THIRD SUPPLEMENT TO AN ESSAY ON THE THEORY OF SYSTEMS OF RAYS

By

William Rowan Hamilton

(Transactions of the Royal Irish Academy, vol. 17, part 1 (1837), pp. 1–144.)

Edited by David R. Wilkins

2001

NOTE ON THE TEXT

The *Third Supplement to an Essay on the Theory of Systems of Rays* by William Rowan Hamilton was originally published in volume 17 of the *Transactions of the Royal Irish Academy.* It is included in *The Mathematical Papers of Sir William Rowan Hamilton, Volume I: Geometrical Optics*, edited for the Royal Irish Academy by A. W. Conway and J. L. Synge, and published by Cambridge University Press in 1931.

There are a small number of obvious typographical errors in the original publication which are corrected in this edition (following the edition edited by Conway and Synge). These include the following:—

— in the article numbered 3, following equation (R), the line

$$\left(\frac{\delta V}{\delta x}\right), \quad \left(\frac{\delta V}{\delta y}\right), \quad \left(\frac{\delta V}{\delta y}\right),$$

in the original publication has been corrected to read

$$\left(\frac{\delta V}{\delta x}\right), \quad \left(\frac{\delta V}{\delta y}\right), \quad \left(\frac{\delta V}{\delta z}\right),$$

— in article 10, equations (H^6), the final term $\left(\delta\beta - v'\,\delta'\dfrac{\delta\Omega}{\delta v}\right)$ in the original publication has been corrected to read $\left(\delta\gamma - v'\,\delta'\dfrac{\delta\Omega}{\delta v}\right)$;

— in article 23, equation (L^{16}), $\dfrac{\delta^2 W}{\delta\sigma\,\delta\tau}$ has been replaced on the second line of the equation by $\dfrac{\delta^2 W}{\delta\tau\,\delta v}$;

— in article 29, equations (W^{19}), τ has been replaced by τ^2.

The table of contents in the original publication differred in places from the headings of the sections in the main body of the paper. This edition follows the edition edited by Conway and Synge in revising the table of contents so as to agree with the section headings in the body of the paper. Accordingly the following changes have been made to the table of contents:—

— in the description of article 16, the words 'guiding surface' have been replaced by 'guiding paraboloid';

— in the description of article 17, the word 'corresponding' has been replaced by 'respective';

— the description of article 25 in the original publication is 'Principal rays and principal foci for straight or curved systems. General method of determining the arrangement and aberrations of the rays, near a principal focus, or other point of vergency.'

A small number of changes have been made to the spelling and punctuation of the original publication.

David R. Wilkins

Dublin, June 1999
Edition corrected October 2001

Third Supplement to an Essay on the Theory of Systems of Rays. By WILLIAM R. HAMILTON, *A. B., M. R. I. A., M. R. Ast. Soc. London, M. G. Soc. Dublin, Hon. M. Soc. Arts for Scotland, Hon. M. Portsmouth Lit. and Phil. Soc., Member of the British Association for the Advancement of Science, Fellow of the American Academy of Arts and Sciences, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

Read January 23, 1832 and October 22, 1832.

[*Transactions of the Royal Irish Academy*, vol. 17, part 1 (1837), pp. 1–144.]

INTRODUCTION.

The present Supplement contains a system of general methods for the solution of Optical Problems, together with some general results, deduced from the fundamental formula and view of Optics, which have been proposed in my former memoirs. The copious analytical headings, prefixed to the several numbers, and collected in the Table of Contents, will sufficiently explain the plan of the present communication; and it is only necessary to say a few words here, respecting some of the principal results.

Of these the theory of external and internal conical refraction, deduced by my general methods from the principles of Fresnel, will probably be thought the least undeserving of attention. It is right, therefore, to state that this theory had been deduced, and was communicated to a general meeting of the Royal Irish Academy, not at the earlier, but at the later of the two dates prefixed to the present Supplement. After making this communication to the Academy, in October, 1832, I requested Professor Lloyd to examine the question experimentally, and to try whether he could perceive any such phenomena in biaxal crystals, as my theory of conical refraction had led me to expect. The experiments of Professor Lloyd, confirming my theoretical expectations, have been published by him in the numbers of the London and Edinburgh Philosophical Magazine, for the months of February and March, 1833; and they will be found with fuller details in the present Volume of the Irish Transactions.

I am informed that James Mac Cullagh, Esq., F.T.C.D., who published in the last preceding Volume of these Transactions a series of elegant Geometrical Illustrations of Fresnel's theory, has, since he heard of the experiments of Professor Lloyd, employed his own geometrical methods to confirm my results respecting the existence of those conoidal cusps and circles of contact on Fresnel's wave, from which I had been led to the expectation of conical refraction. And on my lately mentioning to him that I had connected these cusps and circles on Fresnel's wave, with circles and cusps of the same kind on a certain other surface discovered by M. Cauchy, by a general theory of reciprocal surfaces, which I stated last year at a general meeting of the Royal Irish Academy, Mr. Mac Cullagh said that he had arrived independently at similar results, and put into my hands a paper on the subject, which I have not yet been able to examine, but which will (I hope) be soon presented to the Academy, and published in their Transactions.

I ought also to mention, that on my writing in last November to Professor Airy, and communicating to him my results respecting the cusps and circles of Fresnel's wave, and my expectation of conical refraction which had not then been verified, Professor Airy replied that he had long been aware of the existence of the conoidal cusps, which indeed it is surprising that Fresnel did not perceive. Professor Airy, however, had not perceived the existence of the circles of contact, nor had he drawn from either cusps or circles any theory of conical refraction.

The latter theory was deduced, by my general methods, from the hypothesis of transversal vibrations in a luminous ether, which hypothesis seems to have been first proposed by Dr. Young, but to have been independently framed and far more perfectly developed by Fresnel; and from Fresnel's other principle, of the existence of three rectangular axes of elasticity within a biaxal crystallised medium. The verification, therefore, of this theory of conical refraction, by the experiments of Professor Lloyd, must be considered as affording a new and important probability in favour of Fresnel's views: that is, a new encouragement to reason from those views, in combining and predicting appearances.

The length to which the present Supplement has already extended, obliges me to reserve, for a future communication, many other results deduced by my general methods from the principle of the characteristic function: and especially a general theory of the focal lengths and aberrations of optical instruments of revolution.

WILLIAM R. HAMILTON.

Observatory, *June*, 1833.

CONTENTS OF THE THIRD SUPPLEMENT.

15. Discussion of the four problems. Elements of arrangement of near luminous paths. Axis and constant of chromatic dispersion. Axis of curvature of ray. Guiding paraboloid, and constant of deviation. Guiding planes, and conjugate guiding axes.

16. Application of the elements of arrangement. Connexion of the two final vergencies, and planes of vergency, and guiding lines, with the two principal curvatures and planes of curvature of the guiding paraboloid, and with the constant of deviation. The planes of curvature are the planes of extreme projection of the final ray-lines.

17. Second application of the elements. Arrangement of the near final-ray lines from an oblique plane. Generalisation of the theory of the guiding paraboloid and constant of deviation. General theory of deflexures of surfaces. Circles and axes of deflexure. Rectangular planes and axes of extreme deflexure. Deflected lines, passing through these axes, and having the centres of deflexure for their respective foci by projection. Conjugate planes of deflexure, and indicating cylinder of deflexion.

18. Construction of the new auxiliary paraboloid, (or of an osculating hyperboloid,) and of the new constant of deviation, for ray-lines from an oblique plane, by the former elements of arrangement.

19. Condition of intersection of two near final ray-lines. Conical locus of the near final points in a variable medium which satisfy this condition. Investigations of MALUS. Illustration of the condition of intersection, by the theory of the auxiliary paraboloid, for ray-lines from an oblique plane.

20. Other geometrical illustrations of the condition of intersection, and of the elements of arrangement. Composition of partial deviations. Rotation round the axis of curvature of a final ray.

21. Relations between the elements of arrangement, depending only on the extreme points, directions, and colour of a given luminous path, and on the extreme media. In a final uniform medium, ordinary or extraordinary, the two planes of vergency are conjugate planes of deflexure of any surface of a certain class determined by the final medium; and also of a certain analogous surface determined by the whole combination. Relations between the visible magnitudes and distortions of any two small objects viewed from each other through any optical combination. Interchangeable eye-axes and object-axes of distortion. Planes of no distortion.

22. Calculation of the elements of arrangement, for arbitrary axes of co-ordinates.

23. The general linear expressions for the arrangement of near rays, fail at a point of vergency. Determination of these points, and of their loci, the caustic surfaces, in a straight or curved system, by the methods of the present Supplement.

24. Connexion of the conditions of initial and final intersection of two near paths of light, polygon or curved, with the maxima or minima of the time or action-function $V + V_{\prime} = \Sigma \int v\, ds$. Separating planes, transition planes, and transition points, suggested by these maxima and minima. The separating planes divide the near points of less from those of greater action, and they contain the directions of osculation or intersection of the surfaces for which V and $V_{\prime}$ are constant; the transition planes touch the caustic pencils, and the transition points are on the caustic curves. Extreme osculating waves, or action surfaces. Law of osculation. Analogous theorems for sudden reflection or refraction.

25. Formulæ for the principal foci and principal rays of a straight or curved system, ordinary or extraordinary. General method of investigating the arrangement and aberrations

of the rays, near a principal focus, or other point of vergency.

26. Combination of the foregoing view of optics with the undulatory theory of light. The quantities σ, τ, υ, or

$$\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z},$$

that is, the partial differential coefficients of the first order of the characteristic function V, taken with respect to the final co-ordinates, are, in the undulatory theory of light, the components of normal slowness of propagation of a wave. The fundamental formula (A) may easily be explained and proved by the principles of the same theory.

27. Theory of FRESNEL. New formulae, founded on that theory, for the velocities and polarisations of a plane wave or wave-element. New method of deducing the equation of FRESNEL'S curved wave, propagated from a point in a uniform medium with three unequal elasticities. Lines of single ray-velocity, and of single normal-velocity, discovered by FRESNEL.

28. New properties of FRESNEL'S wave. This wave has four conoidal cusps, at the ends of the lines of single ray-velocity; it has also four circles of contact, of which each is contained on a touching plane of single normal-velocity. The lines of single ray-velocity may therefore be called cusp-rays; and the lines of single normal-velocity may be called normals of circular contact.

29. New consequences of FRESNEL'S principles. It follows from those principles, that crystals of sufficient biaxal energy ought to exhibit two kinds of conical refraction, an external and an internal: a cusp-ray giving an external cone of rays, and a normal of circular contact being connected with an internal cone.

30. Theory of conical polarisation. Lines of vibration. These lines, on FRESNEL'S wave, are the intersections of two series of concentric and co-axal ellipsoids.

31. In any uniform medium, the curved wave propagated from a point is connected with a certain other surface, which may be called the surface of components, by relations discovered by M. CAUCHY, and by some new relations connected with a general theorem of reciprocity. This new theorem of reciprocity gives a new construction for the wave in any undulatory theory of light: and it connects the cusps and circles of contact on FRESNEL'S wave, with circles and cusps of the same kind upon the surface of components.

THIRD SUPPLEMENT.

Fundamental Formula of Mathematical Optics. Design of the present Supplement.

1. When light is considered as propagated, according to that known general law which is called the law of least action, or of swiftest propagation, along any curved or polygon ray, ordinary or extraordinary, describing each element of that ray $ds = \sqrt{(dx^2 + dy^2 + dz^2)}$ with a molecular velocity or undulatory slowness v, which is supposed to depend, in the most general case, on the nature of the medium, the position and direction of the element, and the colour of the light, having only a finite number of values when these are given, and being therefore a function of the three rectangular co-ordinates, or marks of position, x, y, z, the three differential ratios or cosines of direction,

$$\alpha = \frac{dx}{ds}, \quad \beta = \frac{dy}{ds}, \quad \gamma = \frac{dx}{ds},$$

and a chromatic index or measure of colour, χ, the form of which function v depends on and characterises the medium; then if we denote as follows the variation of this function,

$$\delta v = \frac{\delta v}{\delta x}\delta x + \frac{\delta v}{\delta y}\delta y + \frac{\delta v}{\delta z}\delta z + \frac{\delta v}{\delta \alpha}\delta \alpha + \frac{\delta v}{\delta \beta}\delta \beta + \frac{\delta v}{\delta \gamma}\delta \gamma + \frac{\delta v}{\delta \chi}\delta \chi,$$

and if, by the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we determine

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

so as to satisfy the condition

$$\alpha\frac{\delta v}{\delta \alpha} + \beta\frac{\delta v}{\delta \beta} + \gamma\frac{\delta v}{\delta \gamma} = v,$$

namely, by making v homogeneous of the first dimension with respect to α, β, γ; it has been shown, in my First Supplement, that the variation of the definite integral $V = \int v\, ds$, considered as a function, which I have called the *Characteristic Function* of the final and initial co-ordinates, that is, the *variation of the action, or the time, expended by light of any one colour, in going from one variable point to another*, is

$$\delta V = (\delta \int v\, ds =)\ \frac{\delta v}{\delta \alpha}\delta x - \frac{\delta v'}{\delta \alpha'}\delta x' + \frac{\delta v}{\delta \beta}\delta y - \frac{\delta v'}{\delta \beta'}\delta y' + \frac{\delta v}{\delta \gamma}\delta z - \frac{\delta v'}{\delta \gamma'}\delta z' : \tag{A}$$

the accented being the initial quantities. This general equation, (A), which I have called the *Equation of the Characteristic Function*, involves very various and extensive consequences, and appears to me to include the whole of mathematical optics. I propose, in the present Supplement, to offer some additional remarks and methods, connected with the characteristic function V, and the fundamental formula (A); and in particular to point out a new view of the auxiliary function W, introduced in my former memoirs, and a new auxiliary function T, which may be employed with advantage in many optical researches: I shall also give some other general transformations and applications of the fundamental formula, and shall speak of the connection of my view of optics with the undulatory theory of light.

Fundamental Problem of Mathematical Optics, and Solution by the Fundamental Formula. Partial Differential Equations, respecting the Characteristic Function V, and common to all optical combinations. Deduction of the Medium-Functions Ω, v, from this Characteristic Function V. Remarks on the new symbols σ, τ, υ.

2. It may be considered as a *fundamental problem* in Mathematical Optics, to which all others are reducible, *to determine, for any proposed combination of media, the law of dependence of the two extreme directions of a curved or polygon ray, ordinary or extraordinary, on the positions of the two extreme points which are visually connected by that ray, and on the colour of the light*: that is, in our present notation, to determine the law of dependence of the extreme *direction-cosines* $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, on the extreme co-ordinates $x\ y\ z\ x'\ y'\ z'$, and on the chromatic index χ. This fundamental problem is resolved by our fundamental formula (A); or by the six following equations into which (A) resolves itself, and which express the law of dependence required:

$$\left.\begin{array}{lll} \dfrac{\delta V}{\delta x} = \dfrac{\delta v}{\delta \alpha}; & \dfrac{\delta V}{\delta y} = \dfrac{\delta v}{\delta \beta}; & \dfrac{\delta V}{\delta z} = \dfrac{\delta v}{\delta \gamma}; \\ -\dfrac{\delta V}{\delta x'} = \dfrac{\delta v'}{\delta \alpha'}; & -\dfrac{\delta V}{\delta y'} = \dfrac{\delta v'}{\delta \beta'}; & -\dfrac{\delta V}{\delta z'} = \dfrac{\delta v'}{\delta \gamma'}. \end{array}\right\} \tag{B}$$

These equations appear to require, for their application to any proposed combination, not only the knowledge of the form of the *Characteristic Function* V, that is, the law of dependence of the action or time on the extreme positions and on the colour, but also on the knowledge of the forms of the functions v, v', that is, the optical properties of the final and initial media; but these final and initial *medium-functions* v, v', may themselves be deduced from the one characteristic function V, by reasonings of the following kind.

