Cohomology of groups, abelian Sylow subgroups and splendid equivalences

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Abstract

Let G be a finite group and let R be a complete discrete valuation domain of characteristic 0 with residue field k of characteristic p and let S be R or k. The cohomology rings $H^*(K,S)$ for subgroups K of G together with restriction to subgroups of G, transfer from subgroups of G and conjugation by elements of G gives $H^*(-,S)$ the structure of a Mackey functor. Moreover, the group $HSplen_S(K)$ of splendid auto-equivalences of the bounded derived category of finitely generated SG-modules fixing the trivial module acts S-linearly on $H^*(K,S)$. In this note we study the compatibility of these structures and get some consequences when G has an abelian Sylow p subgroup. In particular we see that in case G has an abelian Sylow p subgroup, then $HSplen_R(G)$ acts by automorphisms of the Sylow subgroup on the cohomology.

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Let G be a finite group and let R be a commutative ring, considered as a trivial RG-module. The bounded derived category $D^b(RG)$ of finitely generated RG-modules is used in modular representation theory of finite groups to provide a geometric framework for the classical conjectures like Dade's conjecture or Alperin's conjecture [3, 4]. These two are consequences of Broué's conjecture [1, 4] which states that in case $R = k$ is a field of characteristic p and G is a finite group with abelian Sylow p subgroup P, the derived categories of the principal block $B_0(kG)$ of kG and the principal block $B_0(kN_G(P))$ of $kN_G(P)$ are equivalent. Besides the above conjectures of Alperin and Dade a positive answer to Broué's conjecture implies for example that the K-theory, the cyclic and the Hochschild (co-)homology of the principal blocks of kG and of $kN_G(P)$ coincide. For an account of other consequences and most known results see [4].

If there is an equivalence between two derived categories, an immediate question is, how many equivalences there are. This way, one is lead to the definition of the group $TrPic_R(B_0(RG))$ of auto-equivalences of the derived category $D^b(B_0(RG))$ (see [8]). This group comes into the play from a very different approach as well. The Mirror symmetry conjecture of Kontsevitch imply that symplectic automorphisms of a symplectic manifold with vanishing first Chern class induce auto-equivalences of the derived category of sheaves of the mirror Calabi-Yau manifold. From there as well one is lead to the group of auto-equivalences of the derived category (see [9]).

Studying a group is done most naturally by studying its modules. So, one should look for a natural module on which these groups act on. From many points of view the derived category is the right object to consider homology. Since the derived categories, we are interested in, are derived categories of group rings, in the context of auto-equivalences of group rings the natural module we asked for is the cohomology of groups.

In the present note we construct a module structure on $H^*(G,R)$ coming from interpreting this object in the derived category. Let A be an R-algebra. Bernhard Keller proved [2] that $D^b(B_0(RG)) \simeq D^b(A)$ as triangulated categories if and only if there is an $X \in D^b(A \otimes_R B_0(RG)^{op})$ so that $X \otimes^{\mathbb{L}}_{B_0(RG)} -$ is an equivalence. Such an X is called a two-sided tilting complex. In [11] it is proved that if R is hereditary, then A is R-projective again and by [2] the inverse equivalence is again a derived tensor product by a complex of bimodules. For a complex X note by $[X]$ its isomorphism class in the derived category. Set [8]

$$TrPic_R(B_0(RG)) := \{[X] |\ X \in D^b(B_0(RG) \otimes B_0(RG)^{op}) \text{ is a 2-sided tilting complex}\}$$

and $HD_R(G) := \{[X] \in TrPic_R(B_0(RG)) |\ X \otimes_{B_0(RG)} R \simeq R\}$. It is shown (cf. [12]) that the group cohomology $H^*(G,R)$ is an $R\,HD_R(G)$-module by composing the following morphisms: Take X with

$[X] \in HD_R(G)$. Then X acts on $H^*(G,R)$ by the composition F_X of the following maps

$$\begin{array}{ccc} H^n(G,R) & & H^n(G,R) \\ \| & & \| \\ Hom_{D^b(RG)}(R,R[n]) & \longrightarrow Hom_{D^b(RG)}(X \otimes_{RG} R, X \otimes_{RG} R[n]) \simeq & Hom_{D^b(RG)}(R,R[n]) \end{array} .$$

In [12] it is shown that F_X does not depend on the isomorphisms chosen.

1 Splendid Equivalences

Let now R be a complete discrete valuation ring with field of fractions K of characteristic 0 and residue field k of characteristic p. Let P be a p-Sylow subgroup of G and let $\Delta : G \longrightarrow G \times G$ be the codiagonal $\Delta(g) = (g, g^{-1})$.

