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NATURAL OPERATORS LIFTING FUNCTIONS TO BUNDLE FUNCTORS ON FIBERED MANIFOLDS

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Abstract. The complete description of all natural operators lifting real valued functions to bundle functors on fibered manifolds is given. The full collection of all natural operators lifting projectable real valued functions to bundle functors on fibered manifolds is presented.

0. Various natural operators lifting smooth real valued functions are used practically in all papers in which problems of prolongations of geometric structures have been studied, see [2], [8], [9], [10], e.t.c. Thus the problem of the classification of such natural operators is very important.

The above problem has been studied in papers [1], [3], [5], [6] and [7]. For example, in [5], we determined all natural operators lifting a smooth function $f : M \to \mathbf{R}$ into a smooth function $\Lambda(f) : G(M) \to \mathbf{R}$, where $G : \mathcal{M}f \to \mathcal{FM}$ is a bundle functor on manifolds. All of them have the form $\Lambda(f) = h_\Lambda \circ G(f)$ for some (uniquely determined by Λ) smooth function $h_\Lambda : G(\mathbf{R}) \to \mathbf{R}$.

In this paper we obtain quite similar classifications of all natural transformations lifting functions or projectable functions to bundle functors on fibered manifolds.

The definitions of bundle functors and natural operators can be found in the fundamental monograph of Kolář, Michor, Slovák [4].

All considered manifolds are assumed to be finite dimensional, without boundaries and smooth, i.e. of class $\mathcal{C}^\infty$. Mappings between manifolds are assumed to be smooth, i.e. of class $\mathcal{C}^{\cdot}$

1. Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor on fibered manifolds. Let m and n be two non-negative integers. Let $F^{(m,n)} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ denote the restriction of F to the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and locally invertible fibre respecting mappings.

We study the problem how a mapping $f : X \to \mathbf{R}$, where $\pi : X \to Y$ is a fibered manifold from $\mathcal{FM}_{m,n}$, induces canonically a mapping $\Lambda_\pi(f) : F^{(m,n)}(\pi) \to \mathbf{R}$.

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This problem is reflected in the concept of natural operators $T^{(0,0)} \rightsquigarrow T^{(0,0)}F^{(m,n)}$ in the sense of [4].

1.1. Example. We fix a one-point manifold pt. Let $\lambda : F(pt_{\mathbf{R}}) \to \mathbf{R}$ be a mapping, where $pt_{\mathbf{R}} : \mathbf{R} \to pt$ is the fibered manifold.

For any fibered manifold $\pi : X \to Y$ and any mapping $f : X \to \mathbf{R}$ we define a mapping $f^{(\lambda)} : F(\pi) \to \mathbf{R}$ as follows. We consider the mapping $f : X \to \mathbf{R}$ as the fibered mapping $f : \pi \to pt_{\mathbf{R}}$. We define $f^{(\lambda)} : F(\pi) \to \mathbf{R}$ to be the composition

$$f^{(\lambda)} : \ F(\pi) \xrightarrow{F(f)} F(pt_{\mathbf{R}}) \xrightarrow{\lambda} \mathbf{R} \ .$$

The family $\Lambda^{(\lambda)} = \{\Lambda^{(\lambda)}_\pi\}$ of functions $\Lambda^{(\lambda)}_\pi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(F(\pi))$, $f \to f^{(\lambda)}$, for any fibered manifold $\pi : X \to Y$ from $\mathcal{FM}_{m,n}$ is a natural operator $T^{(0,0)} \rightsquigarrow T^{(0,0)}F^{(m,n)}$.

The main result of this item is the following classification theorem.

1.2. Theorem. *Let F, m, n $F^{(m,n)}$, pt and $pt_{\mathbf{R}}$ be as above. Let $\Lambda : T^{(0,0)} \rightsquigarrow T^{(0,0)}F^{(m,n)}$ be a natural operator. If $n \geq 1$, then there exists one and only one mapping $\lambda : F(pt_{\mathbf{R}}) \to \mathbf{R}$ such that $\Lambda = \Lambda^{(\lambda)}$.*

The proof of this theorem will occupy the rest of this item.