Whatever be the nature of the final medium, that is, whatever be the law of dependence of v on the position, direction, colour, we have supposed, in deducing the general formula (A), that the expression of this dependence has been so prepared as to make the medium-function v homogeneous of the first dimension relatively to the direction-cosines α, β, γ; the partial differential coefficients

$$\frac{\delta v}{\delta \alpha},\quad \frac{\delta v}{\delta \beta},\quad \frac{\delta v}{\delta \gamma},$$

of this homogeneous function, are themselves homogeneous, but of the dimension zero; that is, they are functions of the two ratios

$$\frac{\alpha}{\gamma},\quad \frac{\beta}{\gamma},$$

involving also, in general, the co-ordinates $x\ y\ z$, and the chromatic index χ: if then we conceive the two ratios

$$\frac{\alpha}{\gamma},\quad \frac{\beta}{\gamma},$$

to be eliminated between the three first of equations (B), and if, in like manner, we conceive

$$\frac{\alpha'}{\gamma'},\quad \frac{\beta'}{\gamma'},$$

to be eliminated between the last three equations (B), we see that such eliminations would give two partial differential equations of the first order, between the characteristic function V and the co-ordinates and colour, of the form

$$\left.\begin{aligned} 0 &= \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, x, y, z, \chi\right), \\ 0 &= \Omega'\left(-\frac{\delta V}{\delta x'}, -\frac{\delta V}{\delta y'}, -\frac{\delta V}{\delta z'}, x', y', z', \chi\right), \end{aligned}\right\} \tag{C}$$

which both conduct to the following general equation, of the second order and third degree, common to all optical combinations,

$$\begin{aligned} &\frac{\delta^2 V}{\delta x\,\delta x'}\,\frac{\delta^2 V}{\delta y\,\delta y'}\,\frac{\delta^2 V}{\delta z\,\delta z'} + \frac{\delta^2 V}{\delta x\,\delta y'}\,\frac{\delta^2 V}{\delta y\,\delta z'}\,\frac{\delta^2 V}{\delta z\,\delta x'} + \frac{\delta^2 V}{\delta x\,\delta z'}\,\frac{\delta^2 V}{\delta y\,\delta x'}\,\frac{\delta^2 V}{\delta z\,\delta y'} \\ = &\frac{\delta^2 V}{\delta z\,\delta x'}\,\frac{\delta^2 V}{\delta y\,\delta y'}\,\frac{\delta^2 V}{\delta x\,\delta z'} + \frac{\delta^2 V}{\delta z\,\delta y'}\,\frac{\delta^2 V}{\delta y\,\delta z'}\,\frac{\delta^2 V}{\delta x\,\delta x'} + \frac{\delta^2 V}{\delta z\,\delta z'}\,\frac{\delta^2 V}{\delta y\,\delta x'}\,\frac{\delta^2 V}{\delta x\,\delta y'}. \end{aligned} \tag{D}$$

If now we put, for abridgment,

$$\left.\begin{aligned} \frac{\delta V}{\delta x} &= \sigma, & \frac{\delta V}{\delta y} &= \tau, & \frac{\delta V}{\delta z} &= \upsilon, \\ -\frac{\delta V}{\delta x'} &= \sigma', & -\frac{\delta V}{\delta y'} &= \tau', & -\frac{\delta V}{\delta z'} &= \upsilon', \end{aligned}\right\} \tag{E}$$

and if between the first three of these equations (E) we eliminate two of the three initial co-ordinates x' y' z', it is easy to perceive, by (C) or (D), that in every optical combination the third co-ordinate will disappear; and similarly that between the three last equations (E) we can eliminate all the three final co-ordinates, by eliminating any two of them; and that these eliminations will conduct to the relations (C) under the form

$$\left.\begin{aligned} 0 &= \Omega(\sigma, \tau, \upsilon, x, y, z, \chi), \\ 0 &= \Omega'(\sigma', \tau', \upsilon', x', y', z', \chi), \end{aligned}\right\} \tag{F}$$

which can thus be obtained, by differentiation and elimination, from the characteristic function V alone: and which, as we are about to see, determine the forms of v, v', that is, the properties of the extreme media. Comparing the differentials of the relations (F), with the following, that is, with the conditions of homogeneity of v, v', prepared by the definitions (E) and by the relations (B),

$$\left.\begin{aligned} v &= \alpha\frac{\delta v}{\delta \alpha} + \beta\frac{\delta v}{\delta \beta} + \gamma\frac{\delta v}{\delta \gamma} = \alpha\sigma + \beta\tau + \gamma\upsilon, \\ v' &= \alpha'\frac{\delta v'}{\delta \alpha'} + \beta'\frac{\delta v'}{\delta \beta'} + \gamma'\frac{\delta v'}{\delta \gamma'} = \alpha'\sigma' + \beta'\tau' + \gamma'\upsilon', \end{aligned}\right\} \tag{G}$$

and with their differentials, that is with

$$\left.\begin{aligned}\alpha\,\delta\sigma+\beta\,\delta\tau+\gamma\,\delta\upsilon&=\frac{\delta\upsilon}{\delta x}\delta x+\frac{\delta\upsilon}{\delta y}\delta y+\frac{\delta\upsilon}{\delta z}\delta z+\frac{\delta\upsilon}{\delta\chi}\delta\chi,\\ \alpha'\,\delta\sigma'+\beta'\,\delta\tau'+\gamma'\,\delta\upsilon'&=\frac{\delta\upsilon'}{\delta x'}\delta x'+\frac{\delta\upsilon'}{\delta y'}\delta y'+\frac{\delta\upsilon'}{\delta z'}\delta z'+\frac{\delta\upsilon'}{\delta\chi}\delta\chi,\end{aligned}\right\}\tag{H}$$

we find

$$\left.\begin{aligned}\frac{\alpha}{\upsilon}&=\frac{\delta\Omega}{\delta\sigma}, & \frac{\beta}{\upsilon}&=\frac{\delta\Omega}{\delta\tau}, & \frac{\gamma}{\upsilon}&=\frac{\delta\Omega}{\delta\upsilon},\\ \frac{\alpha'}{\upsilon'}&=\frac{\delta\Omega'}{\delta\sigma'}, & \frac{\beta'}{\upsilon'}&=\frac{\delta\Omega'}{\delta\tau'}, & \frac{\gamma'}{\upsilon'}&=\frac{\delta\Omega'}{\delta\upsilon'},\end{aligned}\right\}\tag{I}$$

and also

$$\left.\begin{aligned}-\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta x}&=\frac{\delta\Omega}{\delta x}, & -\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta y}&=\frac{\delta\Omega}{\delta y}, & -\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta z}&=\frac{\delta\Omega}{\delta z}, & -\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta\chi}&=\frac{\delta\Omega}{\delta\chi},\\ -\frac{1}{\upsilon'}\frac{\delta\upsilon'}{\delta x'}&=\frac{\delta\Omega'}{\delta x'}, & -\frac{1}{\upsilon'}\frac{\delta\upsilon'}{\delta y'}&=\frac{\delta\Omega'}{\delta y'}, & -\frac{1}{\upsilon'}\frac{\delta\upsilon'}{\delta z'}&=\frac{\delta\Omega'}{\delta z'}, & -\frac{1}{\upsilon'}\frac{\delta\upsilon'}{\delta\chi}&=\frac{\delta\Omega'}{\delta\chi},\end{aligned}\right\}\tag{K}$$

if we so prepare the expressions of the relations (F) as to have

$$\left.\begin{aligned}\sigma\frac{\delta\Omega}{\delta\sigma}+\tau\frac{\delta\Omega}{\delta\tau}+\upsilon\frac{\delta\Omega}{\delta\upsilon}&=1,\\ \sigma'\frac{\delta\Omega'}{\delta\sigma'}+\tau'\frac{\delta\Omega'}{\delta\tau'}+\upsilon'\frac{\delta\Omega'}{\delta\upsilon'}&=1;\end{aligned}\right\}\tag{L}$$

which can be done by putting those relations under the form

$$\left.\begin{aligned}0&=(\sigma^2+\tau^2+\upsilon^2)^{\frac{1}{2}}\omega-1=\Omega,\\ 0&=(\sigma'^2+\tau'^2+\upsilon'^2)^{\frac{1}{2}}\omega'-1=\Omega';\end{aligned}\right\}\tag{M}$$

in which ω, ω', that is $(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, and $(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, are to be expressed as functions respectively of $\sigma(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\tau(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\upsilon(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, x, y, z, χ, and of $\sigma'(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, $\tau'(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, $\upsilon'(\sigma'^2+\tau'^2+\upsilon'^2)^{-\frac{1}{2}}$, x', y', z', χ. After this preparation the partial differential coefficients

$$\frac{\delta\Omega}{\delta\sigma},\quad\frac{\delta\Omega}{\delta\tau},\quad\frac{\delta\Omega}{\delta\upsilon},$$

are homogeneous of dimension zero relatively to σ, τ, υ; and in like manner

$$\frac{\delta\Omega'}{\delta\sigma'},\quad\frac{\delta\Omega'}{\delta\tau'},\quad\frac{\delta\Omega'}{\delta\upsilon'},$$

are homogeneous of dimension zero relatively to σ', τ', υ'; if, therefore, between the three first equations (I), we eliminate any two of the three final quantities σ, τ, υ, the third will disappear; and similarly all the three initial quantities σ', τ', υ', can be eliminated together, between the last three of the equations (I): and by these eliminations we shall be conducted to two relations of the form

$$\left.\begin{aligned} 0 &= \Psi(\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}, x, y, z, \chi), \\ 0 &= \Psi'(\frac{\alpha'}{v'}, \frac{\beta'}{v'}, \frac{\gamma'}{v'}, x', y', z', \chi), \end{aligned}\right\} \qquad \text{(N)}$$

which determine the forms of the final and initial medium-functions v, v'; so that these forms can be deduced from the form of the characteristic function V. We can therefore reduce to the study of this one function V, that general problem of mathematical optics which has been already mentioned.

The partial differential coefficients of the characteristic function V, taken with respect to the co-ordinates x, y, z, are of continual occurrence in the optical methods of my present and former memoirs; I have therefore thought it useful to denote them in this Supplement by separate symbols σ, τ, υ, and I shall show in a future number their meanings in the undulatory theory: namely, that they denote, in it, the components of normal slowness of propagation of a wave.

Connection of the Characteristic Function V, with the Formation and Integration of the General Equations of a Curved Ray, Ordinary or Extraordinary.

3. It may be considered as a particular case of the foregoing general problem, to determine general forms for the differential equations of a curved ray, ordinary or extraordinary; that is, to connect the general changes of direction with those of position, in the passage of light through a variable medium. The following forms,

$$d\frac{\delta v}{\delta \alpha} = \frac{\delta v}{\delta x} ds, \quad d\frac{\delta v}{\delta \beta} = \frac{\delta v}{\delta y} ds, \quad d\frac{\delta v}{\delta \gamma} = \frac{\delta v}{\delta z} ds, \qquad \text{(O)}$$

(which are of the second order, because α, β, γ, α', β', γ', are defined by the equations

$$\left.\begin{aligned} \alpha &= \frac{dx}{ds}, & \beta &= \frac{dy}{ds}, & \gamma &= \frac{dz}{ds}, \\ \alpha' &= \frac{dx'}{ds'}, & \beta' &= \frac{dy'}{ds'}, & \gamma' &= \frac{dz'}{ds'}, \end{aligned}\right\} \qquad \text{(P)}$$

the symbol d referring, throughout the present Supplement, to motion along a ray, while δ refers to arbitrary infinitesimal changes of position, direction, and colour, and ds' being the initial element of the ray,) were deduced, in the First Supplement, by the Calculus of Variations, from the law of least action. The same forms (O), which are equivalent to but two distinct equations, may be deduced from the fundamental formula (A), by the properties of the characteristic function V. For, if we differentiate the first equation (C), (which involves the coefficients of this function V, and was deduced from the formula (A),) with reference to each of the three co-ordinates x, y, z, considered as three independent variables, and with reference to the index of colour χ, we find, by the foregoing number,

$$\left.\begin{aligned}\alpha\frac{\delta^2 V}{\delta x^2} + \beta\frac{\delta^2 V}{\delta x\,\delta y} + \gamma\frac{\delta^2 V}{\delta x\,\delta z} &= \frac{\delta \upsilon}{\delta x},\\ \alpha\frac{\delta^2 V}{\delta x\,\delta y} + \beta\frac{\delta^2 V}{\delta y^2} + \gamma\frac{\delta^2 V}{\delta y\,\delta z} &= \frac{\delta \upsilon}{\delta y},\\ \alpha\frac{\delta^2 V}{\delta x\,\delta z} + \beta\frac{\delta^2 V}{\delta y\,\delta z} + \gamma\frac{\delta^2 V}{\delta z^2} &= \frac{\delta \upsilon}{\delta z},\\ \alpha\frac{\delta^2 V}{\delta x\,\delta \chi} + \beta\frac{\delta^2 V}{\delta y\,\delta \chi} + \gamma\frac{\delta^2 V}{\delta z\,\delta \chi} &= \frac{\delta \upsilon}{\delta \chi},\end{aligned}\right\} \tag{Q}$$

and the three first of the equations (Q), by the general relations (B), which were themselves deduced from (A), and by the meanings (P) of α, β, γ, may easily be transformed to (O). The differential equations (O) may also be regarded as the limits of the following,

$$\sigma - \sigma' = \left(\frac{\delta V}{\delta x}\right), \quad \tau - \tau' = \left(\frac{\delta V}{\delta y}\right), \quad \upsilon - \upsilon' = \left(\frac{\delta V}{\delta z}\right), \tag{R}$$

in which

$$\left(\frac{\delta V}{\delta x}\right), \quad \left(\frac{\delta V}{\delta y}\right), \quad \left(\frac{\delta V}{\delta z}\right),$$

are obtained by differentiating V considered as a function of the seven variables x, y, z, Δx, Δy, Δz, χ, if $\Delta x = x - x'$, $\Delta y = y - y'$, $\Delta z = z - z'$; the variation of V, when so considered, being by (A), and by the definitions (E),

$$\begin{aligned}\delta V = (\sigma - \sigma')\delta x + (\tau - \tau')\delta y + (\upsilon - \upsilon')\delta z\\ + \left(\frac{\delta V}{\delta \Delta x}\right)\delta\Delta x + \left(\frac{\delta V}{\delta \Delta y}\right)\delta\Delta y + \left(\frac{\delta V}{\delta \Delta z}\right)\delta\Delta z + \frac{\delta V}{\delta \chi}\delta\chi,\end{aligned} \tag{S}$$

in which

$$\left(\frac{\delta V}{\delta \Delta x}\right) = \sigma', \quad \left(\frac{\delta V}{\delta \Delta y}\right) = \tau', \quad \left(\frac{\delta V}{\delta \Delta z}\right) = \upsilon'. \tag{T}$$

