Alexander polynomial and Koszul resolution

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Abstract: We show that a certain version the knot invariant constructed by Gomez and Sierra from the quantum Heisenberg algebra and its infinite dimensional irreducible representation is the inverse of the Alexander polynomial.

Introduction

Let H be a ribbon quasitriangular Hopf algebra, with R-matrix R and $\rho : H \to End(V)$ be a (finite dimensional) irreducible representation. Let R_ρ be the composition of the image of the R-matrix in the representation with the flip operator: $R_\rho = P\rho \otimes \rho(R)$. Following Turaev ([T]), if there exists an invertible element u in H satisfying some conditions (i.e. $\rho(u)^{\otimes 2}$ commutes with R_ρ and the partial traces $Tr_2(1 \otimes \rho(u) R_\rho) = xy$, and $Tr_2(1 \otimes \rho(u) R_\rho^{-1}) = x^{-1}y$ are non zero scalars), one constructs an invariant of links as follows. Represent your link as the closure of a braid $\alpha \in B_n$, with B_n the braid group on n strands. The element R_ρ provides, for each n a representation π_n of B_n in the tensor power $V^{\otimes n}$, commuting with the action of H. Denoting by $w(\alpha)$ the writhe of α, the expression

$$x^{-w(\alpha)} y^{-n} Tr(\rho(u)^{\otimes n} \pi_n(\alpha))$$

depends only on the closure of α and is an invariant. The value of the invariant on the unknot is $y^{-1} Tr(\rho(u))$.

An important remark is that in fact, the partial trace $Tr_{2,\ldots,n}(1 \otimes \rho(u)^{\otimes n-1} \pi_n(\alpha))$ is an intertwiner for the action of $H : V \to V$, and so is a scalar as the representation is assumed to be irreducible. Taking the full trace on the n components just amounts to multiplying this scalar by $Tr(\rho(u))$, the so-called quantum dimension of V.

There are at least 2 situations in which one may want to use this remark to compute the invariant from a (rescaled form) of the partial trace:

(1) if the quantum dimension of V is zero; in the case of a knot, this amounts to considering the 1-1 tangle associated with it;

(2) if V is infinite dimensional; but one still has to give a meaning to the partial traces.

I. The quantum Heisenberg algebra and the Gomez-Sierra invariant [G-S]

I.1. Definitions:

a) The quantum Heisenberg algebra is the algebra generated by elements a, a^*, E and N, with the following relations: E is central, $[N, a] = -a$, $[N, a^*] = a^*$, $[a, a^*] = \frac{(q^E - q^{-E})}{(q - q^{-1})}$.

It has a Hopf algebra structure, with coproduct given by: E and N are primitive, $\Delta(a) = a \otimes q^E + 1 \otimes a$, $\Delta(a^*) = a^* \otimes 1 + q^{-E} \otimes a^*$, and the four generators are in the kernel of the augmentation.

b) As shown by Gomez and Sierra, it is a quasitriangular Hopf algebra, with special element $u = 1$ (as the square of the antipode is the identity), and universal R-matrix :

$$R = q^{-(E \otimes N + N \otimes E)} exp((q - q^{-1}) a \otimes a^*).$$

c) The irreducible representations are infinite dimensional, and are parametrized by the eigenvalues of E and N. The one we shall be interested in is the following: in a suitable orthonormal basis, ($|r\rangle$, r a non negative integer), $a|r\rangle = \sqrt{r}|r-1\rangle$, $a^*|r\rangle = \sqrt{r+1}|r+1\rangle$, E acts by the identity, $N|r\rangle = r|r\rangle$.
So the space V of the representation is naturally graded by the the nonnegative integers, this gradation being induced by the action of N, and as a graded vector space, is naturally isomorphic to the algebra of polynomials in one variable (or, if one prefers, to the symmetric algebra on a one-dimensional vector space).

I.2. The invariant:

Assuming that, for $\alpha \in B_n$, the partial traces $Tr_{2,\ldots,n}(\pi_n(\alpha))$ make sense in some way as a power series in q, Gomez and Sierra define $T(\alpha) = q^{n-1-w(\alpha)} Tr_{2,\ldots,n}(\pi_n(\alpha))$, which depends only on the closure of α and is formally an invariant. As they show, the partial trace could also be taken with respect to the first $n-1$ factors.
By very clever computations, they check on several examples that this indeed makes sense, and that $T(\alpha)$ is equal to the inverse of the Alexander polynomial. Moreover, they conjecture that this is true in general.