Let $q : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ denote the projection onto first factor. (It is a fibered manifold from $\mathcal{FM}_{m,n}$.) Let $x^1, ..., x^m, y^1, ..., y^n : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ denote the usual coordinates.

1.3. Lemma. *Assume that $n \geq 1$. If $\Lambda', \Lambda'' : T^{(0,0)} \rightsquigarrow T^{(0,0)}F^{(m,n)}$ are natural operators such that $\Lambda'_q(y^1) = \Lambda''_q(y^1)$, then $\Lambda' = \Lambda''$.*

Proof. We have to show that $\Lambda'_\pi = \Lambda''_\pi$ for any fibered manifold $\pi : X \to Y$ from $\mathcal{FM}_{m,n}$. By the naturality of Λ' and Λ'' with respect to fibered manifold charts, we can assume that $\pi = q : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$.

Let $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be a mapping and let $v_o \in F_{(x_o,y_o)}(q)$, $(x_o, y_o) \in \mathbf{R}^m \times \mathbf{R}^n$. It remains to show that $\Lambda'_q(f)(v_o) = \Lambda''_q(f)(v_o)$. By the regularity of Λ' and Λ'', we can assume that $\frac{\partial f}{\partial y^1}(x_o, y_o) \neq 0$. Then $\varphi = (x^1, ..., x^m, f, y^2, ..., y^n) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$ is a fibered manifold chart defined near (x_o, y_o). Now, by the naturality of Λ' and Λ'' with respect to φ and the assumption of the lemma we obtain

$$\Lambda'_q(f)(v_o) = \Lambda'_q(y^1)(F(\varphi)(v_o)) = \Lambda''_q(y^1)(F(\varphi)(v_o)) = \Lambda''_q(f)(v_o) \ .$$

The proof of the lemma is complete. □

Proof of Theorem 1.2. At first we prove the existence part of the theorem. Define $\lambda : F(pt_{\mathbf{R}}) \to \mathbf{R}$ to be the composition

$$\lambda : \ F(pt_{\mathbf{R}}) \xrightarrow{F(i)} F(q) \xrightarrow{\Lambda_q(y^1)} \mathbf{R} \ ,$$

where $i : pt_{\mathbf{R}} \to q$ is determined by $i : \mathbf{R} \to \mathbf{R}^m \times \mathbf{R}^n$, $i(t) := (0, ..., 0, t, 0, ..., 0) \in \mathbf{R}^m \times \mathbf{R}^n$, t in $(m+1)$-position. We prove that $\Lambda = \Lambda^{(\lambda)}$.

For any $t \in \mathbf{R}$ we define $a_t = (tx^1, ..., tx^m, y^1, ty^2, ..., ty^n) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$. We see that: $y^1 \circ a_t = y^1$ for any $t \in \mathbf{R}$, $a_0 = i \circ y^1$ and the mappings a_t for $t \neq 0$ are fibered isomorphisms $q \to q$. Then by the naturality of Λ with respect to the a_t and by the regularity of F we obtain

$$\Lambda_q(y^1) = \Lambda_q(y^1 \circ a_t) = \Lambda_q(y^1) \circ F(a_t) \xrightarrow{t \to 0} \Lambda_q(y^1) \circ F(i \circ y^1) \ ,$$

i.e. $\Lambda_q(y^1) = \Lambda_q(y^1) \circ F(i) \circ F(y^1)$, where the last y^1 is considered as the fibered mapping $q \to pt_{\mathbf{R}}$. Then $\Lambda_q(y^1) = \Lambda_q(y^1) \circ F(i) \circ F(y^1) = \lambda \circ F(y^1) = \Lambda_q^{(\lambda)}(y^1)$. Consequently, $\Lambda = \Lambda^{(\lambda)}$ because of Lemma 1.3.

