Eight-Dimensional Real Quadratic Division Algebras

Ernst Dieterich

Abstract

Given a euclidean vector space V, a linear map $\eta : V \wedge V \to V$ is called *dissident* in case $v, w, \eta(v \wedge w)$ are linearly independent whenever so are $v, w \in V$. The problem of classifying all real quadratic division algebras is reduced to the problem of classifying all eight-dimensional real quadratic division algebras, and further to the problem of classifying all dissident maps $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$. Should all of these satisfy $\eta = \varepsilon\pi$ for a vector product π on $\mathbb{R}^7$ and a positive-definite endomorphism ε of $\mathbb{R}^7$, then the latter problem would be solved. This *strong factorization* property however, even though it does hold for all dissident maps in lower dimensions, is shown to fail in dimension 7. It is replaced by a weak factorization property for which a proof is announced. Evidence is given for the conjecture that even weak factorization will suffice to accomplish the complete classification of all eight-dimensional real quadratic division algebras.

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1 Introduction

By a real *algebra* we mean a real vector space A, endowed with a bilinear multiplication $A \times A \to A$, $(x, y) \mapsto xy$. A real *division* algebra is understood to be a real algebra A satisfying $0 < \dim A < \infty$ and having no zero divisors (i.e. $xy = 0$ only if $x = 0$ or $y = 0$). The old and challenging problem of classifying all real division algebras up to isomorphism has as yet admitted only partial solutions, the most significant of which we proceed to indicate.

The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ (Hamilton 1843) and the octonions $\mathbb{O}$ (Graves 1843, Cayley 1845) constitute the oldest known examples of real division algebras. Classical results of Frobenius [7] and Zorn [11] assert that $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ classifies all associative real division algebras, whereas $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ classifies all alternative real division algebras. Adams' celebrated formula for the span of a sphere [1] implies that each real division algebra has dimension 1,2,4 or 8.

A real algebra A is called *quadratic* in case $0 < \dim A$, there exists an identity element $1 \in A$ and each $x \in A$ satisfies an equation $x^2 = \alpha x + \beta 1$ with real coefficients α, β. Every alternative real division algebra is quadratic. The classification of all quadratic real division algebras seems to be within reach. It is based on their intimate relationship with dissident triples which we proceed to recall. A *dissident triple* (V, ξ, η) consists of a finite-dimensional euclidean vector space $V = (V, \langle\ \rangle)$, a linear form $\xi : V \wedge V \to \mathbb{R}$ and a dissident map $\eta : V \wedge V \to V$. The category $\mathcal{D}$ of all dissident triples and the category $\mathcal{Q}$ of all quadratic real division algebras are related by a functor $\mathcal{G} : \mathcal{D} \to \mathcal{Q}$ which is defined on objects by $\mathcal{G}(V, \xi, \eta) = \mathbb{R} \times V$, with multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w))$, and on morphisms by $\mathcal{G}(\sigma) = \mathbb{I}_{\mathbb{R}} \times \sigma$. 'Frobenius Lemma' [7],[8] and 'Osborns Theorem' [10] are predecessors of the following result.

Proposition 1.1 *[3, pp. 15, 17], [5]. The functor $\mathcal{G} : \mathcal{D} \to \mathcal{Q}$ is an equivalence of categories.*

Intending to exploit this equivalence, let $n \in \{1, 2, 4, 8\}$ and denote by $\mathcal{Q}_n$ the full subcategory of $\mathcal{Q}$ which is formed by all objects of dimension n. Likewise, set $m = n - 1$ and denote by $\mathcal{D}_m$ the full subcategory of $\mathcal{D}$ which is formed by all objects of dimension m. The (1,2,4,8)-Theorem and Proposition 1.1 reduce the classification problem for $\mathcal{Q}$ to the classification problem for $\mathcal{D}_m$, where $m \in \{0, 1, 3, 7\}$. Clearly $\{(0, o, o)\}$ classifies $\mathcal{D}_0$ and $\{(\mathbb{R}, o, o)\}$ classifies $\mathcal{D}_1$, so $\{\mathbb{R}\}$ classifies $\mathcal{Q}_1$ and $\{\mathbb{C}\}$ classifies $\mathcal{Q}_2$. In case $m \in \{3, 7\}$ we are faced with the key subproblem of classifying all dissident maps $\mathbb{R}^m \wedge \mathbb{R}^m \to \mathbb{R}^m$ up to $O(\mathbb{R}^m)$-action. It is partially solved by the following result, taking into account that the set of all vector products on $\mathbb{R}^m$ forms one orbit under this group action.