Note that, besides the fact that V is infinite dimensional, there is a good reason why the full trace will never make sense: $V^{\otimes n}$ contains an infinite dimensional subspace fixed by all elements of B_n!
Denote by $\Delta^{(m)}$ the iterated comultiplication (with $\Delta^{(2)} = \Delta$). Then, for all k, $\Delta^{(n)}((a^*)^k)$ commutes with all $\pi_n(\alpha)$, and as $|0\rangle^{\otimes n}$ is fixed by $\pi_n(\alpha)$, the linear span of $\Delta^{(n)}((a^*)^k)(|0\rangle^{\otimes n})$, $k \in \mathbf{N}$ is fixed.
This fact will be completely clarified when we make the connection with the Burau representation.

I.3. Connection with the Burau representation:

For our purposes, it will be convenient to define the Burau representation as follows:
Let X an n-dimensional vector space, with basis $(x_1, \dots, x_n)$. The Burau representation of B_n in X is given by the action of its standard generators $\sigma_1, \dots, \sigma_{n-1}$:
for $i = 1, \dots, n-1$, $\sigma_i(x_j) = x_j$, if $j \neq i, i+1$, $\sigma_i(x_i) = (1-q^{-2})x_i + q^{-1}x_{i+1}$, $\sigma_i(x_{i+1}) = q^{-1}x_i$.
Then, by functoriality, B_n acts on all symmetric or exterior powers of X, in particular it acts on the symmetric algebra $S^\cdot(X)$ of X. If v is an endomorphism of X, we shall denote by $S(v)$ the corresponding endomorphism of $S^\cdot(X)$.

Proposition: *Under the isomorphism of $V^{\otimes n}$ with $S^\cdot(X)$ sending $(a^*)^{k_1}|0\rangle \otimes \dots \otimes (a^*)^{k_n}|0\rangle$ to $x_1^{k_1} \dots x_n^{k_n}$, the representation π_n of B_n is nothing but the representation obtained by functoriality from the Burau representation.*

Proof: It is enough to check on the generators, and this brings back to the case $n = 2$. Denote $\pi_2(\sigma)$ as σ. Now,

$$\sigma((a^*)^k|0\rangle \otimes (a^*)^l|0\rangle) = \sigma((a^*)^k \otimes (a^*)^l)|0\rangle \otimes |0\rangle = \sigma((a^* \otimes 1))^k(1 \otimes a^*)^l)|0\rangle \otimes |0\rangle$$

$$= (\sigma(a^* \otimes 1)\sigma^{-1})^k(\sigma(1 \otimes a^*)\sigma^{-1})^l)|0\rangle \otimes |0\rangle.$$

And it is now an easy computation to see that, in the representation, the action by conjugation by σ leaves the span of the images of $(a^* \otimes 1)$and $(1 \otimes a^*)$ invariant and acts in it via the 2×2 Burau matrix.

Note that the vector $y = x_1 + q^{-1}x_2 + \dots + q^{-(n-1)}x_n$ of X is fixed by B_n. One introduces the reduced Burau representation ψ as the representation obtained on the quotient $Y = X/ < y >$. It might be sometimes convenient to lift it to a subspace of X supplement to the line generated by y. We choose $X' = span(x_1, \dots, x_{n-1})$.
Observe that now $V^{\otimes n}$ becomes isomorphic to $S^\cdot(X') \otimes S^\cdot(y)$, and we have another way to take a partial trace: take the trace with respect to the $S^\cdot(X')$ factor.

Theorem: *Let $\alpha \in B_n$ such that the Alexander polynomial of its closure is not zero. Then:*
1) the partial trace of $\pi_n(\alpha)$ with respect to the $S^\cdot(X')$ factor makes sense, is a scalar operator and this scalar is equal to the trace of α acting in the symmetric algebra $S^\cdot(Y)$ on the reduced Burau representation;
2) the quantity $q^{n-1-w(\alpha)}Tr_{S^\cdot(Y)}(S\psi(\alpha))$ depends only on the closure of α and is the inverse of the Alexander polynomial of this closure.

The Theorem will follow from the facts explained in the next parts of the paper.