It remains to prove the uniqueness part of the theorem. Assume that $\Lambda = \Lambda^{(\lambda)}$ for some $\lambda : F(pt_{\mathbf{R}}) \to \mathbf{R}$. Then $\Lambda_q(y^1) = \Lambda_q^{(\lambda)}(y^1) = \lambda \circ F(y^1)$, where the last y^1 is considered as the fibered mapping $q \to pt_{\mathbf{R}}$. Then (since $y^1 \circ i : pt_{\mathbf{R}} \to pt_{\mathbf{R}}$ is the identity fibered mapping) we have

$$\lambda = \lambda \circ F(y^1 \circ i) = \lambda \circ F(y^1) \circ F(i) = \Lambda_q(y^1) \circ F(i) \ .$$

The proof of the theorem is complete. □

2. If $\pi : X \to Y$ is a fibered manifold, a mapping $f : X \to \mathbf{R}$ is called *projectable with respect to* π if there exists a mapping $\underline{f} : Y \to \mathbf{R}$ such that $f = \underline{f} \circ \pi$.

Let $F : \mathcal{FM} \to \mathcal{FM}$, m, n and $F^{(m,n)} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be as in Item 1.

We study the problem how a projectable with respect to π mapping $f : X \to \mathbf{R}$, where $\pi : X \to Y$ is a fibered manifold from $\mathcal{FM}_{m,n}$, induces canonically a mapping $\mathcal{K}_\pi(f) : F^{(m,n)}(\pi) \to \mathbf{R}$. This problem is reflected in the concept of natural operators $T^{(0,0)}_{proj} \rightsquigarrow T^{(0,0)}F^{(m,n)}$.

2.1. Example. Let $\kappa : F(id_{\mathbf{R}}) \to \mathbf{R}$ be a mapping, where $id_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$ is the fibered manifold (the identity map).

For any fibered manifold $\pi : X \to Y$ and any projectable with respect to π mapping $f : X \to \mathbf{R}$ we define a mapping $f^{<\kappa>} : F(\pi) \to \mathbf{R}$ as follows. We consider the mapping f as the fibered mapping $f : \pi \to id_{\mathbf{R}}$ over $\underline{f} : Y \to \mathbf{R}$, where $\underline{f}$ is such that $f = \underline{f} \circ \pi$. We define $f^{<\kappa>} : F(\pi) \to \mathbf{R}$ to be the composition

$$f^{<\kappa>} : \ F(\pi) \xrightarrow{F(f)} F(id_{\mathbf{R}}) \xrightarrow{\kappa} \mathbf{R} \ .$$

The family $\mathcal{K}^{<\kappa>} = \{\mathcal{K}^{<\kappa>}_\pi\}$ of functions $\mathcal{K}^{<\kappa>}_\pi : \{f = \underline{f} \circ \pi \in \mathcal{C}^\infty(X) | \ \underline{f} \in \mathcal{C}^\infty(Y)\} \to \mathcal{C}^\infty(F(\pi))$, $f \to f^{<\kappa>}$, for any fibered manifold $\pi : X \to Y$ from $\mathcal{FM}_{m,n}$ is a natural operator $T^{(0,0)}_{proj} \rightsquigarrow T^{(0,0)}F^{(m,n)}$.

The main result of this item is the following classification theorem.

2.2. Theorem. *Let F, m, n and $F^{(m,n)}$ be as above. Let $\mathcal{K} : T^{(0,0)}_{proj} \rightsquigarrow T^{(0,0)}F^{(m,n)}$ be a natural operator. If $m \geq 1$, then there exists one and only one mapping $\kappa : F(id_{\mathbf{R}}) \to \mathbf{R}$ such that $\mathcal{K} = \mathcal{K}^{<\kappa>}$.*

The proof of this theorem will occupy the rest of this item.

Let $q : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and $x^1, ..., x^m, y^1, ..., y^n : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be as in Item 1. We see that $x^1, ..., x^m : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ are projectable with respect to q.

2.3. Lemma. *Assume that $m \geq 1$. If $\mathcal{K}', \mathcal{K}'' : T^{(0,0)}_{proj} \rightsquigarrow T^{(0,0)}F^{(m,n)}$ are natural operators such that $\mathcal{K}'_q(x^1) = \mathcal{K}''_q(x^1)$, then $\mathcal{K}' = \mathcal{K}''$.*

Proof. We have to show that $\mathcal{K}'_\pi = \mathcal{K}''_\pi$ for any fibered manifold $\pi : X \to Y$ from $\mathcal{FM}_{m,n}$. By the naturality of $\mathcal{K}'$ and $\mathcal{K}''$ with respect to fibered manifold charts, we can assume that $\pi = q : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$.