Proposition 1.2 *[3, p. 19], [5]. Let V be a euclidean vector space of dimension $m \in \{3, 7\}$.*
(i) If $\pi : V \wedge V \to V$ is a vector product and $\varepsilon : V \to V$ is a definite endomorphism, then $\varepsilon\pi : V \wedge V \to V$ is a dissident map.
(ii) If $\pi, \pi' : V \wedge V \to V$ are vector products and $\varepsilon, \varepsilon' : V \to V$ are definite endomorphisms, then $\varepsilon\pi = \varepsilon'\pi'$ iff $(\varepsilon, \pi) = (\varepsilon', \pi')$ or $(\varepsilon, \pi) = (-\varepsilon', -\pi')$.
(iii) If $m = 3$ and $\pi : V \wedge V \to V$ is a fixed vector product, then $\varepsilon \mapsto \varepsilon\pi$ is a bijection between the set of all definite endomorphisms $\varepsilon : V \to V$ and the set of all dissident maps $V \wedge V \to V$.

If $m = 3$, choose your favourite vector product $\pi : \mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R}^3$. For each $x \in \mathbb{R}^3$, denote by A_x the antisymmetric matrix representing the endomorphism $\pi(x \wedge ?) : \mathbb{R}^3 \to \mathbb{R}^3$ in the standard basis. Let $\mathcal{K}_3 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T}$, where $\mathcal{T} = \{d \in \mathbb{R}^3 \mid 0 < d_1 \leq d_2 \leq d_3\}$. For each $d \in \mathcal{T}$, denote by D_d the diagonal matrix with diagonal sequence d. Define an equivalence relation on $\mathcal{K}_3$ by $(x, y, d) \sim (x', y', d')$ iff $d = d'$ and $(Sx, Sy) = (x', y')$ for some $S \in SO_3(\mathbb{R})$ satisfying $SD_dS^T = D_d$. Now we can formulate a first consequence of Propositions 1.1 and 1.2, providing a complete description of the isoclasses of $\mathcal{Q}_4$ in terms of equivalence classes of configurations in $\mathbb{R}^3$, formed by a pair of points and an ellipsoid in normal position.

Proposition 1.3 *[4, p. 944]. The map* $\Phi_3 : \mathcal{K}_3 \to \mathcal{Q}_4$, $(x, y, d) \mapsto \mathbb{R} \times \mathbb{R}^3$, $(\alpha, v)(\beta, w) = (\alpha\beta - v^T w + v^T A_x w \ , \ \alpha w + \beta v + (A_y + D_d)\pi(v \wedge w))$, *induces a bijection* $\overline{\Phi}_3 : \mathcal{K}_3/\sim \ \tilde{\to} \ \mathcal{Q}_4/\simeq$.

A cross-section $\mathcal{C}$ for $\mathcal{K}_3/\sim$ is given in [2, pp. 17-18]. From Proposition 1.3 we infer that $\Phi_3(\mathcal{C})$ classifies $\mathcal{Q}_4$.

If $m = 7$, choose any vector product $\pi : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$. Let $\mathcal{K}_7$ be the set of all matrix pairs $(X, E) \in \mathbb{R}^{7\times 7} \times \mathbb{R}^{7\times 7}$ such that X is antisymmetric and E is positive-definite. Define an equivalence relation on $\mathcal{K}_7$ by $(X, E) \sim (X', E')$ iff $(SXS^T, SES^T) = (X', E')$ for some $S \in O_7(\mathbb{R})$ stabilizing π. Imitating as far as possible the reasoning applied for $m = 3$, one obtains the following construction of isoclasses of $\mathcal{Q}_8$.

Proposition 1.4 *[3, p. 20],[5]. The map* $\Phi_7 : \mathcal{K}_7 \to \mathcal{Q}_8$, $(X, E) \mapsto \mathbb{R} \times \mathbb{R}^7$, $(\alpha, v)(\beta, w) = (\alpha\beta - v^T w + v^T X w \ , \ \alpha w + \beta v + E\pi(v \wedge w))$, *induces an injection* $\overline{\Phi}_7 : \mathcal{K}_7/\sim \ \hookrightarrow \ \mathcal{Q}_8/\simeq$.

2 On the factorization of dissident maps

The last piece that is missing to make Propositions 1.3 and 1.4 a complete classification of $\mathcal{Q}$ is the surjectivity of $\overline{\Phi}_7$. Although the argument for the surjectivity of $\overline{\Phi}_3$, namely the bijectivity of $\pi : \mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R}^3$, does not generalize to $\overline{\Phi}_7$, one still might hope that $\overline{\Phi}_7$ is surjective for other reasons. In what follows we shall outline an argument proving that this hope fails to be true. And we shall indicate a first step in closing the gap that is left open by the non-surjectivity of $\overline{\Phi}_7$.