If we differentiate the first equation (C) relatively to x', y', z', we find, by the foregoing number,

$$\left.\begin{aligned}0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta x'} + \beta\frac{\delta^2 V}{\delta y\,\delta x'} + \gamma\frac{\delta^2 V}{\delta z\,\delta x'},\\ 0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta y'} + \beta\frac{\delta^2 V}{\delta y\,\delta y'} + \gamma\frac{\delta^2 V}{\delta z\,\delta y'},\\ 0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta z'} + \beta\frac{\delta^2 V}{\delta y\,\delta z'} + \gamma\frac{\delta^2 V}{\delta z\,\delta z'},\end{aligned}\right\} \tag{U}$$

of which, in virtue of (D), any two include the third, and which may be put by (P) under the form

$$0 = d\frac{\delta V}{\delta x'}; \quad 0 = d\frac{\delta V}{\delta y'}; \quad 0 = d\frac{\delta V}{\delta z'}; \tag{V}$$

and these differential equations (V) of the first order, in which the initial co-ordinates and the colour are constant, belong to the ray, and may be regarded as integrals of (O). They have, themselves, for integrals

$$\frac{\delta V}{\delta x'} = \text{const.}, \quad \frac{\delta V}{\delta y'} = \text{const.}, \quad \frac{\delta V}{\delta z'} = \text{const.}, \tag{W}$$

the constants being, by (B), the values of the initial quantities

$$-\frac{\delta v'}{\delta \alpha'}, \quad -\frac{\delta v'}{\delta \beta'}, \quad -\frac{\delta v'}{\delta \gamma'}.$$

In like manner, by differentiating the last equation (C), we find the following equations, which are analogous to (Q) and (U),

$$\left.\begin{aligned}
\alpha' \frac{\delta^2 V}{\delta x'^2} + \beta' \frac{\delta^2 V}{\delta x' \, \delta y'} + \gamma' \frac{\delta^2 V}{\delta x' \, \delta z'} &= -\frac{\delta v'}{\delta x'}, \\
\alpha' \frac{\delta^2 V}{\delta x' \, \delta y'} + \beta' \frac{\delta^2 V}{\delta y'^2} + \gamma' \frac{\delta^2 V}{\delta y' \, \delta z'} &= -\frac{\delta v'}{\delta y'}, \\
\alpha' \frac{\delta^2 V}{\delta x' \, \delta z'} + \beta' \frac{\delta^2 V}{\delta y' \, \delta z'} + \gamma' \frac{\delta^2 V}{\delta z'^2} &= -\frac{\delta v'}{\delta z'}, \\
\alpha' \frac{\delta^2 V}{\delta x' \, \delta \chi} + \beta' \frac{\delta^2 V}{\delta y' \, \delta \chi} + \gamma' \frac{\delta^2 V}{\delta z' \, \delta \chi} &= -\frac{\delta v'}{\delta \chi},
\end{aligned}\right\} \tag{X}$$

and

$$\left.\begin{aligned}
0 &= \alpha' \frac{\delta^2 V}{\delta x \, \delta x'} + \beta' \frac{\delta^2 V}{\delta x \, \delta y'} + \gamma' \frac{\delta^2 V}{\delta x \, \delta z'}, \\
0 &= \alpha' \frac{\delta^2 V}{\delta y \, \delta x'} + \beta' \frac{\delta^2 V}{\delta y \, \delta y'} + \gamma' \frac{\delta^2 V}{\delta y \, \delta z'}, \\
0 &= \alpha' \frac{\delta^2 V}{\delta z \, \delta x'} + \beta' \frac{\delta^2 V}{\delta z \, \delta y'} + \gamma' \frac{\delta^2 V}{\delta z \, \delta z'},
\end{aligned}\right\} \tag{Y}$$

The second members of the three first equations (X) vanish when the initial medium is uniform, and those of the three first equations (Q) when the final medium is so; and in this latter case, of a final uniform medium, the final portion of the ray is straight, and in its whole extent we have not only the equations (W) but also the following,

$$\frac{\delta V}{\delta x} = \text{const.}, \quad \frac{\delta V}{\delta y} = \text{const.}, \quad \frac{\delta V}{\delta z} = \text{const.}, \tag{Z}$$

the constants being by (B) those functions of the final direction-cosines and of the colour which we have denoted by

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

and which are here independent of the co-ordinates. In general, if we consider the final co-ordinates and the colour as constant, the relations (Z) between the initial co-ordinates are forms for the equations of a ray. And though we have hitherto considered rectangular co-ordinates only, yet we shall show in a future number that there are analogous results for oblique and even for polar co-ordinates.

Transformations of the Fundamental Formula. New View of the Auxiliary Function W; New Auxiliary Function T. Deductions of the Characteristic and Auxiliary Functions, V, W, T, each from each. General Theorem of Maxima and Minima, which includes the details of such deductions. Remarks on the respective advantages of the Characteristic and Auxiliary Functions.

4. The fundamental equation (A) may be put under the form

$$\delta V = \sigma\,\delta x - \sigma'\,\delta x' + \tau\,\delta y - \tau'\,\delta y' + \upsilon\,\delta z - \upsilon'\,\delta z' + \frac{\delta V}{\delta \chi}\,\delta\chi, \tag{A$'$}$$

employing the definitions (E), and introducing the variation of colour; it admits also of the two following general transformations,

$$\delta W = x\,\delta\sigma + y\,\delta\tau + z\,\delta\upsilon + \sigma'\,\delta x' + \tau'\,\delta y' + \upsilon'\,\delta z' - \frac{\delta V}{\delta \chi}\,\delta\chi, \tag{B$'$}$$

and

$$\delta T = x\,\delta\sigma - x'\,\delta\sigma' + y\,\delta\tau - y'\,\delta\tau' + z\,\delta\upsilon - z'\,\delta\upsilon' - \frac{\delta V}{\delta \chi}\,\delta\chi, \tag{C$'$}$$

in which

$$W = -V + x\sigma + y\tau + z\upsilon, \tag{D$'$}$$

and

$$T = W - x'\sigma' - y'\tau' - z'\upsilon'. \tag{E$'$}$$

In the two foregoing Supplements, the quantity W was introduced, and was considered as a function of the final direction-cosines α, β, γ, the final medium being regarded as uniform, and the luminous origin and colour as given; we shall now take another and a more general view of this auxiliary function W, and shall consider it as depending, by (B$'$), for all optical combinations, on the seven quantities $\sigma\ \tau\ \upsilon\ x'\ y'\ z'\ \chi$. In like manner, we shall consider the new auxiliary function T as depending, by the new transformation (C$'$), on the seven quantities $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ \chi$. The forms of these auxiliary functions W, T, are connected with each other, and with the characteristic function V, by relations of which the knowledge is important, in the theory of optical systems. Let us therefore consider how the form of each of the three functions V, W, T, can be deduced from the form of either of the other two.

These deductions may all be affected by suitable applications of the three forms (A$'$) (B$'$) (C$'$), of our fundamental equation (A), together with the definitions (D$'$) (E$'$), as we shall soon see more in detail, by means of the following remarks.

When the form of the characteristic function V is known, and it is required to deduce the form of the auxiliary function W, we are to eliminate the three final co-ordinates x, y, z, between the equation (D$'$) and the three first of the equations (E); and similarly when it is required to deduce the form of T from that of V, we are to eliminate the six final and initial co-ordinates $x\ y\ z\ x'\ y'\ z'$ between the six equations (E), (which are all included in the formula (A$'$),) and the following,

$$T = -V + x\sigma - x'\sigma' + y\tau - y'\tau' + z\upsilon - z'\upsilon' : \tag{F$'$}$$

and if it be required to deduce the form of T from that of W, we are to eliminate the three initial co-ordinates x' y' z', between the equation (E′) and the three following general equations,

$$\sigma' = \frac{\delta W}{\delta x'}, \quad \tau' = \frac{\delta W}{\delta y'}, \quad \upsilon' = \frac{\delta W}{\delta z'}. \tag{G′}$$

But when it is required to deduce reciprocally V from T or from W, or W from T, we must distinguish between the cases of variable and of uniform media; because we must then use the equations into which (B′) and (C′) resolve themselves, and this resolution, when the extreme media are not both variable, requires the consideration of the connexion that then exists between the quantities σ τ υ σ' τ' υ' χ: which circumstance also, of a connexion between these variable quantities, leave a partial indeterminateness in the forms of T and W as deduced from V, and in the form of T as deduced from W, for the case of uniform media.

When the final medium is variable, then σ, τ, υ, χ, may in general vary independently, and the equation (B′) gives

$$\frac{\delta W}{\delta \sigma} = x, \quad \frac{\delta W}{\delta \tau} = y, \quad \frac{\delta W}{\delta \upsilon} = z, \quad \frac{\delta W}{\delta \chi} = -\frac{\delta V}{\delta \chi}; \tag{H′}$$

and, in this case, V can in general be deduced from W by eliminating σ, τ, υ, between the equation (D′), and the three first equations (H′). But if the final medium be uniform, then σ, τ, υ, χ, are connected by the first of the relations (F), from which, in this case, the final co-ordinates disappear; and instead of the four equations (H′) we have the three following

$$\frac{\dfrac{\delta W}{\delta \sigma} - x}{\dfrac{\delta \Omega}{\delta \sigma}} = \frac{\dfrac{\delta W}{\delta \tau} - y}{\dfrac{\delta \Omega}{\delta \tau}} = \frac{\dfrac{\delta W}{\delta \upsilon} - z}{\dfrac{\delta \Omega}{\delta \upsilon}} = \frac{\dfrac{\delta W}{\delta \chi} + \dfrac{\delta V}{\delta \chi}}{\dfrac{\delta \Omega}{\delta \chi}}; \tag{I′}$$

by means of the two first of which, combined with the relation already mentioned, namely,

$$0 = \Omega(\sigma, \tau, \upsilon, \chi), \tag{K′}$$

which depends on, and characterises, the nature of the final uniform medium, we can eliminate σ, τ, υ, from equation (D′), and so deduce V from W.

In like manner, if both the extreme media be variable, then the seven quantities σ τ υ σ' τ' υ' χ may in general vary independently, and the equation (C′) resolves itself into the seven following,

$$\frac{\delta T}{\delta \sigma} = x, \quad \frac{\delta T}{\delta \tau} = y, \quad \frac{\delta T}{\delta \upsilon} = z, \quad \frac{\delta T}{\delta \chi} = -\frac{\delta V}{\delta \chi}, \quad \frac{\delta T}{\delta \sigma'} = -x', \quad \frac{\delta T}{\delta \tau'} = -y', \quad \frac{\delta T}{\delta \upsilon'} = -z', \tag{L′}$$

by the three first and three last of which we can eliminate σ τ υ σ' τ' υ' from (F′), and so deduce V from T. And in the same case, or even in the case when only the initial medium is variable, the three last of the equations (L′) are true, and suffice to eliminate σ', τ', υ', from (E′), and so to deduce W from T.

But if the final medium be uniform, the initial being still variable, then σ, τ, υ, χ, are connected by the relation (K′), while σ' τ' υ' remain independent; and instead of the four first equations (L′) we have the three following,

$$\frac{\dfrac{\delta T}{\delta \sigma} - x}{\dfrac{\delta \Omega}{\delta \sigma}} = \frac{\dfrac{\delta T}{\delta \tau} - y}{\dfrac{\delta \Omega}{\delta \tau}} = \frac{\dfrac{\delta T}{\delta \upsilon} - z}{\dfrac{\delta \Omega}{\delta \upsilon}} = \frac{\dfrac{\delta T}{\delta \chi} + \dfrac{\delta V}{\delta \chi}}{\dfrac{\delta \Omega}{\delta \chi}}; \tag{M′}$$

by the two first of which, combined with the relation (K′), and with the three last equations (L′), we can eliminate σ, τ, υ, σ', τ', υ', from (F′), and so deduce V from T.

If both the extreme media be uniform, we have then not only the relation (K′) for the final medium, but also an analogous relation

$$0 = \Omega'(\sigma', \tau', \upsilon', \chi) \tag{N′}$$

for the initial; and instead of the seven equations (L′), we have the two first of the equations (M′), and the two following,

$$\frac{\dfrac{\delta T}{\delta \sigma'} + x'}{\dfrac{\delta \Omega}{\delta \sigma'}} = \frac{\dfrac{\delta T}{\delta \tau'} + y'}{\dfrac{\delta \Omega}{\delta \tau'}} = \frac{\dfrac{\delta T}{\delta \upsilon'} + z'}{\dfrac{\delta \Omega}{\delta \upsilon'}}, \tag{O′}$$

together with this equation,

$$\frac{\delta T}{\delta \chi} + \frac{\delta V}{\delta \chi} = \lambda \frac{\delta \Omega}{\delta \chi} + \lambda' \frac{\delta \Omega'}{\delta \chi}, \tag{P′}$$

in which λ is the common value of the three first equated quantities in (M′), and λ' is the common value of the three equated quantities in (O′). And in this case, by means of the two equations (O′), and the two that remain of (M′), combined with the two relations (K′) (N′), we can eliminate σ, τ, υ, σ', τ', υ', from (F′), and so deduce V from T: while, in the same case, or even if the initial medium alone be uniform, we are to deduce W from T, by eliminating σ', τ', υ', between the equations (E′) (N′) (O′).