(1.1) A two-sided tilting complex X with isomorphism class in $TrPic_R(RG)$ is called splendid [5] provided all homogeneous components of X are ΔP-projective p-permutation modules, projective as $B_0(RG)$-modules from the left and from the right, and provided $Hom_{B_0(RG)^{op}}(X,X) \simeq B_0(RG) \simeq Hom_{B_0(RG)}(X,X)$ in the homotopy category of complexes of $B_0(RG)$-bimodules. Let $SplenPic_R(G)$ be the group (!) of homotopy equivalence classes (X) of splendid tilting complexes of $B_0(RG) \otimes_R B_0(RG)^{op}$-modules. Then, there is a natural group homomorphism $\varphi : SplenPic_R(G) \longrightarrow TrPic_R(B_0(RG)$ by taking isomorphism classes of the objects in the derived category instead of in the homotopy category. Let $HSplen_R(RG) := \varphi^{-1}(HD_R(G)) \cap SplenPic_R(RG)$. We use similar notations for k as base ring.

(1.2) A second major ingredient in what follows is the Brauer construction. For any p-subgroup Q of G and any kG-module M set $M(Q) := M^Q / \sum_{R<Q, R \neq Q} Tr_R^Q M^R$ where M^Q denotes Q-fixed points and Tr is the transfer map [10]. The mapping $M \mapsto M(Q)$ is functorial.

We shall use the Brauer functor in the following way: Results of Rickard [5] imply that $-(\Delta Q) : SplenPic_k(G) \longrightarrow SplenPic_k(C_G(Q))$ is a homomorphism of groups.

We come to the main theorem.

Theorem 1 *Let G be a finite group, let k be a field of characteristic p and let Q be a p-subgroup of G. For any $(X) \in HSplen_R(G)$ with $(X(\Delta Q)) \in HSplen_R(C_G(Q))$ we have*

$$F_{X(\Delta Q)} \circ res^G_{C_G(Q)} = res^G_{C_G(Q)} \circ F_X \text{ and } F_X \circ tr^G_{C_G(Q)} = tr^G_{C_G(Q)} \circ F_{X(\Delta Q)}.$$

The proof of the theorem is based essentially on the isomorphism $X(\Delta Q) \otimes_{kC_G(Q)} k \simeq X^G(Q)$ for elements $(X) \in SplenPic_k(G)$. In order to show that the action of X commutes with the transfer one uses in addition an abstract variant of Frobenius reciprocity. Showing that the action of X commutes with restriction is more straightforward.

We shall be concerned with functorial properties of the above constructions.

(1.3) Let $\mathfrak{P}_{(P,G)}$ be a set of p-subgroups of G, partially ordered by inclusion and let $\mathcal{S}ub_G$ be the set of subgroups of G partially ordered by inclusion. Then, $\mathfrak{C}_{(P,G)} := \{C_G(Q) |\ Q \in \mathfrak{P}_{(P,G)}\}$ is a partially ordered set as well, and $C_G(-)$ is an inclusion reversing mapping $\mathfrak{P}_{(P,G)} \longrightarrow \mathcal{S}ub_G$ with image $\mathfrak{C}_{(P,G)}$. A partially ordered set $(S, \leq)$ may be seen as category with objects being the elements of the set and the set of morphisms from one element x to another y of S is a singleton if $x \leq y$ and empty otherwise. A group sheaf is then a contravariant functor of $(S, \leq)$ to the category of groups. Suppose that each element of $\mathfrak{P}_{p,G}$ is abelian. Then the functor given by $SplenPic_R(-)$ on objects and the Brauer functor on morphisms is a group sheaf over the partially ordered set $\mathfrak{C}_{(P,G)}$. In fact, $Q_1 \subseteq Q_2 \Rightarrow C_G(Q_1) \supseteq C_G(Q_2)$ and this inclusion is mapped to the homomorphism $SplenPic_R(C_G(Q_1)) \longrightarrow SplenPic_R(C_G(Q_2))$ given by $(X) \mapsto (X(\Delta Q_2))$.