Given a finite-dimensional euclidean vector space $V = (V, \langle\ \rangle)$, we denote by $\mathbb{E}(V)$ the set of all two-dimensional subspaces of V. Each linear map $\varphi : V \wedge V \to V$ induces a map

$$\varphi_{\mathbb{E}} : \mathbb{E}(V) \to \mathbb{R}_{\geq 0} \ , \ \varphi_{\mathbb{E}}(P) = \langle \varphi(v \wedge w), \varphi(v \wedge w)\rangle \ ,$$

where (v, w) is any orthonormal basis for $P \in \mathbb{E}(V)$. Apart from being dissident, or even a vector product, we now introduce two more properties for linear maps $\varphi : V \wedge V \to V$. We call φ *plane-constant* iff $\varphi_{\mathbb{E}}$ is constant with positive value. We call φ a *weak vector product* iff it satisfies $\langle \varphi(u \wedge v), w\rangle = \langle u, \varphi(v \wedge w)\rangle$ for all $(u, v, w) \in V^3$ and $\varphi(v \wedge w) \neq 0$ for all non-proportional pairs $(v, w) \in V^2$. Note that each vector product is a plane-constant weak vector product, and each weak vector product is dissident.

Whereas the map Φ_7 depends on a chosen vector product on $\mathbb{R}^7$, the set of isoclasses $\overline{\Phi}_7(\mathcal{K}_7/\sim)$ no longer does it. The surjectivity of $\overline{\Phi}_7$ can now be characterized as follows.

Proposition 2.1 *[6]. The following assertions are equivalent:*
(i) The injective map $\overline{\Phi}_7 : \mathcal{K}_7/\sim \ \hookrightarrow \ \mathcal{Q}_8/\simeq$ *is surjective.*
(ii) Every dissident map $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ *admits strong factorization.*
(iii) Every weak vector product $\varrho : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ *is plane-constant.*

The group $GL(\mathbb{R}^7)$ acts on the set $\mathrm{Wec}(\mathbb{R}^7)$ of all weak vector products on $\mathbb{R}^7$ by

$$\mathrm{Wec}(\mathbb{R}^7) \times GL(\mathbb{R}^7) \to \mathrm{Wec}(\mathbb{R}^7), \ \varrho \cdot \varepsilon = \varepsilon^* \varrho(\varepsilon \wedge \varepsilon),$$

where ε^* denotes the adjoint of ε. Restricting this group action to the set $\mathrm{Vec}(\mathbb{R}^7)$ of all vector products on $\mathbb{R}^7$, we obtain the map

$$\mathrm{Vec}(\mathbb{R}^7) \times GL(\mathbb{R}^7) \to \mathrm{Wec}(\mathbb{R}^7), \ \pi \cdot \varepsilon = \varepsilon^* \pi(\varepsilon \wedge \varepsilon)$$

which we view as a means to construct weak vector products out of a given vector product. Now we apply this construction to the 'standard' vector product π induced from $\mathbb{O}$ on its purely imaginary hyperplane (cf. [9]) and, accordingly, having the multiplication table

π	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

in the standard basis $e_1, \dots, e_7$ of $\mathbb{R}^7$. Defining $\varepsilon \in GL(\mathbb{R}^7)$ by $\varepsilon(e_i) = \varepsilon_i e_i$ for all $i \in \{1, \dots, 7\}$, where $0 < \varepsilon_1 < \dots < \varepsilon_7$ is any chosen positive ascending septuple, we obtain the weak vector product $\varrho = \pi \cdot \varepsilon$. Its induced map $\varrho_{\mathbb{E}}$ satisfies strict inequalities such as e.g.

$$\varrho_{\mathbb{E}}(\mathbb{R}e_1 \oplus \mathbb{R}e_2) = (\varepsilon_1 \varepsilon_2 \varepsilon_3)^2 < (\varepsilon_1 \varepsilon_4 \varepsilon_5)^2 = \varrho_{\mathbb{E}}(\mathbb{R}e_1 \oplus \mathbb{R}e_4) \ .$$

Hence ϱ is not plane-constant. From Proposition 2.1 we infer that $\overline{\Phi}_7$ is not surjective. Moreover, not every dissident map on $\mathbb{R}^7$ admits strong factorization. However, combining ideas from real algebraic geometry with matrix formulae drawn from linear statistical inference theory, we are able to prove that each dissident map on $\mathbb{R}^7$ does admit *weak factorization*, in the following sense.

Proposition 2.2 *[6]. For each dissident map $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ there exist a weak vector product $\varrho : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ and a positive-definite endomorphism $\varepsilon : \mathbb{R}^7 \to \mathbb{R}^7$ such that $\eta = \varepsilon\varrho$.*

Weak factorization reduces the problem of classifying $\mathcal{Q}_8$ essentially to the problem of classifying all weak vector products on $\mathbb{R}^7$ up to $O(\mathbb{R}^7)$-action. The very same map $\mathrm{Vec}(\mathbb{R}^7) \times GL(\mathbb{R}^7) \to \mathrm{Wec}(\mathbb{R}^7)$ which just destroyed our hope that Φ_7 might completely classify $\mathcal{Q}_8$ will be an important tool for the solution of this remaining problem.

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Matematiska institutionen, Uppsala universitet, Box 480, S-751 06 Uppsala, Sweden

E-mail: Ernst.Dieterich@math.uu.se

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