When all the media of the combination are not only uniform, but bounded by plane surfaces, which happens in investigations respecting prisms, ordinary or extraordinary, then of the seven quantities σ, τ, υ, σ', τ', υ', χ, only three are independent; two other relations existing besides (K′) and (N′), which may be thus denoted,

$$\left.\begin{aligned} 0 &= \Omega''(\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi), \\ 0 &= \Omega'''(\sigma, \tau, \upsilon, \sigma', \tau', \upsilon', \chi); \end{aligned}\right\} \tag{Q′}$$

because, in this case, the initial direction, and the colour, determine the final direction. In this case, we may still treat the variations of σ, τ, υ, σ', τ', υ', χ, as independent, in δT, by introducing the variations of the four conditions (K′) (N′) (Q′), multiplied by factors λ, λ', λ'', λ''', that is by putting

$$\delta T = x\,\delta\sigma - x'\,\delta\sigma' + y\,\delta\tau - y'\,\delta\tau' + z\,\delta\upsilon - z'\,\delta\upsilon' - \frac{\delta V}{\delta \chi}\,\delta\chi + \lambda\,\delta\Omega + \lambda'\,\delta\Omega' + \lambda''\,\delta\Omega'' + \lambda'''\,\delta\Omega''' : \tag{R′}$$

an equation which decomposes itself into the seven following,

$$\left.\begin{aligned}
\frac{\delta T}{\delta \sigma} - x &= \lambda \frac{\delta \Omega}{\delta \sigma} + \lambda'' \frac{\delta \Omega''}{\delta \sigma} + \lambda''' \frac{\delta \Omega'''}{\delta \sigma}, \\
\frac{\delta T}{\delta \tau} - y &= \lambda \frac{\delta \Omega}{\delta \tau} + \lambda'' \frac{\delta \Omega''}{\delta \tau} + \lambda''' \frac{\delta \Omega'''}{\delta \tau}, \\
\frac{\delta T}{\delta \upsilon} - z &= \lambda \frac{\delta \Omega}{\delta \upsilon} + \lambda'' \frac{\delta \Omega''}{\delta \upsilon} + \lambda''' \frac{\delta \Omega'''}{\delta \upsilon}, \\
\frac{\delta T}{\delta \sigma'} + x' &= \lambda' \frac{\delta \Omega'}{\delta \sigma'} + \lambda'' \frac{\delta \Omega''}{\delta \sigma'} + \lambda''' \frac{\delta \Omega'''}{\delta \sigma'}, \\
\frac{\delta T}{\delta \tau'} + y' &= \lambda' \frac{\delta \Omega'}{\delta \tau'} + \lambda'' \frac{\delta \Omega''}{\delta \tau'} + \lambda''' \frac{\delta \Omega'''}{\delta \tau'}, \\
\frac{\delta T}{\delta \upsilon'} + z' &= \lambda' \frac{\delta \Omega'}{\delta \upsilon'} + \lambda'' \frac{\delta \Omega''}{\delta \upsilon'} + \lambda''' \frac{\delta \Omega'''}{\delta \upsilon'}, \\
\frac{\delta T}{\delta \chi} + \frac{\delta V}{\delta \chi} &= \lambda \frac{\delta \Omega}{\delta \chi} + \lambda' \frac{\delta \Omega'}{\delta \chi} + \lambda'' \frac{\delta \Omega''}{\delta \chi} + \lambda''' \frac{\delta \Omega'''}{\delta \chi},
\end{aligned}\right\} \qquad (\mathrm{S}')$$

between the six first of which, and the five equations marked (F′) (K′) (N′) (Q′), we can eliminate the ten quantities σ, τ, υ, σ', τ', υ', λ, λ', λ'', λ''', and thus deduce the relation between V, x, y, z, x', y', z', χ, from that between T, σ, τ, υ, σ', τ', υ', χ. It is easy to extend this method to other cases, in which there exists a mutual dependence, expressed by any number of equations, between the seven quantities σ, τ, υ, σ', τ', υ', χ.

And all the foregoing details respecting the mutual deductions of the function V, W, T, may be summed up in this one rule or theorem: that each of these three functions may be deduced from either of the other two, by using one of the three equations (D′) (E′) (F′) and by making the sought function a maximum or minimum with respect to the variables that are to be eliminated. For example we may deduce T from V, by making the expression (F′) a maximum or minimum with respect to the initial and final co-ordinates.

An optical combination is more perfectly characterised by the original function V, than by either of the two connected auxiliary functions W, T; because V enables us to determine the properties of the extreme media, which W and T do not; but there is an advantage in using these latter functions when the extreme media are uniform and known, because the known relations which in this case exist, of the forms (K′) and (N′), (together with the other relations (Q′) which arise when the combination is prismatic,) leave fewer independent variables in the auxiliary than in the original function. At the same time, as has been already remarked, and will be afterwards more fully shown, the existence of relations between the variables produces a partial indeterminateness in the forms of the auxiliary functions, from which the characteristic function V is free, but which is rather advantageous than the contrary, because it permits us to introduce suppositions and transformations, that contribute to elegance or simplicity.

General Transformations, by the Auxiliary Functions W, T, of the Partial Differential Equations in V. Other Partial Differential Equations in V, for Extreme Uniform Media. Integration of these Equations, by the Functions W, T.

5. Another advantage of the auxiliary functions W, T, is that they serve to transform,

and in the case of extreme uniform media to integrate, the partial differential equations (C), which the characteristic function V must satisfy. In fact, if the final medium be variable, the first of the two partial differential equations (C) may be put by the foregoing number under the two following forms,

$$\left.\begin{aligned} 0 &= \Omega\left(\sigma, \tau, \upsilon, \frac{\delta W}{\delta \sigma}, \frac{\delta W}{\delta \tau}, \frac{\delta W}{\delta \upsilon}, \chi\right), \\ 0 &= \Omega\left(\sigma, \tau, \upsilon, \frac{\delta T}{\delta \sigma}, \frac{\delta T}{\delta \tau}, \frac{\delta T}{\delta \upsilon}, \chi\right); \end{aligned}\right\} \tag{T$'$}$$

and if the initial medium be variable, the second of the two partial differential equations (C) may be put under these two forms,

$$\left.\begin{aligned} 0 &= \Omega'\left(\frac{\delta W}{\delta x'}, \frac{\delta W}{\delta y'}, \frac{\delta W}{\delta z'}, x', y', z', \chi\right), \\ 0 &= \Omega'\left(\sigma', \tau', \upsilon', -\frac{\delta T}{\delta \sigma'}, -\frac{\delta T}{\delta \tau'}, -\frac{\delta T}{\delta \upsilon'}, \chi\right); \end{aligned}\right\} \tag{U$'$}$$

of which indeed the first is general. But if the final medium be uniform, then W remains an arbitrary function of the four variables σ, τ, υ, χ, which are in this case connected with each other by the relation (K$'$); and the two equations (D$'$) (K$'$), together with the two first of those marked (I$'$), compose a system, which is a form for the integral of the partial differential equation

$$0 = \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, \chi\right), \tag{V$'$}$$

to which the first equation (C) in this case reduces itself. In like manner, if both the extreme media be uniform, in which case the second equation (C) reduces itself to the form

$$0 = \Omega'\left(-\frac{\delta V}{\delta x'}, -\frac{\delta V}{\delta y'}, -\frac{\delta V}{\delta z'}, \chi\right), \tag{W$'$}$$

the system of partial differential equations (V$'$) (W$'$) has for integral the system composed of the equations (F$'$) (K$'$) (N$'$) (O$'$), and the two first equations (M$'$), in which T is considered an arbitrary function of σ, τ, υ, σ', τ', υ', χ. It will be found that these integrals are extensively useful, in the study of optical combinations.

The two partial differential equations, (V$'$) (W$'$), of the first order, are themselves integrals of the two following, of the second order,

$$\begin{aligned} &\frac{\delta^2 V}{\delta x^2}\frac{\delta^2 V}{\delta y^2}\frac{\delta^2 V}{\delta z^2} + 2\frac{\delta^2 V}{\delta x\,\delta y}\frac{\delta^2 V}{\delta y\,\delta z}\frac{\delta^2 V}{\delta z\,\delta x} \\ &\qquad = \frac{\delta^2 V}{\delta x^2}\left(\frac{\delta^2 V}{\delta y\,\delta z}\right)^2 + \frac{\delta^2 V}{\delta y^2}\left(\frac{\delta^2 V}{\delta z\,\delta x}\right)^2 + \frac{\delta^2 V}{\delta z^2}\left(\frac{\delta^2 V}{\delta x\,\delta y}\right)^2, \end{aligned} \tag{X$'$}$$

and

$$\frac{\delta^2 V}{\delta x'^2}\frac{\delta^2 V}{\delta y'^2}\frac{\delta^2 V}{\delta z'^2} + 2\frac{\delta^2 V}{\delta x'\,\delta y'}\frac{\delta^2 V}{\delta y'\,\delta z'}\frac{\delta^2 V}{\delta z'\,\delta x'}$$
$$= \frac{\delta^2 V}{\delta x'^2}\left(\frac{\delta^2 V}{\delta y'\,\delta z'}\right)^2 + \frac{\delta^2 V}{\delta y'^2}\left(\frac{\delta^2 V}{\delta z'\,\delta x'}\right)^2 + \frac{\delta^2 V}{\delta z'^2}\left(\frac{\delta^2 V}{\delta x'\,\delta y'}\right)^2, \qquad (\mathrm{Y}')$$

which are obtained by elimination from (Q) and (X), after making

$$\left.\begin{array}{lll} \dfrac{\delta v}{\delta x} = 0, & \dfrac{\delta v}{\delta y} = 0, & \dfrac{\delta v}{\delta z} = 0; \\ \dfrac{\delta v'}{\delta x'} = 0, & \dfrac{\delta v'}{\delta y'} = 0, & \dfrac{\delta v'}{\delta z'} = 0. \end{array}\right\} \qquad (\mathrm{Z}')$$

The system of the three first of these six equations (Z′), or the partial differential equation of the second order (X′), or its integral of the first order (V′), expresses that the final medium is uniform; and the uniformity of the initial medium is, in like manner, expressed by the three last equations (Z′), or by the partial differential equation (Y′), or by its integral of the first order (W′). The integral systems of equations, also, which we have already assigned, express properties peculiar to optical combinations that have one or both of the extreme media uniform.

The first equation (U′) has for transformation the second equation (U′), when the initial medium is variable; and it has for integral, when the initial medium is uniform, the system (E′) (N′) (O′), by which, in that case, W is deduced from the arbitrary function T: while, in the same case, of an initial uniform medium, the first equation (U′) becomes of the form

$$0 = \Omega'\left(\frac{\delta W}{\delta x'}, \frac{\delta W}{\delta y'}, \frac{\delta W}{\delta z'}, \chi\right), \qquad (\mathrm{A}^2)$$

and is an integral of the following equation of the second order, analogous to (Y′),

$$\frac{\delta^2 W}{\delta x'^2}\frac{\delta^2 W}{\delta y'^2}\frac{\delta^2 W}{\delta z'^2} + 2\frac{\delta^2 W}{\delta x'\,\delta y'}\frac{\delta^2 W}{\delta y'\,\delta z'}\frac{\delta^2 W}{\delta z'\,\delta x'}$$
$$= \frac{\delta^2 W}{\delta x'^2}\left(\frac{\delta^2 W}{\delta y'\,\delta z'}\right)^2 + \frac{\delta^2 W}{\delta y'^2}\left(\frac{\delta^2 W}{\delta z'\,\delta x'}\right)^2 + \frac{\delta^2 W}{\delta z'^2}\left(\frac{\delta^2 W}{\delta x'\,\delta y'}\right)^2. \qquad (\mathrm{B}^2)$$

When the final medium is variable, the function W satisfies the following partial differential equation, analogous to the general equation (D),

$$\frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 W}{\delta\upsilon\,\delta z'} + \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 W}{\delta\upsilon\,\delta x'} + \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 W}{\delta\upsilon\,\delta y'}$$
$$= \frac{\delta^2 W}{\delta\upsilon\,\delta x'}\frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 W}{\delta\sigma\,\delta z'} + \frac{\delta^2 W}{\delta\upsilon\,\delta y'}\frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 W}{\delta\sigma\,\delta x'} + \frac{\delta^2 W}{\delta\upsilon\,\delta z'}\frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 W}{\delta\sigma\,\delta y'}; \qquad (\mathrm{C}^2)$$

and when both the extreme media are variable, the function T satisfies the following analogous equation,

$$\begin{aligned}&\frac{\delta^2 T}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 T}{\delta\tau\,\delta\tau'}\frac{\delta^2 T}{\delta\upsilon\,\delta\upsilon'}+\frac{\delta^2 T}{\delta\sigma\,\delta\tau'}\frac{\delta^2 T}{\delta\tau\,\delta\upsilon'}\frac{\delta^2 T}{\delta\upsilon\,\delta\sigma'}+\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon'}\frac{\delta^2 T}{\delta\tau\,\delta\sigma'}\frac{\delta^2 T}{\delta\upsilon\,\delta\tau'}\\ &\quad=\frac{\delta^2 T}{\delta\upsilon\,\delta\sigma'}\frac{\delta^2 T}{\delta\tau\,\delta\tau'}\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon'}+\frac{\delta^2 T}{\delta\upsilon\,\delta\tau'}\frac{\delta^2 T}{\delta\tau\,\delta\upsilon'}\frac{\delta^2 T}{\delta\sigma\,\delta\sigma'}+\frac{\delta^2 T}{\delta\upsilon\,\delta\upsilon'}\frac{\delta^2 T}{\delta\tau\,\delta\sigma'}\frac{\delta^2 T}{\delta\sigma\,\delta\tau'}.\end{aligned}\tag{D^2}$$

General Deductions and Transformations of the Differential and Integral Equations of a Curved or Straight Ray, Ordinary or Extraordinary, by the Auxiliary Functions W, T.

6. The auxiliary functions W, T, give new equations for the initial and final portions of a curved or polygon ray. Thus the function W gives generally the following equations, between the final quantities σ, τ, υ, analogous to the equations (W),

$$\frac{\delta W}{\delta x'}=\text{const.},\qquad \frac{\delta W}{\delta y'}=\text{const.},\qquad \frac{\delta W}{\delta z'}=\text{const.},\tag{E^2}$$

in which x' y' z' are the co-ordinates of some fixed point on the initial portion, and the constants are, by (G′), the corresponding values of the initial quantities σ', τ', υ'. The equations (E^2) have for differentials the following,

$$\left.\begin{aligned}0&=\frac{\delta^2 W}{\delta\sigma\,\delta x'}\,d\sigma+\frac{\delta^2 W}{\delta\tau\,\delta x'}\,d\tau+\frac{\delta^2 W}{\delta\upsilon\,\delta x'}\,d\upsilon,\\ 0&=\frac{\delta^2 W}{\delta\sigma\,\delta y'}\,d\sigma+\frac{\delta^2 W}{\delta\tau\,\delta y'}\,d\tau+\frac{\delta^2 W}{\delta\upsilon\,\delta y'}\,d\upsilon,\\ 0&=\frac{\delta^2 W}{\delta\sigma\,\delta z'}\,d\sigma+\frac{\delta^2 W}{\delta\tau\,\delta z'}\,d\tau+\frac{\delta^2 W}{\delta\upsilon\,\delta z'}\,d\upsilon;\end{aligned}\right\}\tag{F^2}$$

d still referring to motion along a ray: and if we combine these with the following,

$$\left.\begin{aligned}0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\,\delta x'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\,\delta x'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\,\delta x'},\\ 0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\,\delta y'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\,\delta y'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\,\delta y'},\\ 0&=\frac{\delta v}{\delta x}\frac{\delta^2 W}{\delta\sigma\,\delta z'}+\frac{\delta v}{\delta y}\frac{\delta^2 W}{\delta\tau\,\delta z'}+\frac{\delta v}{\delta z}\frac{\delta^2 W}{\delta\upsilon\,\delta z'},\end{aligned}\right\}\tag{G^2}$$

which are obtained by differentiating the first equation (T′) relatively to the initial co-ordinates x' y' z', and by attending to the relations (K), we see that for a curved ray the differentials $d\sigma$, $d\tau$, $d\upsilon$, are proportional to

$$\frac{\delta v}{\delta x},\quad \frac{\delta v}{\delta y},\quad \frac{\delta v}{\delta z};$$

and from this proportionality, combined with the relation

$$\alpha\,\delta\sigma + \beta\,\delta\tau + \gamma\,\delta\upsilon = \left(\alpha\frac{\delta\upsilon}{\delta x} + \beta\frac{\delta\upsilon}{\delta y} + \gamma\frac{\delta\upsilon}{\delta z}\right) ds, \tag{H^2}$$

which results from (H) and (P), we can easily infer the equations (O): these differential equations (O) for the final portion of a curved ray, which can be extended to the initial portion by merely accenting the symbols, may therefore be deduced from the consideration of the auxiliary function W. The equations (O) for a curved ray, may also be deduced from the function W, by combining the differentials d of the three first equations (H′), with the partial differentials of the first equation (T′), taken with respect to σ, τ, υ.