(1.4) The cohomology of groups $H^*(-,R)$ together with restriction res on morphisms is a contravariant functor $\mathcal{S}ub_G \longrightarrow R-mod$ and $H^*(-,R)$ together with transfer tr is a covariant functor $\mathcal{S}ub_G \longrightarrow R-mod$. Moreover, for any $g \in G$ and any $K < G$ there is an R-linear mapping $c_g : H^*(K,R) \longrightarrow H^*({}^gK,R)$. This mapping is a natural transformation for $(H^*(-,R), res)$ as well as for $(H^*(-,R), tr)$. Each c_g acts as the identity on $H^*(K,R)$ if $g \in K$ and $c_g c_h = c_{gh}$. Furthermore, one has the Mackey formula $res^H_L tr^H_K = \sum_{x \in L \backslash H/K} tr^L_{L \cap {}^xK} c_x res^K_{{}^xL \cap K}$ for subgroups $L, K \leq H \leq G$.

All these properties may be subsumed by saying that $(H^*(-,R), res, tr, c)$ is a *Mackey functor* (see [10, section 53]). A natural transformation $\eta : H^*(-,R) \longrightarrow H^*(-,R)$ *with respect to both res and tr* is called a morphism of Mackey functors.

(1.5) Let S be a partially ordered set and F be a group sheaf on S. Let M be a sheaf of R-modules on S. Then, we say that F acts on M if there is a natural transformation $F \times M \longrightarrow M$ satisfying the usual properties of a group action locally for any fixed $x \in S$. This means that we demand for this action in addition to the usual axioms of a group action to be natural with respect to the order relation: More precisely, if $x < y$, then the natural diagram

$$\begin{array}{ccc} F(x) \times M(x) & \longrightarrow & M(x) \\ \uparrow & & \uparrow \\ F(y) \times M(y) & \longrightarrow & M(y) \end{array}$$

is commutative. We say that F acts on a *covariant functor* $N : S \longrightarrow R - mod$ if F acts in the above sense on the sheaf $N : S \longrightarrow (R - mod)^{op}$. We define an action of a group sheaf F on a Mackey functor (M, res, tr, c) as an action of F on (M, res) and on (M, tr). Remark that for technical reasons we consider $(M, tr) : X \longrightarrow (R - mod)^{op}$ being valued in the opposite category. Note that a particular case is the constant sheaf where $F(x)$ equals a fixed group Γ for any $x \in S$.

Define for any partially ordered set of abelian p-subgroups $\mathfrak{P}_{p,G}$ of the finite group G the following:

$$HSplen_{k,\mathfrak{P}_{p,G}}(C_G(Q)) := \{(X) \in HSplen_k(C_G(Q)) \mid Q \leq Q' \in \mathfrak{P}_{p,G} \Rightarrow (X(\Delta Q')) \in HSplen_k(C_G(Q'))\}$$

It is clear that $G = C_G(1)$. With this definition we have the

Theorem 2 *Let $\mathfrak{P}_{p,G}$ be a partially ordered set of p-subgroups of the finite group G and let $\mathfrak{C}_{p,G} := \{C_G(Q) \mid Q \in \mathfrak{P}_{p,G}\}$.*

If $\mathfrak{C}_{p,G}$ is closed under intersection and conjugation, then the constant sheaf $HSplen_{k,\mathfrak{P}_{p,G}}(G)$ acts as morphisms of Mackey functors on $H^*(-,k)$.

Suppose that each element of $\mathfrak{P}_{p,G}$ is abelian. Let Res: $\mathfrak{C}_{p,G} \longrightarrow k - \mathfrak{Mod}$ *and* Trans: $\mathfrak{C}_{p,G} \longrightarrow (k - \mathfrak{Mod})^{op}$ *be the two functors which are identical on objects: $H^*(-,k) : \mathfrak{C}_{p,G} \longrightarrow k - \mathfrak{Mod}$ while on morphisms* $\mathsf{Res}(C_G(Q_1) \leq C_G(Q_2)) := res^{C_G(Q_2)}_{C_G(Q_1)}$ *and* $\mathsf{Trans}(C_G(Q_1) \leq C_G(Q_2)) := tr^{C_G(Q_2)}_{C_G(Q_1)}$ *Then, $HSplen_{k,\mathfrak{P}_{p,G}}(-) : \mathfrak{C}_{p,G} \longrightarrow \mathfrak{Group}$ acts by natural transformations on* Res *and on* Trans.

Observe that the second statement makes sense since we assume that the groups in $\mathfrak{P}_{p,G}$ are abelian. This implies that for $Q_2 > Q_1$ one has $Q_2 \leq C_G(Q_1)$ for all $Q_1, Q_2 \in \mathfrak{P}_{p,G}$.