Let $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be a projectable with respect to q mapping and let $v_o \in F_{(x_o,y_o)}(q)$, $(x_o, y_o) \in \mathbf{R}^m \times \mathbf{R}^n$. It remains to show that $\mathcal{K}'_q(f)(v_o) = \mathcal{K}''_q(f)(v_o)$. By the regularity of $\mathcal{K}'$ and $\mathcal{K}''$, we can assume that $\frac{\partial f}{\partial x^1}(x_o, y_o) \neq 0$. Then, since f is projectable with respect to q, $\varphi = (f, x^2, ..., x^m, y^1, ..., y^n) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$ is a fibered manifold chart defined near (x_o, y_o). Now, by the naturality of $\mathcal{K}'$ and $\mathcal{K}''$ with respect to φ and the assumption of the lemma we obtain

$$\mathcal{K}'_q(f)(v_o) = \mathcal{K}'_q(x^1)(F(\varphi)(v_o)) = \mathcal{K}''_q(x^1)(F(\varphi)(v_o)) = \mathcal{K}''_q(f)(v_o) \ .$$

The proof of the lemma is complete. □

Proof of Theorem 2.2. At first we prove the existence part of the theorem. Define $\kappa : F(id_{\mathbf{R}}) \to \mathbf{R}$ to be the composition

$$\kappa : \ F(id_{\mathbf{R}}) \xrightarrow{F(j)} F(q) \xrightarrow{\mathcal{K}_q(x^1)} \mathbf{R} \ ,$$

where $j : id_{\mathbf{R}} \to q$ is determined by $j : \mathbf{R} \to \mathbf{R}^m \times \mathbf{R}^n$, $j(t) := (t, 0, ..., 0) \in \mathbf{R}^m \times \mathbf{R}^n$, t in 1-position. We prove that $\mathcal{K} = \mathcal{K}^{<\kappa>}$.

For any $t \in \mathbf{R}$ we define $b_t = (x^1, tx^2, ..., tx^m, ty^1, ..., ty^n) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$. We see that: $x^1 \circ b_t = x^1$ for any $t \in \mathbf{R}$, $b_0 = j \circ x^1$ and the mappings b_t for $t \neq 0$ are fibered isomorphisms $q \to q$. Then by the naturality of $\mathcal{K}$ with respect to the b_t and by the regularity of F we obtain

$$\mathcal{K}_q(x^1) = \mathcal{K}_q(x^1 \circ b_t) = \mathcal{K}_q(x^1) \circ F(b_t) \xrightarrow{t \to 0} \mathcal{K}_q(x^1) \circ F(j \circ x^1) \ ,$$

i.e. $\mathcal{K}_q(x^1) = \mathcal{K}_q(x^1) \circ F(j) \circ F(x^1)$, where the last x^1 is considered as the fibered mapping $q \to id_{\mathbf{R}}$. Then $\mathcal{K}_q(x^1) = \mathcal{K}_q(x^1) \circ F(j) \circ F(x^1) = \kappa \circ F(x^1) = \mathcal{K}^{<\kappa>}_q(x^1)$. Consequently, $\mathcal{K} = \mathcal{K}^{<\kappa>}$ because of Lemma 2.3.

It remains to prove the uniqueness part of the theorem. Assume that $\mathcal{K} = \mathcal{K}^{<\kappa>}$ for some $\kappa : F(id_{\mathbf{R}}) \to \mathbf{R}$. Then $\mathcal{K}_q(x^1) = \mathcal{K}^{<\kappa>}_q(x^1) = \kappa \circ F(x^1)$, where the last

x^1 is considered as the fibered mapping $q \to id_{\mathbf{R}}$. Then (since $x^1 \circ j : id_{\mathbf{R}} \to id_{\mathbf{R}}$ is the identity fibered mapping) we have

$$\kappa = \kappa \circ F(x^1 \circ j) = \kappa \circ F(x^1) \circ F(j) = \mathcal{K}_q(x^1) \circ F(j) \ .$$

The proof of the theorem is complete. □

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