The same auxiliary function W gives for the final straight portion of a polygon ray, the two first equations (I′), which may be thus written,

$$\frac{1}{\alpha}\left(x - \frac{\delta W}{\delta\sigma}\right) = \frac{1}{\beta}\left(y - \frac{\delta W}{\delta\tau}\right) = \frac{1}{\gamma}\left(z - \frac{\delta W}{\delta\upsilon}\right): \tag{I^2}$$

these equations may also be put under the form

$$\left.\begin{aligned} x\frac{\delta\sigma}{\delta\theta} + y\frac{\delta\tau}{\delta\theta} + z\frac{\delta\upsilon}{\delta\theta} &= \frac{\delta W}{\delta\theta},\\ x\frac{\delta\sigma}{\delta\phi} + y\frac{\delta\tau}{\delta\phi} + z\frac{\delta\upsilon}{\delta\phi} &= \frac{\delta W}{\delta\phi}, \end{aligned}\right\} \tag{K^2}$$

if in virtue of (K′), we consider σ, τ, υ, as functions, each, of χ, and of two other independent variables denoted by θ, ϕ, and consider W as a function of the six independent variables θ, ϕ, χ, x', y', z'. We may choose σ, τ, for the independent variables θ, ϕ, considering υ as, by (K′), a function of σ, τ, χ, such that by (H),

$$\frac{\delta\upsilon}{\delta\sigma} = -\frac{\alpha}{\gamma}, \quad \frac{\delta\upsilon}{\delta\tau} = -\frac{\beta}{\gamma}, \quad \frac{\delta\upsilon}{\delta\chi} = \frac{1}{\gamma}\frac{\delta\upsilon}{\delta\chi}, \tag{L^2}$$

and considering W as a function of the six independent variables σ, τ, χ, x', y', z'; and then the equations (I^2) or (K^2), for the final straight portion of a polygon ray, ordinary or extraordinary, will take these simpler forms, which we shall have frequent occasion to employ,

$$x - \frac{\alpha}{\gamma}z = \frac{\delta W}{\delta\sigma}; \quad y - \frac{\beta}{\gamma}z = \frac{\delta W}{\delta\tau}. \tag{M^2}$$

The other auxiliary function, T, gives the following equations between σ, τ, υ, for the final portion, straight or curved, when the initial medium is variable,

$$\frac{\delta T}{\delta\sigma'} = \text{const.}, \quad \frac{\delta T}{\delta\tau'} = \text{const.}, \quad \frac{\delta T}{\delta\upsilon'} = \text{const.}, \tag{N^2}$$

in which σ', τ', υ', belong to some point on the initial portion, and in which the constants are, by (L′), the negatives of the co-ordinates of that point; it gives, in like manner, for the initial portion, when the final medium is variable, the following equations between σ', τ', υ',

$$\frac{\delta T}{\delta\sigma} = \text{const.}, \quad \frac{\delta T}{\delta\tau} = \text{const.}, \quad \frac{\delta T}{\delta\upsilon} = \text{const.}, \tag{O^2}$$

σ, τ, υ, belonging to some point upon the final portion, and the constants being the co-ordinates of that point: and from these equations we might deduce the differential equations (O), by processes analogous to those already mentioned. When both the extreme media are uniform, and therefore both the extreme portions straight, we have, for those straight portions, the following equations, deduced from (M′) (O′) (I),

$$\left.\begin{aligned} \frac{1}{\alpha}\left(x - \frac{\delta T}{\delta\sigma}\right) = \frac{1}{\beta}\left(y - \frac{\delta T}{\delta\tau}\right) = \frac{1}{\gamma}\left(z - \frac{\delta T}{\delta\upsilon}\right), \\ \frac{1}{\alpha'}\left(x' + \frac{\delta T}{\delta\sigma'}\right) = \frac{1}{\beta'}\left(y' + \frac{\delta T}{\delta\tau'}\right) = \frac{1}{\gamma'}\left(z' + \frac{\delta T}{\delta\upsilon'}\right); \end{aligned}\right\} \tag{P^2}$$

which may be thus transformed,

$$\left.\begin{aligned} 0 &= x\frac{\delta\sigma}{\delta\theta} + y\frac{\delta\tau}{\delta\theta} + z\frac{\delta\upsilon}{\delta\theta} - \frac{\delta T}{\delta\theta}, \\ 0 &= x\frac{\delta\sigma}{\delta\phi} + y\frac{\delta\tau}{\delta\phi} + z\frac{\delta\upsilon}{\delta\phi} - \frac{\delta T}{\delta\phi}, \\ 0 &= x'\frac{\delta\sigma'}{\delta\theta'} + y'\frac{\delta\tau'}{\delta\theta'} + z'\frac{\delta\upsilon'}{\delta\theta'} + \frac{\delta T}{\delta\theta'}, \\ 0 &= x'\frac{\delta\sigma'}{\delta\phi'} + y'\frac{\delta\tau'}{\delta\phi'} + z'\frac{\delta\upsilon'}{\delta\phi'} + \frac{\delta T}{\delta\phi'}, \end{aligned}\right\} \tag{Q^2}$$

if, as before, by virtue of (K′), we consider σ, τ, υ, as functions, each, of χ and of two other independent variables θ, ϕ, considering similarly σ', τ', υ', as functions, each, by (N′), of three independent variables θ', ϕ', χ; and T as a function of the five independent variables θ, ϕ, θ', ϕ', χ. If we choose the independent variables θ, ϕ, so as to coincide with σ, τ, and if in like manner we take σ', τ', for the independent variables θ', ϕ', making, by (H),

$$\frac{\delta\upsilon'}{\delta\sigma'} = -\frac{\alpha'}{\gamma'}, \quad \frac{\delta\upsilon'}{\delta\tau'} = -\frac{\beta'}{\gamma'}, \quad \frac{\delta\upsilon'}{\delta\chi} = \frac{1}{\gamma'}\frac{\delta\upsilon'}{\delta\chi}, \tag{R^2}$$

and considering T as a function of the five independent variables σ, τ, σ', τ', χ, we have the following transformed equations for the extreme straight portions of a polygon ray, ordinary or extraordinary,

$$\left.\begin{aligned} 0 = x - \frac{\alpha}{\gamma}z - \frac{\delta T}{\delta\sigma}; \quad & 0 = y - \frac{\beta}{\gamma}z - \frac{\delta T}{\delta\tau}; \\ 0 = x' - \frac{\alpha'}{\gamma'}z' + \frac{\delta T}{\delta\sigma'}; \quad & 0 = y' - \frac{\beta'}{\gamma'}z' + \frac{\delta T}{\delta\tau'}: \end{aligned}\right\} \tag{S^2}$$

which are analogous to the equations (M^2) and, like them, will often be found useful.

It may be remarked here, that from the differential equations (O) of a curved ray, ordinary or extraordinary, to which, in the present and former numbers, we have been conducted by so many processes, the following may be deduced,

$$\left.\begin{aligned} 0 = \frac{d\sigma}{dV} + \frac{\delta\Omega}{\delta x}; \quad 0 = \frac{d\tau}{dV} + \frac{\delta\Omega}{\delta y}; \quad 0 = \frac{d\upsilon}{dV} + \frac{\delta\Omega}{\delta z}: \\ dW = dT = \left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} + z\frac{\delta v}{\delta z}\right) ds = x\,d\sigma + y\,d\tau + z\,d\upsilon. \end{aligned}\right\} \tag{T^2}$$

We may also remark, that when the final medium is uniform, and when therefore the quantities σ, τ, υ, χ, are connected by a relation (K′), the quantity

$$W(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{n}{2}}$$

may, in general, by means of this relation, be expressed as a function of

$$\frac{\sigma}{\upsilon}, \frac{\tau}{\upsilon}, x', y', z', \chi,$$

and that $T(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{n}{2}}$ may, in like manner, be expressed as a function of

$$\frac{\sigma}{\upsilon}, \frac{\tau}{\upsilon}, \sigma', \tau', \upsilon', \chi;$$

and that therefore W, T, may both be made homogeneous functions, of any assumed dimension n, relatively to σ, τ, υ, so as to satisfy the following conditions

$$\left.\begin{aligned} \sigma\frac{\delta W}{\delta\sigma} + \tau\frac{\delta W}{\delta\tau} + \upsilon\frac{\delta W}{\delta\upsilon} &= nW, \\ \sigma\frac{\delta T}{\delta\sigma} + \tau\frac{\delta T}{\delta\tau} + \upsilon\frac{\delta T}{\delta\upsilon} &= nT. \end{aligned}\right\} \tag{U^2}$$

With this preparation, the two first equations (I′), and the two first equations (M′), which belong to the straight final portion of the ray, may be transformed by (L) to the following,

$$\left.\begin{aligned} x - \frac{\delta\Omega}{\delta\sigma}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\sigma} - nW\frac{\delta\Omega}{\delta\sigma} = \frac{\delta T}{\delta\sigma} - nT\frac{\delta\Omega}{\delta\sigma}, \\ y - \frac{\delta\Omega}{\delta\tau}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\tau} - nW\frac{\delta\Omega}{\delta\tau} = \frac{\delta T}{\delta\tau} - nT\frac{\delta\Omega}{\delta\tau}, \\ z - \frac{\delta\Omega}{\delta\upsilon}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta\upsilon} - nW\frac{\delta\Omega}{\delta\upsilon} = \frac{\delta T}{\delta\upsilon} - nT\frac{\delta\Omega}{\delta\upsilon}. \end{aligned}\right\} \tag{V^2}$$

If then we make $n = 1$, that is if we make W homogeneous of the first dimension relatively to σ, τ, υ, and if we attend to the relation (D′), we see that the equations of this straight final portion may be thus written,

$$x = \frac{\delta W}{\delta\sigma} + V\frac{\delta\Omega}{\delta\sigma}, \quad y = \frac{\delta W}{\delta\tau} + V\frac{\delta\Omega}{\delta\tau}, \quad z = \frac{\delta W}{\delta\upsilon} + V\frac{\delta\Omega}{\delta\upsilon}, \tag{W^2}$$

of which any two include the third, and which we shall often hereafter employ, on account of their symmetry.

In like manner, when the initial medium is uniform, and therefore the initial portion straight, the equations (O′) of this straight portion may be put under the form,

$$\left.\begin{aligned} x' - \frac{\delta\Omega'}{\delta\sigma'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\sigma'} + n'T\frac{\delta\Omega}{\delta\sigma'}, \\ y' - \frac{\delta\Omega'}{\delta\tau'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\tau'} + n'T\frac{\delta\Omega}{\delta\tau'}, \\ z' - \frac{\delta\Omega'}{\delta\upsilon'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta\upsilon'} + n'T\frac{\delta\Omega}{\delta\upsilon'}, \end{aligned}\right\} \tag{X^2}$$

by making T homogeneous of dimension n' relatively to σ', τ', υ', so as to have

$$\sigma' \frac{\delta T}{\delta \sigma'} + \tau' \frac{\delta T}{\delta \tau'} + \upsilon' \frac{\delta T}{\delta \upsilon'} = n'T. \tag{Y^2}$$

If both the extreme media be uniform, and if we make $n = 0$, $n' = 0$, that is, if we express W as a function of

$$\frac{\sigma}{\upsilon}, \frac{\tau}{\upsilon}, x', y', z', \chi,$$

and T as a function of

$$\frac{\sigma}{\upsilon}, \frac{\tau}{\upsilon}, \frac{\sigma'}{\upsilon'}, \frac{\tau'}{\upsilon'}, \chi,$$

we find the following form for the equations of the extreme straight portions of a polygon ray, ordinary or extraordinary, less simple than (S^2), but more symmetric,

$$\left.\begin{aligned}
x - \frac{\delta \Omega}{\delta \sigma}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta \sigma} = \frac{\delta T}{\delta \sigma}, \\
y - \frac{\delta \Omega}{\delta \tau}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta \tau} = \frac{\delta T}{\delta \tau}, \\
z - \frac{\delta \Omega}{\delta \upsilon}(\sigma x + \tau y + \upsilon z) &= \frac{\delta W}{\delta \upsilon} = \frac{\delta T}{\delta \upsilon}, \\
x' - \frac{\delta \Omega'}{\delta \sigma'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta \sigma'}, \\
y' - \frac{\delta \Omega'}{\delta \tau'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta \tau'}, \\
z' - \frac{\delta \Omega'}{\delta \upsilon'}(\sigma' x' + \tau' y' + \upsilon' z') &= -\frac{\delta T}{\delta \upsilon'}.
\end{aligned}\right\} \tag{Z^2}$$

The case of prismatic combinations may be treated as in the fourth number.

General Remarks on the Connexions between the Partial Differential Coefficients of the Second Order of the Functions V, W, T. General Method of investigating those Connexions. Deductions of the Coefficients of V from those of W, when the Final Medium is uniform.

7. It is easy to see, from the manner in which the equations of a ray involve the partial differential coefficients of the first order, of the functions V, W, T, that the partial differential coefficients of the second order, of the same three functions, must present themselves in investigations respecting the geometrical relations between infinitely near rays of a system; and that therefore it must be useful to know the general connexions between these coefficients of the second order. Connexions of this kind, between the coefficients of the second order of the characteristic function V, taken with respect to the final co-ordinates, and those of the auxiliary function W, considered as belonging to a final system of straight rays of a given colour, which issued originally from a given luminous point, were investigated in the First Supplement; but these connexions will now be considered in a more general manner, and will be extended to the new auxiliary function T, which was not introduced before: the new

investigations will differ also from the former, by making W depend on the quantities σ, τ, υ, rather than on α, β, γ.

The general problem of investigating these connexions may be decomposed into many particular problems, according to the way in which we pair the functions, and according as we suppose the extreme media to be uniform or variable; but all these particular problems may be resolved by attending to the following general principle, that the connexions between the partial differential coefficients of the three functions, whether of the second or of higher orders, are to be obtained by differentiating and comparing the equations which connect the three functions themselves: that is, by differentiating and comparing the three forms (A′) (B′) (C') of the fundamental equation (A), and the equations into which these forms (A′) (B′) (C′) resolve themselves.

Thus to deduce the twenty-eight partial differential coefficients of the second order, of the characteristic function V, taken with respect to the extreme co-ordinates and the colour, from the coefficients of the same order of the auxiliary function W, or T, we are to differentiate the equations into which (B′) or (C') resolves itself, together with the relations between the variables on which W or T depends, if any such relations exist; and then by elimination to deduce the variations of the first order of the seven coefficients of the variation (A′) as linear functions of the seven variations of the first order of the extreme co-ordinates and the colour: these seven linear functions will have forty-nine coefficients, of which, however, only twenty-eight will be distinct, and these will be the coefficients sought.