If Q_2 is not abelian it is false in general that $Q_1 < Q_2$ imply that Q_2 centralize Q_1 and the Brauer functor $-(\Delta Q_2)$ is not defined on $SplenPic_R(C_G(Q_1))$.

Note however that we did not have to assume that the Sylow p subgroup of G is abelian.

2 Lifting to characteristic 0 and Abelian Sylow subgroups

In this section we shall give some consequences of Theorem 2 in case G is a finite group with abelian Sylow p subgroup P. We keep the hypotheses on R and k at the beginning of section 1.

(2.1) It is easy to see that the group homomorphism $- \otimes_R k : TrPic_R(B_0(RG)) \longrightarrow TrPic_k(B_0(kG))$ gives rise to a commutative diagram

$$\begin{array}{ccc} TrPic_R(RG) & \longrightarrow & TrPic_k(kG) \\ \uparrow & & \uparrow \\ SplenPic_R(RG) & \stackrel{\sim}{\longrightarrow} & SplenPic_k(kG) \\ \cup & & \cup \\ HSplen_R(G) & \longrightarrow & HSplen_k(G) \end{array}$$

where the middle horizontal morphism is an isomorphism as a consequence of a result of Rickard [5, Theorem 5.2]. This implies that the lower mapping is injective.

(2.2) The Künneth sequence

$$0 \longrightarrow H^i(X \otimes_{RG} R) \otimes_R k \longrightarrow H^i(X \otimes_{RG} k) \longrightarrow Tor_1^R(k, H^{i+1}(X \otimes_{RG} R)) \longrightarrow 0$$

implies that $X \otimes_{RG} R \simeq M \in RG-mod$ with $M \otimes_R k \simeq k$. Since X is a two-sided tilting complex, this is true for $K \otimes_R X$ as well and $K \otimes_R X \otimes_{RG} R \simeq X \otimes_{RG} K \neq 0$. Hence, M is an RG-lattice and since $M \otimes_R k \simeq k$, we conclude that the module M is R-free of rank 1. There are only a finite number of such RG-modules since their G-structure is entirely fixed by their rational (one-dimensional) character. Let $M_1, M_2, \ldots, M_n$ be representatives of the isomorphism classes of such modules. Then, the mapping $R \longrightarrow End_R(M_i)$ induced by scalar multiplication is an isomorphism of RG-modules. Hence, the induced map $H^m(G,R) \longrightarrow H^m(G, End_R(M_i)) \simeq Ext^m_{RG}(M_i, M_i)$ is an isomorphism. Set $\Gamma := \{(X) \in SplenPic_R(RG) | \ (X \otimes_R k) \in HSplen_R(G)\}$. Then,

$$- \otimes_R k : \bigoplus_{i=1}^n Ext^m_{RG}(M_i, M_i) \longrightarrow \bigoplus_{i=1}^n H^m(G,k)$$

is a homomorphism of $R\Gamma$-modules. Any $(X) \in \Gamma$ acts on the right hand side as $(X \otimes_R k)$ and therefore diagonally. It is clear that (X) acts on the left hand side as monomial matrices with the same induced permutation in each degree. For $m = 0$ each of the summands on the right hand side is a copy of k and each of the summands on the left hand side is a copy of R. One concludes that (X) acts diagonally on the left hand side as well. We proved the following

Lemma 2.1 $- \otimes_R k : \ HSplen_R(G) \simeq HSplen_k(G)$.

As a consequence we formulate

Theorem 3 *Let G be a group with abelian Sylow p subgroup P and let R be a complete discrete valuation domain of characteristic 0 and residue field k of characteristic p. Then, $HSplen_{k,\{P\}}(G)$ acts on $H^*(G,k)$ by outer automorphisms of P.*

Proof. Let S be R or k. Let X be a splendid tilting complex with isomorphism class in $HSplen_S(G)$. Then

$$\begin{array}{ccc} H^*(G,S) & \stackrel{res^G_{C_G(P)}}{\longrightarrow} & H^*(C_G(P),S) \\ \downarrow F^*_X & & \downarrow F^*_{X(\Delta P)} \\ H^*(G,S) & \stackrel{res^G_{C_G(P)}}{\longrightarrow} & H^*(C_G(P),S) \end{array}$$

