Then there exists a finitely presented minimal cotilting module if and only if the modules of finite injective dimension form a covariantly finite subcategory of* $\operatorname{mod} \Lambda$. *Moreover, in this case the equivalent conditions of* Theorem 3.3 *are satisfied.*

A similar result has been obtained independently by Happel and Unger for the category of finitely presented Λ-modules.

We do not give the complete proof but sketch the argument. Suppose first that $\mathcal{I}(\operatorname{mod} \Lambda) = \{X \in \operatorname{mod} \Lambda \mid \operatorname{id} X < \infty\}$ is covariantly finite. Using the correspondence for cotilting modules in $\operatorname{mod} \Lambda$, there exists a cotilting module T such that ${}^{\perp}T = {}^{\perp}\mathcal{I}(\operatorname{mod} \Lambda)$ in $\operatorname{mod} \Lambda$; see [2]. The assumption implies that every module of finite injective dimension is a filtered colimit of modules in $\mathcal{I}(\operatorname{mod} \Lambda)$. Using this, one proves that T is minimal. Conversely, if $T_{\min}$ is finitely presented, then one can use Proposition 2.1 to show that $\mathcal{I}(\operatorname{mod} \Lambda)$ is covariantly finite in $\operatorname{mod} \Lambda$.

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