More particularly, if the final medium be variable, and if it be required to deduce the coefficients of the second order of V from those of W, we first obtain expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, as linear functions of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, from the differentials of the three first equations (H′), deduced from (B′), expressions which will necessarily satisfy the first condition (H); we then substitute these expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, in the differentials of the three equations (G′), deduced from (B′), so as to get analogous expressions for $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, which must satisfy the second condition (H); and substituting the same expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, in the differential of the last equation (H′), also deduced from (B′), we get an expression of the same kind for $\delta\dfrac{\delta V}{\delta\chi}$: after which, we have only to compare the expressions so obtained, with the following, that is, with the differentials of the equations into which the formula (A′) resolves itself,

$$\left.\begin{aligned}
\delta\sigma &= \frac{\delta^2 V}{\delta x^2}\,\delta x + \frac{\delta^2 V}{\delta x\,\delta y}\,\delta y + \frac{\delta^2 V}{\delta x\,\delta z}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta x\,\delta z'}\,\delta z' + \frac{\delta^2 V}{\delta x\,\delta \chi}\,\delta \chi, \\
\delta\tau &= \frac{\delta^2 V}{\delta x\,\delta y}\,\delta x + \frac{\delta^2 V}{\delta y^2}\,\delta y + \frac{\delta^2 V}{\delta y\,\delta z}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta y\,\delta z'}\,\delta z' + \frac{\delta^2 V}{\delta y\,\delta \chi}\,\delta \chi, \\
\delta\upsilon &= \frac{\delta^2 V}{\delta x\,\delta z}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta z}\,\delta y + \frac{\delta^2 V}{\delta z^2}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta z\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta z\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta z\,\delta z'}\,\delta z' + \frac{\delta^2 V}{\delta z\,\delta \chi}\,\delta \chi, \\
-\delta\sigma' &= \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta y + \frac{\delta^2 V}{\delta z\,\delta x'}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta x'^2}\,\delta x' + \frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta x'\,\delta z'}\,\delta z' + \frac{\delta^2 V}{\delta x'\,\delta \chi}\,\delta \chi, \\
-\delta\tau' &= \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y + \frac{\delta^2 V}{\delta z\,\delta y'}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta x' + \frac{\delta^2 V}{\delta y'^2}\,\delta y' + \frac{\delta^2 V}{\delta y'\,\delta z'}\,\delta z' + \frac{\delta^2 V}{\delta y'\,\delta \chi}\,\delta \chi, \\
-\delta\upsilon' &= \frac{\delta^2 V}{\delta x\,\delta z'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta z'}\,\delta y + \frac{\delta^2 V}{\delta z\,\delta z'}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta x'\,\delta z'}\,\delta x' + \frac{\delta^2 V}{\delta y'\,\delta z'}\,\delta y' + \frac{\delta^2 V}{\delta z'^2}\,\delta z' + \frac{\delta^2 V}{\delta z'\,\delta \chi}\,\delta \chi, \\
\delta\frac{\delta V}{\delta \chi} &= \frac{\delta^2 V}{\delta x\,\delta \chi}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta \chi}\,\delta y + \frac{\delta^2 V}{\delta z\,\delta \chi}\,\delta z \\
&\quad + \frac{\delta^2 V}{\delta x'\,\delta \chi}\,\delta x' + \frac{\delta^2 V}{\delta y'\,\delta \chi}\,\delta y' + \frac{\delta^2 V}{\delta z'\,\delta \chi}\,\delta z' + \frac{\delta^2 V}{\delta \chi^2}\,\delta \chi.
\end{aligned}\right\} \tag{A^3}$$

But if the final medium be uniform, then σ, τ, υ, χ, are not independent, but related by (K′); and the formula (B′) resolves itself, not into the seven equations (G′) and (H′) but into the six equations (G′) and (I′), the differentials of which are to be combined with the differential of the relation (K′), so as to give the expressions for $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, $\delta\dfrac{\delta V}{\delta\chi}$, which are to be compared with (A^3) as before. And in this case, of a final uniform medium, we may employ, instead of the two first equations (I′), any of the transformations of those equations in the foregoing number; or we may employ the following transformations of (I′),

$$x + z\frac{\delta\upsilon}{\delta\sigma} = \frac{\delta W}{\delta\sigma}; \quad y + z\frac{\delta\upsilon}{\delta\tau} = \frac{\delta W}{\delta\tau}; \quad z\frac{\delta\upsilon}{\delta\chi} = \frac{\delta V}{\delta\chi} + \frac{\delta W}{\delta\chi}: \tag{B^3}$$

in which, W is considered as a function of the six independent variables σ, τ, χ, x', y', z', obtained by substituting for υ its value as a function of σ, τ, χ; the form of which function υ depends on and characterises the properties of the final medium, and is deduced from the relation (K′). It may be useful here to go through the process last indicated, both to explain its nature more fully, and to have its results ready for future researches.

Differentiating therefore the first two equations (B³), we obtain

$$\left.\begin{aligned}\delta x + \frac{\delta\upsilon}{\delta\sigma}\,\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\chi}\,\delta\chi &= \left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right)\delta\tau,\\ \delta y + \frac{\delta\upsilon}{\delta\tau}\,\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\,\delta\chi}\,\delta\chi &= \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right)\delta\tau,\end{aligned}\right\}\tag{C^3}$$

in which we have put for abridgment

$$\left.\begin{aligned}\delta'\frac{\delta W}{\delta\sigma} &= \frac{\delta^2 W}{\delta\sigma\,\delta x'}\,\delta x' + \frac{\delta^2 W}{\delta\sigma\,\delta y'}\,\delta y' + \frac{\delta^2 W}{\delta\sigma\,\delta z'}\,\delta z' + \frac{\delta^2 W}{\delta\sigma\,\delta\chi}\,\delta\chi,\\ \delta'\frac{\delta W}{\delta\tau} &= \frac{\delta^2 W}{\delta\tau\,\delta x'}\,\delta x' + \frac{\delta^2 W}{\delta\tau\,\delta y'}\,\delta y' + \frac{\delta^2 W}{\delta\tau\,\delta z'}\,\delta z' + \frac{\delta^2 W}{\delta\tau\,\delta\chi}\,\delta\chi,\end{aligned}\right\}\tag{D^3}$$

δ' referring only to the variations of the initial co-ordinates and of the colour: and if we put

$$w'' = \left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right)^2,\tag{E^3}$$

the equations (C³) give, by elimination,

$$\left.\begin{aligned}w''\,\delta\sigma = {}&\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right)\left(\delta x + \frac{\delta\upsilon}{\delta\sigma}\,\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\chi}\,\delta\chi\right)\\ &- \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right)\left(\delta y + \frac{\delta\upsilon}{\delta\tau}\,\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\,\delta\chi}\,\delta\chi\right),\\ w''\,\delta\tau = {}&\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right)\left(\delta y + \frac{\delta\upsilon}{\delta\tau}\,\delta z - \delta'\frac{\delta W}{\delta\tau} + z\frac{\delta^2\upsilon}{\delta\tau\,\delta\chi}\,\delta\chi\right)\\ &- \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right)\left(\delta x + \frac{\delta\upsilon}{\delta\sigma}\,\delta z - \delta'\frac{\delta W}{\delta\sigma} + z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\chi}\,\delta\chi\right);\end{aligned}\right\}\tag{F^3}$$

and hence by (A³) we can deduce already, without any farther differentiation,

$$\left.\begin{aligned}\frac{\delta^2 V}{\delta x^2} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\tau^2} - z\frac{\delta^2\upsilon}{\delta\tau^2}\right); & \frac{\delta^2 V}{\delta x\,\delta z} &= \frac{\delta\upsilon}{\delta\sigma}\frac{\delta^2 V}{\delta x^2} + \frac{\delta\upsilon}{\delta\tau}\frac{\delta^2 V}{\delta x\,\delta y};\\ \frac{\delta^2 V}{\delta x\,\delta y} &= -\frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} - z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau}\right); & \frac{\delta^2 V}{\delta y\,\delta z} &= \frac{\delta\upsilon}{\delta\sigma}\frac{\delta^2 V}{\delta x\,\delta y} + \frac{\delta\upsilon}{\delta\tau}\frac{\delta^2 V}{\delta y^2};\\ \frac{\delta^2 V}{\delta y^2} &= \frac{1}{w''}\left(\frac{\delta^2 W}{\delta\sigma^2} - z\frac{\delta^2\upsilon}{\delta\sigma^2}\right); & \frac{\delta^2 V}{\delta z^2} &= \frac{\delta\upsilon}{\delta\sigma}\frac{\delta^2 V}{\delta x\,\delta z} + \frac{\delta\upsilon}{\delta\tau}\frac{\delta^2 V}{\delta y\,\delta z};\end{aligned}\right\}\tag{G^3}$$

observing, in deducing the sixth of these equations (G^3), that by the definitions (E), and by the dependence of υ on σ, τ, χ, we have

$$\delta \frac{\delta V}{\delta z} = (\delta \upsilon =) \, \frac{\delta \upsilon}{\delta \sigma} \, \delta \frac{\delta V}{\delta x} + \frac{\delta \upsilon}{\delta \tau} \, \delta \frac{\delta V}{\delta y} + \frac{\delta \upsilon}{\delta \chi} \, \delta \chi. \tag{H^3}$$

The equations (A^3) (F^3) (H^3) give also

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x \, \delta x'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta x'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta x'} \left(\frac{\delta^2 W}{\delta \tau^2} - z \frac{\delta^2 \upsilon}{\delta \tau^2} \right); \\
\frac{\delta^2 V}{\delta x \, \delta y'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta y'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta y'} \left(\frac{\delta^2 W}{\delta \tau^2} - z \frac{\delta^2 \upsilon}{\delta \tau^2} \right); \\
\frac{\delta^2 V}{\delta x \, \delta z'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta z'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta z'} \left(\frac{\delta^2 W}{\delta \tau^2} - z \frac{\delta^2 \upsilon}{\delta \tau^2} \right); \\
\frac{\delta^2 V}{\delta y \, \delta x'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta x'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta x'} \left(\frac{\delta^2 W}{\delta \sigma^2} - z \frac{\delta^2 \upsilon}{\delta \sigma^2} \right); \\
\frac{\delta^2 V}{\delta y \, \delta y'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta y'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta y'} \left(\frac{\delta^2 W}{\delta \sigma^2} - z \frac{\delta^2 \upsilon}{\delta \sigma^2} \right); \\
\frac{\delta^2 V}{\delta y \, \delta z'} &= \frac{1}{w''} \frac{\delta^2 W}{\delta \sigma \, \delta z'} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) - \frac{1}{w''} \frac{\delta^2 W}{\delta \tau \, \delta z'} \left(\frac{\delta^2 W}{\delta \sigma^2} - z \frac{\delta^2 \upsilon}{\delta \sigma^2} \right); \\
\frac{\delta^2 V}{\delta z \, \delta x'} &= \frac{\delta \upsilon}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta x'} + \frac{\delta \upsilon}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta x'}; \\
\frac{\delta^2 V}{\delta z \, \delta y'} &= \frac{\delta \upsilon}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta y'} + \frac{\delta \upsilon}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta y'}; \\
\frac{\delta^2 V}{\delta z \, \delta z'} &= \frac{\delta \upsilon}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta z'} + \frac{\delta \upsilon}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta z'};
\end{aligned}\right\} \tag{I^3}$$

and

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x \, \delta \chi} &= \frac{1}{w''} \left(\frac{\delta^2 W}{\delta \tau \, \delta \chi} - z \frac{\delta^2 \upsilon}{\delta \tau \, \delta \chi} \right) \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) \\
&\quad - \frac{1}{w''} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \chi} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \chi} \right) \left(\frac{\delta^2 W}{\delta \tau^2} - z \frac{\delta^2 \upsilon}{\delta \tau^2} \right); \\
\frac{\delta^2 V}{\delta y \, \delta \chi} &= \frac{1}{w''} \left(\frac{\delta^2 W}{\delta \sigma \, \delta \chi} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \chi} \right) \left(\frac{\delta^2 W}{\delta \sigma \, \delta \tau} - z \frac{\delta^2 \upsilon}{\delta \sigma \, \delta \tau} \right) \\
&\quad - \frac{1}{w''} \left(\frac{\delta^2 W}{\delta \tau \, \delta \chi} - z \frac{\delta^2 \upsilon}{\delta \tau \, \delta \chi} \right) \left(\frac{\delta^2 W}{\delta \sigma^2} - z \frac{\delta^2 \upsilon}{\delta \sigma^2} \right); \\
\frac{\delta^2 V}{\delta z \, \delta \chi} &= \frac{\delta \upsilon}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta \chi} + \frac{\delta \upsilon}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta \chi} + \frac{\delta \upsilon}{\delta \chi}.
\end{aligned}\right\} \tag{K^3}$$

We have therefore found expressions (G^3) (I^3) (K^3), for eighteen out of the twenty-eight partial differential coefficients of V of the second order; and with respect to nine of the remaining ten, namely all except $\dfrac{\delta^2 V}{\delta\chi^2}$, we may obtain expressions for these by differentiating the three equations (G′), and comparing the differentials with (A³); for thus we find,

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'^2} &= -\frac{\delta^2 W}{\delta x'^2} - \frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta x'} - \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta x'};\\
\frac{\delta^2 V}{\delta y'^2} &= -\frac{\delta^2 W}{\delta y'^2} - \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta y'} - \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta y'};\\
\frac{\delta^2 V}{\delta z'^2} &= -\frac{\delta^2 W}{\delta z'^2} - \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta z'} - \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta z'};\\
\frac{\delta^2 V}{\delta x'\,\delta y'} &= -\frac{\delta^2 W}{\delta x'\,\delta y'} - \frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta y'} - \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta y'};\\
\frac{\delta^2 V}{\delta y'\,\delta z'} &= -\frac{\delta^2 W}{\delta y'\,\delta z'} - \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta z'} - \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta z'};\\
\frac{\delta^2 V}{\delta z'\,\delta x'} &= -\frac{\delta^2 W}{\delta z'\,\delta x'} - \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta x'} - \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta x'};
\end{aligned}\right\}\qquad (\mathrm{L}^3)$$

and

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'\,\delta\chi} &= -\frac{\delta^2 W}{\delta x'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta\chi} - \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta\chi};\\
\frac{\delta^2 V}{\delta y'\,\delta\chi} &= -\frac{\delta^2 W}{\delta y'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta\chi} - \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta\chi};\\
\frac{\delta^2 V}{\delta z'\,\delta\chi} &= -\frac{\delta^2 W}{\delta z'\delta\chi} - \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta\chi} - \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta\chi}:
\end{aligned}\right\}\qquad (\mathrm{M}^3)$$

the equations (G′) give also

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x'\,\delta y'} &= -\frac{\delta^2 W}{\delta x'\,\delta y'} - \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta x'} - \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta x'};\\
\frac{\delta^2 V}{\delta y'\,\delta z'} &= -\frac{\delta^2 W}{\delta y'\,\delta z'} - \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta y'} - \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta y'};\\
\frac{\delta^2 V}{\delta z'\,\delta x'} &= -\frac{\delta^2 W}{\delta z'\,\delta x'} - \frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta z'} - \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta z'};
\end{aligned}\right\}\qquad (\mathrm{N}^3)$$

but these three expressions (N³) agree with the corresponding expressions (L³), because, by (I³),