is a commutative diagram. The image of $res^G_{C_G(P)}$ is the set of stable elements in $H^*(C_G(P),S)$. By definition, $C_G(P)$ acts trivially on P and so every element of $H(P,S)$ is $C_G(P)$-stable. Hence $res_P^{C_G(P)}$ is an isomorphism. Since the index $[G : C_G(P)]$ is invertible in S, the homomorphism $res^G_{C_G(P)}$ is injective. Moreover, P is a normal subgroup of $C_G(P)$ and the quotient $C_G(P)/P$ is a p'-group. So, $C_G(P) = P \times C_G(P)/P$. It is clear that the principal block of $kC_G(P)$ is isomorphic to kP. The isomorphism is induced by the trivial representation of $C_G(P)/P$. By [8], $TrPic_S(SP) = Pic_S(SP) \times C_\infty$ where C_∞ is the group generated by the shift of degree. So, $HSplen_{S,\{P\}}(G)$ acts on $H^*(G,k)$ as $HSplen_S(C_G(P))$ acts on $H^*(C_G(P),k)$ and $HSplen_S(C_G(P)) = HSplen_S(P) \subseteq Out_S(SP)$.

But, $HSplen_R(P) \simeq HSplen_k(P)$ by Lemma 2.1. Roggenkamp and Scott prove (see [6]) that the set of automorphisms of RP which preserve the augmentation equals $Inn(RP) \cdot Aut(P)$. A theorem of Coleman and an improvement due to Leonard Scott (see [7, part I § 2.1, Lemma 2.1]) states that $Aut(P) \cap Inn(RP) = Inn(P)$. Hence $Out_R(RP) \cap HSplen_R(RP) \subseteq Out(P)$. Since P is abelian, $Out(P) = Aut(P)$. This proves the theorem. ■

Example (2.3) : It is clear that the restriction res^G_P maps the action of an outer automorphism α of G to the action of the restriction $\alpha|_P$ to P, where one may modify if necessary α by an inner automorphism so that α fixes P. From that description it is clear that there are automorphisms of P which do not act as any automorphism of G on the image of the restriction map res^G_P. One might ask if the action of any automorphism of P on the image of res^G_P can be realized as the action of an element in the larger group $HSplen_R(G)$. This is not the case. Let D_p be the dihedral group of order $2p$ for a prime number p. Then, for $R = \hat{\mathbb{Z}}_p$, the p-adic integers, the group ring RD_p is a Brauer tree algebra with two edges and exceptional vertex in the middle vertex. By the arguments used in [8] and [12], $TrPic_R(B_0(RD_p))$ is a central extension by an infinite cyclic group of some subgroup of $PSL_2(\mathbb{Z})$. In fact, it is not difficult to see that $TrPic_R(B_0(RD_p))$ is generated by the preimage of the level two congruence subgroup $\Gamma(2)$ of $PSL_2(\mathbb{Z})$ and the standard element ϕ of order 2. Call s^2 and t^2 preimages of the standard generators of $\Gamma(2)$. The stabilizer of the trivial module equals the group $< \phi ts(\phi s)^{-3}, t^2 >$ remarking that $(\phi s)^3$ is shift by 2 degrees. But, as the action of $HD_R(G)$ on $H^*(G,R)$ factors via the natural quotient of $HD_R(G)$ to the group of auto-equivalences of the stable module category (see [12]) and as there are no stable auto-equivalences fixing the trivial module (cf Linckelmann [4, chapter XI]), $< \phi ts(\phi s)^{-3}, t^2 >$ acts trivially on $H^*(D_p, \mathbb{F}_p)$. Nevertheless, $H^*(D_p, \mathbb{F}_p) = H^*(C_p, \mathbb{F}_p)^{C_2} = \mathbb{F}_p[X^2]$ where X is a 2-cocycle in $H^*(C_p, \mathbb{F}_p)$. The action of $Aut(C_p) = C_{p-1}$ is multiplication by a multiplicative generator in degree 2, hence by its square in degree 4. As soon as $p > 3$, there exists $\alpha \in \mathbb{F}_p$ with $\alpha^2 \neq 1$.

(2.4) Let G be a finite group with abelian Sylow p subgroup P. By the above considerations the action of $HSplen_{R,\{P,\{1\}\}}(G)$ on $H(G,R)$ induces a group homomorphism $HSplen_{R,\{P,\{1\}\}}(G) \longrightarrow Aut(P)$.

It might be an interesting question to determine the image of the Brauer functor $-(\Delta P)$. For an abelian group P it is clear that $Aut(P)$ acts faithfully on $H^*(P,R)$, but there is a difficult and open question of S. Jackowski if for any p-group P the group $Out(P)$ acts faithfully on $H^*(P,R)$. The answer is negative if R is replaced by a field k of characteristic p; the cyclic group of order p^2 gives already a counterexample.

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