II. Recollections on the Alexander polynomial

Although not strictly necessary, it is interesting to recall the interpretation of the Alexander polynomial in terms of the quantized enveloping algebra of $gl(1|1)$ as advocated by Kauffman and Saleur ([K-S]). It is a fermionic construction which parallels the bosonic one of Gomez and Sierra.

II.1. Definitions:

a) The quantized enveloping algebra of $gl(1|1)$ is the algebra generated by $\eta, \eta^*, F, N, \varepsilon$, with the following relations: F is central, $[N,\eta] = -\eta$, $[N,\eta^*] = \eta^*$, $\eta^2 = 0$, $(\eta^*)^2 = 0$, $\varepsilon^2 = 1$, ε commutes with F and N, $\varepsilon\eta\varepsilon = -\eta$, $\varepsilon\eta^*\varepsilon = -\eta^*$, $\eta\eta^* + \eta^*\eta = \frac{(q^F - q^{-F})}{(q-q^{-1})}$.

It has a Hopf algebra structure, with coproduct given by: F and N are primitive, ε is grouplike, $\Delta\eta = \eta \otimes 1 + \varepsilon q^F \otimes \eta$, $\Delta\eta^* = \eta^* \otimes q^{-F} + \varepsilon \otimes \eta^*$; in the augmentation, ε goes to 1 and the other generators are in the kernel.

b) It has a quasitriangular structure, with universal R-matrix given by

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes \varepsilon + \varepsilon \otimes 1 - \varepsilon \otimes \varepsilon)(1 + (q - q^{-1})q^F \eta^* \otimes \varepsilon q^{-F}\eta)q^{-(F \otimes N + N \otimes F)},$$

and special element ε.

c) It has an irreducible 2 dimensional representation W, with basis $(|0\rangle, |1\rangle)$, on which F acts as the identity, N is diagonal in this basis, with eigenvalues 0 and 1, ε is diagonal with eigenvalues 1 and -1, $\eta|0\rangle = 0, \eta|1\rangle = |0\rangle$, and η^* is the transpose of η.
We denote by π'_n the ensuing representation of B_n in $W^{\otimes n}$.

II.2. The Alexander polynomial:

a) As the quantum dimension of the representation space is 0, Kauffman and Saleur consider, for $\alpha \in B_n$, the partial trace $Tr_{2,\dots,n}(1 \otimes (\varepsilon)^{\otimes n-1}\pi'_n(\alpha))$ and show that a suitable normalization of it is the Alexander polynomial of the closure of α.

b) Here, the connection with the (reduced) Burau representation is familiar:
As a *mod* 2 graded vector space, $W^{\otimes n}$ is isomorphic to the exterior algebra $\Lambda^\cdot(X)$ and the representation of B_n on this exterior algebra, obtained by functoriality from the Burau representation is equivalent to the representation π'_n .
Writing $\Lambda^\cdot(X)$ as $\Lambda^\cdot(X') \otimes \Lambda(y)$, taking the partial graded trace of π'_n with respect to the first factor leads to scalar which is equal to the graded trace in the exterior algebra $\Lambda^\cdot(Y)$ on the reduced Burau representation, which in turn is nothing but $det(1 - \psi(\alpha))$.

c) In fact, all we need to know is that one computes the Alexander polynomial of the closure of α via the reduced Burau representation as: $q^{-n+1+w(\alpha)}det(1 - \psi(\alpha))$.

III. Koszul resolution and identification of the invariant

Recall that the symmetric and the exterior algebras are known to be Koszul algebras, dual to each other. This means that their graded tensor product, with the Koszul differential, is a resolution of the ground field: this complex is the direct sum, for each homogenous component of total degree p, of finite length subcomplexes: $\oplus_{i+j=p} S^j \otimes \Lambda^i$, and for $p > 0$ the subcomplex is acyclic.
The identification of the (modified) Gomez-Sierra invariant with the Alexander polynomial is now a consequence of the specialization at $t = 1$, of the following general fact: for any endomorphism v of Y, one has an equality of formal power series in t:

$$(\Sigma t^p Tr S^p(v))(\Sigma(-t)^k Tr \Lambda^k(v)) = 1,$$

as a result of adding the identities: $t^p \Sigma_{k=0}^{k=p} (-1)^k Tr S^{p-k}(v) Tr \Lambda^k(v) = 0$ coming from the Lefschetz principle applied to each of the short exacts sequences making the Koszul resolution.

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