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta y'} + \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta y'} &= \frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta x'} + \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta x'};\\
\frac{\delta^2 W}{\delta\sigma\,\delta y'}\frac{\delta^2 V}{\delta x\,\delta z'} + \frac{\delta^2 W}{\delta\tau\,\delta y'}\frac{\delta^2 V}{\delta y\,\delta z'} &= \frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta y'} + \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta y'};\\
\frac{\delta^2 W}{\delta\sigma\,\delta z'}\frac{\delta^2 V}{\delta x\,\delta x'} + \frac{\delta^2 W}{\delta\tau\,\delta z'}\frac{\delta^2 V}{\delta y\,\delta x'} &= \frac{\delta^2 W}{\delta\sigma\,\delta x'}\frac{\delta^2 V}{\delta x\,\delta z'} + \frac{\delta^2 W}{\delta\tau\,\delta x'}\frac{\delta^2 V}{\delta y\,\delta z'}.
\end{aligned}\right\}\qquad (\mathrm{O}^3)$$

Finally, with respect to the twenty-eighth coefficient $\dfrac{\delta^2 V}{\delta\chi^2}$, this may be obtained by differentiating the third equation (B^3), which gives

$$\frac{\delta^2 V}{\delta\chi^2} = z\frac{\delta^2 \upsilon}{\delta\chi^2} - \frac{\delta^2 W}{\delta\chi^2} + \left(z\frac{\delta^2\upsilon}{\delta\sigma\,\delta\chi} - \frac{\delta^2 W}{\delta\sigma\,\delta\chi}\right)\frac{\delta^2 V}{\delta x\,\delta\chi} + \left(z\frac{\delta^2\upsilon}{\delta\tau\,\delta\chi} - \frac{\delta^2 W}{\delta\tau\,\delta\chi}\right)\frac{\delta^2 V}{\delta y\,\delta\chi}. \tag{P^3}$$

And if we would generalise the twenty-eight expressions (G^3) (I^3) (K^3) (L^3) (M^3) (P^3), so as to render them independent of the particular supposition, that W has been made, by a previous elimination of υ, a function involving only the six independent variables σ, τ, χ, x', y', z', we may do so by suitably generalising fifteen out of the twenty-one coefficients of W, of the second order, which result from the foregoing suppositions; that is by leaving unchanged the six that are formed by differentiating only with respect to x', y', z', but changing $\dfrac{\delta^2 W}{\delta\sigma^2}$, &c., to the following more general expressions $\left[\dfrac{\delta^2 W}{\delta\sigma^2}\right]$, &c.;

$$\left.\begin{aligned}
\left[\frac{\delta^2 W}{\delta\sigma^2}\right] &= \frac{\delta^2 W}{\delta\sigma^2} + 2\frac{\delta^2 W}{\delta\sigma\,\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\sigma}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma^2};\\
\left[\frac{\delta^2 W}{\delta\tau^2}\right] &= \frac{\delta^2 W}{\delta\tau^2} + 2\frac{\delta^2 W}{\delta\tau\,\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\tau}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\tau^2};\\
\left[\frac{\delta^2 W}{\delta\chi^2}\right] &= \frac{\delta^2 W}{\delta\chi^2} + 2\frac{\delta^2 W}{\delta\chi\,\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta\upsilon}{\delta\chi}\right)^2 + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\chi^2};\\
\left[\frac{\delta^2 W}{\delta\sigma\,\delta\tau}\right] &= \frac{\delta^2 W}{\delta\sigma\,\delta\tau} + \frac{\delta^2 W}{\delta\tau\,\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\sigma\,\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\sigma}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\,\delta\chi}\right] &= \frac{\delta^2 W}{\delta\sigma\,\delta\chi} + \frac{\delta^2 W}{\delta\chi\,\delta\upsilon}\frac{\delta\upsilon}{\delta\sigma} + \frac{\delta^2 W}{\delta\sigma\,\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\sigma}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\sigma\,\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\tau\,\delta\chi}\right] &= \frac{\delta^2 W}{\delta\tau\,\delta\chi} + \frac{\delta^2 W}{\delta\chi\,\delta\upsilon}\frac{\delta\upsilon}{\delta\tau} + \frac{\delta^2 W}{\delta\tau\,\delta\upsilon}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta^2 W}{\delta\upsilon^2}\frac{\delta\upsilon}{\delta\tau}\frac{\delta\upsilon}{\delta\chi} + \frac{\delta W}{\delta\upsilon}\frac{\delta^2\upsilon}{\delta\tau\,\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\sigma\,\delta x'}\right] &= \frac{\delta^2 W}{\delta\sigma\,\delta x'} + \frac{\delta^2 W}{\delta\upsilon\,\delta x'}\frac{\delta\upsilon}{\delta\sigma}; \qquad \left[\frac{\delta^2 W}{\delta\tau\,\delta x'}\right] = \frac{\delta^2 W}{\delta\tau\,\delta x'} + \frac{\delta^2 W}{\delta\upsilon\,\delta x'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\,\delta y'}\right] &= \frac{\delta^2 W}{\delta\sigma\,\delta y'} + \frac{\delta^2 W}{\delta\upsilon\,\delta y'}\frac{\delta\upsilon}{\delta\sigma}; \qquad \left[\frac{\delta^2 W}{\delta\tau\,\delta y'}\right] = \frac{\delta^2 W}{\delta\tau\,\delta y'} + \frac{\delta^2 W}{\delta\upsilon\,\delta y'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\sigma\,\delta z'}\right] &= \frac{\delta^2 W}{\delta\sigma\,\delta z'} + \frac{\delta^2 W}{\delta\upsilon\,\delta z'}\frac{\delta\upsilon}{\delta\sigma}; \qquad \left[\frac{\delta^2 W}{\delta\tau\,\delta z'}\right] = \frac{\delta^2 W}{\delta\tau\,\delta z'} + \frac{\delta^2 W}{\delta\upsilon\,\delta z'}\frac{\delta\upsilon}{\delta\tau};\\
\left[\frac{\delta^2 W}{\delta\chi\,\delta x'}\right] &= \frac{\delta^2 W}{\delta\chi\,\delta x'} + \frac{\delta^2 W}{\delta\upsilon\,\delta x'}\frac{\delta\upsilon}{\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\chi\,\delta y'}\right] &= \frac{\delta^2 W}{\delta\chi\,\delta y'} + \frac{\delta^2 W}{\delta\upsilon\,\delta y'}\frac{\delta\upsilon}{\delta\chi};\\
\left[\frac{\delta^2 W}{\delta\chi\,\delta z'}\right] &= \frac{\delta^2 W}{\delta\chi\,\delta z'} + \frac{\delta^2 W}{\delta\upsilon\,\delta z'}\frac{\delta\upsilon}{\delta\chi}:
\end{aligned}\right\} \tag{Q^3}$$

obtained by differentiating the three corresponding expressions of the first order,

$$\left[\frac{\delta W}{\delta \sigma}\right] = \frac{\delta W}{\delta \sigma} + \frac{\delta W}{\delta \upsilon}\frac{\delta \upsilon}{\delta \sigma}; \quad \left[\frac{\delta W}{\delta \tau}\right] = \frac{\delta W}{\delta \tau} + \frac{\delta W}{\delta \upsilon}\frac{\delta \upsilon}{\delta \tau}; \quad \left[\frac{\delta W}{\delta \chi}\right] = \frac{\delta W}{\delta \chi} + \frac{\delta W}{\delta \upsilon}\frac{\delta \upsilon}{\delta \chi}, \qquad (\mathrm{R}^3)$$

which are to be substituted in (B^3), in place of

$$\frac{\delta W}{\delta \sigma}, \quad \frac{\delta W}{\delta \tau}, \quad \frac{\delta W}{\delta \chi}.$$

Deduction of the Coefficients of W from those of V. Homogeneous Transformations.

8. Reciprocally, if it be required to deduce the partial differential coefficients of W, of the second order, from those of V, in the case of a final variable medium, we have only to compare the expressions for

$$\delta x, \quad \delta y, \quad \delta z, \quad \delta \sigma', \quad \delta \tau', \quad \delta \upsilon', \quad -\delta \frac{\delta V}{\delta \chi},$$

as linear functions of $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, deduced from the equations (A^3), with those that are obtained by differentiating the seven equations (G') (H'), into which (B') resolves itself: that is, with the developed expressions for the variations of

$$\frac{\delta W}{\delta \sigma}, \quad \frac{\delta W}{\delta \tau}, \quad \frac{\delta W}{\delta \upsilon}, \quad \frac{\delta W}{\delta x'}, \quad \frac{\delta W}{\delta y'}, \quad \frac{\delta W}{\delta z'}, \quad \frac{\delta W}{\delta \chi}.$$

But if the final medium be uniform, then (B') no longer furnishes the seven equations (G') (H'), nor can δx, δy, δz, themselves, but only certain combinations of them, be deduced from (A^3); and the auxiliary function W is no longer completely determined in form, by the mere knowledge of the form of the characteristic function V, with which it is connected; because, in this case, the seven variables on which W depends, are not independent of each other, four of them being connected by the relations (K'), by means of which relation the dependence of W on the seven may be changed in an infinite variety of ways, while the dependence of V on its seven variables, and the properties of the optical combination, remain unaltered. Accordingly this indeterminateness of W, as deduced from V, in the case of a final uniform medium, produces an indeterminateness, in the same case, in the partial differential coefficients of W; and whereas W, considered as a function of seven variables, has thirty-five partial differential coefficients of the first and second orders, we have only twenty-seven relations between these thirty-five coefficients, unless we make some particular supposition respecting the form of W; such as the supposition, already mentioned, that one of the related variables, for example υ, has been removed by a previous elimination, which gives the eight conditions,

$$\left.\begin{array}{l} \dfrac{\delta W}{\delta \upsilon} = 0, \quad \dfrac{\delta^2 W}{\delta \sigma\, \delta \upsilon} = 0, \quad \dfrac{\delta^2 W}{\delta \tau\, \delta \upsilon} = 0, \quad \dfrac{\delta^2 W}{\delta \upsilon^2} = 0, \quad \dfrac{\delta^2 W}{\delta \upsilon\, \delta \chi} = 0, \\ \dfrac{\delta^2 W}{\delta \upsilon\, \delta x'} = 0, \quad \dfrac{\delta^2 W}{\delta \upsilon\, \delta y'} = 0, \quad \dfrac{\delta^2 W}{\delta \upsilon\, \delta z'} = 0. \end{array}\right\} \qquad (\mathrm{S}^3)$$

This last supposition removes the indeterminateness of W itself, and therefore of its partial differential coefficients; of which, for the two first orders, eight vanish by (S^3), and the remaining twenty-seven are determined, (when the variables and coefficients of V are known,) by the six equations (G'), (B^3), the three left-hand equations (G^3), the six first (I^3), the two first (K^3), and the ten (L^3) (M^3) (P^3); in resolving which equations it is useful to observe, that by (E^3) and (G^3),

$$\frac{1}{w''} = \frac{\delta^2 V}{\delta x^2}\frac{\delta^2 V}{\delta y^2} - \left(\frac{\delta^2 V}{\delta x\,\delta y}\right)^2. \tag{T^3}$$

And the twenty-seven expressions thus found for the coefficients of W of the two first orders, on the supposition of a previous elimination of one of the seven related variables, may be generalised, by (Q^3) and (R^3), into the twenty-seven relations already mentioned as existing between the thirty-five coefficients on any other supposition; which supposition, if it be sufficient to determine the form of W, will give the eight remaining conditions analogous to the conditions (S^3), that are necessary to determine the coefficients sought.

If, for example, we determine W by supposing it made homogeneous of the first dimension with respect to σ, τ, υ, we shall have the eight following conditions,

$$\sigma\frac{\delta W}{\delta\sigma} + \tau\frac{\delta W}{\delta\tau} + \upsilon\frac{\delta W}{\delta\upsilon} = W, \tag{U^3}$$

and

$$\left.\begin{aligned}
\sigma\frac{\delta^2 W}{\delta\sigma^2} + \tau\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + \upsilon\frac{\delta^2 W}{\delta\sigma\,\delta\upsilon} &= 0,\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + \tau\frac{\delta^2 W}{\delta\tau^2} + \upsilon\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} &= 0,\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta\upsilon} + \tau\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + \upsilon\frac{\delta^2 W}{\delta\upsilon^2} &= 0,\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta x'} + \tau\frac{\delta^2 W}{\delta\tau\,\delta x'} + \upsilon\frac{\delta^2 W}{\delta\upsilon\,\delta x'} &= \frac{\delta W}{\delta x'},\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta y'} + \tau\frac{\delta^2 W}{\delta\tau\,\delta y'} + \upsilon\frac{\delta^2 W}{\delta\upsilon\,\delta y'} &= \frac{\delta W}{\delta y'},\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta z'} + \tau\frac{\delta^2 W}{\delta\tau\,\delta z'} + \upsilon\frac{\delta^2 W}{\delta\upsilon\,\delta z'} &= \frac{\delta W}{\delta z'},\\
\sigma\frac{\delta^2 W}{\delta\sigma\,\delta\chi} + \tau\frac{\delta^2 W}{\delta\tau\,\delta\chi} + \upsilon\frac{\delta^2 W}{\delta\upsilon\,\delta\chi} &= \frac{\delta W}{\delta\chi},
\end{aligned}\right\} \tag{V^3}$$

to be combined with the twenty-seven which are independent of the form of W, and are deduced by the general method already mentioned. But this supposition of homogeneity appears to deserve a separate investigation, on account of the symmetry of the processes and results to which it leads.

Let us therefore resume the equations

$$x = \frac{\delta W}{\delta\sigma} + V\frac{\delta\Omega}{\delta\sigma},\quad y = \frac{\delta W}{\delta\tau} + V\frac{\delta\Omega}{\delta\tau},\quad z = \frac{\delta W}{\delta\upsilon} + V\frac{\delta\Omega}{\delta\upsilon}, \tag{W^2}$$

which were deduced in the sixth number from the homogeneous form that we now assign to W, and which are to be combined with the following

$$0 = \frac{\delta W}{\delta \chi} + \frac{\delta V}{\delta \chi} + V \frac{\delta \Omega}{\delta \chi}, \tag{W^3}$$

and with the general equations of the fourth number,

$$\sigma' = \frac{\delta W}{\delta x'}, \quad \tau' = \frac{\delta W}{\delta y'}, \quad \upsilon' = \frac{\delta W}{\delta z'} : \tag{G$'$}$$

and let us eliminate

$$\delta x, \quad \delta y, \quad \delta z, \quad \delta \sigma', \quad \delta \tau', \quad \delta \upsilon', \quad \delta \frac{\delta V}{\delta \chi},$$

by (A^3), from the differentials of these seven equations, (W^2) (G$'$) (W^3), that is from the seven following,

$$\left.\begin{array}{c}
\delta \frac{\delta W}{\delta \sigma} + V \, \delta \frac{\delta \Omega}{\delta \sigma} = \delta x - \frac{\delta \Omega}{\delta \sigma} \, \delta V, \\
\delta \frac{\delta W}{\delta \tau} + V \, \delta \frac{\delta \Omega}{\delta \tau} = \delta y - \frac{\delta \Omega}{\delta \tau} \, \delta V, \\
\delta \frac{\delta W}{\delta \upsilon} + V \, \delta \frac{\delta \Omega}{\delta \upsilon} = \delta z - \frac{\delta \Omega}{\delta \upsilon} \, \delta V, \\
\delta \frac{\delta W}{\delta x'} = \delta \sigma', \quad \delta \frac{\delta W}{\delta y'} = \delta \tau', \quad \delta \frac{\delta W}{\delta z'} = \delta \upsilon', \\
\delta \frac{\delta W}{\delta \chi} + V \delta \frac{\delta \Omega}{\delta \chi} = -\delta \frac{\delta V}{\delta \chi} - \frac{\delta \Omega}{\delta \chi} \, \delta V.
\end{array}\right\} \tag{X^3}$$

This elimination gives

$$\left.\begin{aligned}
\lambda^{(1)}\,\delta\Omega &= -\delta\sigma + \delta'\frac{\delta V}{\delta x} + \frac{\delta^2 V}{\delta x^2}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta x\,\delta y}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta x\,\delta z}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(2)}\,\delta\Omega &= -\delta\tau + \delta'\frac{\delta V}{\delta y} + \frac{\delta^2 V}{\delta x\,\delta y}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y^2}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta y\,\delta z}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(3)}\,\delta\Omega &= -\delta\upsilon + \delta'\frac{\delta V}{\delta z} + \frac{\delta^2 V}{\delta x\,\delta z}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\,\delta z}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta z^2}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(4)}\,\delta\Omega &= \delta\frac{\delta W}{\delta x'} + \delta'\frac{\delta V}{\delta x'} + \frac{\delta^2 V}{\delta x\,\delta x'}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\,\delta x'}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta z\,\delta x'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(5)}\,\delta\Omega &= \delta\frac{\delta W}{\delta y'} + \delta'\frac{\delta V}{\delta y'} + \frac{\delta^2 V}{\delta x\,\delta y'}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\,\delta y'}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta z\,\delta y'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(6)}\,\delta\Omega &= \delta\frac{\delta W}{\delta z'} + \delta'\frac{\delta V}{\delta z'} + \frac{\delta^2 V}{\delta x\,\delta z'}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \frac{\delta^2 V}{\delta y\,\delta z'}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right)\\
&\qquad + \frac{\delta^2 V}{\delta z\,\delta z'}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);\\
\lambda^{(7)}\,\delta\Omega &= \delta\frac{\delta W}{\delta\chi} + V\,\delta\frac{\delta\Omega}{\delta\chi} + \delta'\frac{\delta V}{\delta\chi} + \frac{\delta^2 V}{\delta x\,\delta\chi}\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right)\\
&\qquad + \frac{\delta^2 V}{\delta y\,\delta\chi}\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right) + \frac{\delta^2 V}{\delta z\,\delta\chi}\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right);
\end{aligned}\right\}\qquad (\mathrm{Y}^3)$$

if we put for abridgment

$$\left.\begin{aligned}
\delta' \frac{\delta V}{\delta x} &= \frac{\delta^2 V}{\delta x \, \delta x'} \, \delta x' + \frac{\delta^2 V}{\delta x \, \delta y'} \, \delta y' + \frac{\delta^2 V}{\delta x \, \delta z'} \, \delta z' + \frac{\delta^2 V}{\delta x \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta y} &= \frac{\delta^2 V}{\delta y \, \delta x'} \, \delta x' + \frac{\delta^2 V}{\delta y \, \delta y'} \, \delta y' + \frac{\delta^2 V}{\delta y \, \delta z'} \, \delta z' + \frac{\delta^2 V}{\delta y \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta z} &= \frac{\delta^2 V}{\delta z \, \delta x'} \, \delta x' + \frac{\delta^2 V}{\delta z \, \delta y'} \, \delta y' + \frac{\delta^2 V}{\delta z \, \delta z'} \, \delta z' + \frac{\delta^2 V}{\delta z \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta x'} &= \frac{\delta^2 V}{\delta x'^2} \, \delta x' + \frac{\delta^2 V}{\delta x' \, \delta y'} \, \delta y' + \frac{\delta^2 V}{\delta x' \, \delta z'} \, \delta z' + \frac{\delta^2 V}{\delta x' \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta y'} &= \frac{\delta^2 V}{\delta x' \, \delta y'} \, \delta x' + \frac{\delta^2 V}{\delta y'^2} \, \delta y' + \frac{\delta^2 V}{\delta y' \, \delta z'} \, \delta z' + \frac{\delta^2 V}{\delta y' \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta z'} &= \frac{\delta^2 V}{\delta x' \, \delta z'} \, \delta x' + \frac{\delta^2 V}{\delta y' \, \delta z'} \, \delta y' + \frac{\delta^2 V}{\delta z'^2} \, \delta z' + \frac{\delta^2 V}{\delta z' \, \delta \chi} \, \delta \chi, \\
\delta' \frac{\delta V}{\delta \chi} &= \frac{\delta^2 V}{\delta x' \, \delta \chi} \, \delta x' + \frac{\delta^2 V}{\delta y' \, \delta \chi} \, \delta y' + \frac{\delta^2 V}{\delta z' \, \delta \chi} \, \delta z' + \frac{\delta^2 V}{\delta \chi^2} \, \delta \chi,
\end{aligned}\right\} \tag{Z^3}$$

using δ' as in the notation (D^3); and if we observe that the partial differential equation of the fifth number,

$$0 = \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, \chi\right), \tag{V$'$}$$

gives

$$\left.\begin{aligned}
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x^2} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta x \, \delta y} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta x \, \delta z}, \\
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta y} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y^2} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta y \, \delta z}, \\
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta z} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta z} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta z^2}, \\
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta x'} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta x'} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta z \, \delta x'}, \\
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta y'} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta y'} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta z \, \delta y'}, \\
0 &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta z'} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta z'} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta z \, \delta z'}, \\
-\frac{\delta \Omega}{\delta \chi} &= \frac{\delta \Omega}{\delta \sigma} \frac{\delta^2 V}{\delta x \, \delta \chi} + \frac{\delta \Omega}{\delta \tau} \frac{\delta^2 V}{\delta y \, \delta \chi} + \frac{\delta \Omega}{\delta \upsilon} \frac{\delta^2 V}{\delta z \, \delta \chi}.
\end{aligned}\right\} \tag{A^4}$$

We have introduced, in the equations (Y^3), the terms $\lambda^{(1)} \, \delta\Omega, \ldots \lambda^{(7)} \, \delta\Omega$, that we may treat as independent the variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, which are connected by the condition $\delta\Omega = 0$.

To determine the multipliers $\lambda^{(1)}, \ldots \lambda^{(7)}$, we are to observe that in deducing the foregoing equations, the relation $\Omega = 0$ between the four variables σ, τ, υ, χ, has been supposed to have been so expressed, by the method mentioned in the second number, that the function Ω

when increased by unity becomes homogeneous of the first dimension with respect to σ, τ, υ; in such a manner that we have identically, for all values of the four variables σ, τ, υ, χ,

$$\sigma \frac{\delta \Omega}{\delta \sigma} + \tau \frac{\delta \Omega}{\delta \tau} + \upsilon \frac{\delta \Omega}{\delta \upsilon} = \Omega + 1, \tag{B^4}$$

and therefore,

$$\left.\begin{aligned}
\sigma \frac{\delta^2 \Omega}{\delta \sigma^2} + \tau \frac{\delta^2 \Omega}{\delta \sigma \, \delta \tau} + \upsilon \frac{\delta^2 \Omega}{\delta \sigma \, \delta \upsilon} &= 0, \\
\sigma \frac{\delta^2 \Omega}{\delta \sigma \, \delta \tau} + \tau \frac{\delta^2 \Omega}{\delta \tau^2} + \upsilon \frac{\delta^2 \Omega}{\delta \tau \, \delta \upsilon} &= 0, \\
\sigma \frac{\delta^2 \Omega}{\delta \sigma \, \delta \upsilon} + \tau \frac{\delta^2 \Omega}{\delta \tau \, \delta \upsilon} + \upsilon \frac{\delta^2 \Omega}{\delta \upsilon^2} &= 0, \\
\sigma \frac{\delta^2 \Omega}{\delta \sigma \, \delta \chi} + \tau \frac{\delta^2 \Omega}{\delta \tau \, \delta \chi} + \upsilon \frac{\delta^2 \Omega}{\delta \upsilon \, \delta \chi} &= \frac{\delta \Omega}{\delta \chi}.
\end{aligned}\right\} \tag{C^4}$$

Hence, and from the conditions (V^3), relative to the homogeneity of the function W, it is easy to infer that the multipliers have the following values;

$$\left.\begin{aligned}
&\lambda^{(1)} = -\sigma; \quad \lambda^{(2)} = -\tau; \quad \lambda^{(3)} = -\upsilon; \\
&\lambda^{(4)} = \sigma'; \quad \lambda^{(5)} = \tau'; \quad \lambda^{(6)} = \upsilon'; \quad \lambda^{(7)} = -\frac{\delta V}{\delta \chi}:
\end{aligned}\right\} \tag{D^4}$$

attending to (G$'$) and (W^3). If we substitute these values of the multipliers, in the seven equations (Y^3), we may decompose each of those equations into seven others, by treating the seven variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, as independent; and thus obtain forty-nine equations of the first degree, of which however only twenty-eight are distinct, for the determination of the twenty-eight partial differential coefficients of the second order, of W considered as a function of σ, τ, υ, x', y', z', χ, which relatively to σ, τ, υ, is homogeneous of the first dimension: the corresponding coefficients of the first order being determined by the seven equations (G$'$) (W^2) (W^3).

Instead of calculating in this manner the coefficients of W of the second order, by eliminating between the equations into which the system (Y^3) may be decomposed, it is simpler to eliminate between the equations (Y^3) themselves, and thus to obtain expressions for the variations

$$\delta \frac{\delta W}{\delta \sigma}, \ldots \delta \frac{\delta W}{\delta \chi}$$

of the coefficients of the first order, from which expressions the coefficients of the second order will then immediately result. Eliminating, therefore, between the three first equations (Y^3), in order to get expressions for the three variations

$$\delta \frac{\delta W}{\delta \sigma}, \quad \delta \frac{\delta W}{\delta \tau}, \quad \delta \frac{\delta W}{\delta \upsilon},$$

we find, after some symmetric reductions,

$$\left.\begin{aligned}
\delta\frac{\delta W}{\delta\sigma} &= -V\,\delta\frac{\delta\Omega}{\delta\sigma} + \frac{1}{v^2V''}\left(\tau\frac{\delta^2V}{\delta x\,\delta z} - \upsilon\frac{\delta^2V}{\delta x\,\delta y}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
&\quad -\frac{\delta\Omega}{\delta\sigma}\,\delta'V + \frac{1}{v^2V''}\left(\tau\frac{\delta^2V}{\delta y\,\delta z} - \upsilon\frac{\delta^2V}{\delta y^2}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&\quad +\frac{1}{v^2V''}\left(\tau\frac{\delta^2V}{\delta z^2} - \upsilon\frac{\delta^2V}{\delta y\,\delta z}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};\\
\delta\frac{\delta W}{\delta\tau} &= -V\,\delta\frac{\delta\Omega}{\delta\tau} + \frac{1}{v^2V''}\left(\upsilon\frac{\delta^2V}{\delta x^2} - \sigma\frac{\delta^2V}{\delta x\,\delta z}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
&\quad -\frac{\delta\Omega}{\delta\tau}\,\delta'V + \frac{1}{v^2V''}\left(\upsilon\frac{\delta^2V}{\delta x\,\delta y} - \sigma\frac{\delta^2V}{\delta y\,\delta z}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&\quad +\frac{1}{v^2V''}\left(\upsilon\frac{\delta^2V}{\delta x\,\delta z} - \sigma\frac{\delta^2V}{\delta z^2}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};\\
\delta\frac{\delta W}{\delta\upsilon} &= -V\,\delta\frac{\delta\Omega}{\delta\upsilon} + \frac{1}{v^2V''}\left(\sigma\frac{\delta^2V}{\delta x\,\delta y} - \tau\frac{\delta^2V}{\delta x^2}\right)\left\{\upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
&\quad -\frac{\delta\Omega}{\delta\upsilon}\,\delta'V + \frac{1}{v^2V''}\left(\sigma\frac{\delta^2V}{\delta y^2} - \tau\frac{\delta^2V}{\delta x\,\delta y}\right)\left\{\sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&\quad +\frac{1}{v^2V''}\left(\sigma\frac{\delta^2V}{\delta y\,\delta z} - \tau\frac{\delta^2V}{\delta x\,\delta z}\right)\left\{\tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};
\end{aligned}\right\} \tag{E^4}$$

in which,

$$\left.\begin{aligned}
V'' &= \frac{\delta^2V}{\delta x^2}\frac{\delta^2V}{\delta y^2} - \left(\frac{\delta^2V}{\delta x\,\delta y}\right)^2 + \frac{\delta^2V}{\delta y^2}\frac{\delta^2V}{\delta z^2} - \left(\frac{\delta^2V}{\delta y\,\delta z}\right)^2 + \frac{\delta^2V}{\delta z^2}\frac{\delta^2V}{\delta x^2} - \left(\frac{\delta^2V}{\delta z\,\delta x}\right)^2,\\
\text{and}\quad \frac{1}{v^2} &= \left(\frac{\delta\Omega}{\delta\sigma}\right)^2 + \left(\frac{\delta\Omega}{\delta\tau}\right)^2 + \left(\frac{\delta\Omega}{\delta\upsilon}\right)^2,
\end{aligned}\right\} \tag{F^4}$$

v having the same meaning as before: δ' also referring, as before, to the variations of x' y' z' χ, alone, and V'' having the same meaning as in the First Supplement. In effecting this elimination, we have attended to the forms of the functions W, Ω which give

$$\sigma\left(\delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}\right) + \tau\left(\delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}\right) + \upsilon\left(\delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon}\right) = -\delta'V; \tag{G^4}$$

we have also employed the equations (A^4), which give, by (F^4),

$$\left.\begin{aligned}
&\frac{\delta^2V}{\delta y^2}\frac{\delta^2V}{\delta z^2} - \left(\frac{\delta^2V}{\delta y\,\delta z}\right)^2 = V''v^2\left(\frac{\delta\Omega}{\delta\sigma}\right)^2; \quad \frac{\delta^2V}{\delta x\,\delta y}\frac{\delta^2V}{\delta z\,\delta x} - \frac{\delta^2V}{\delta x^2}\frac{\delta^2V}{\delta y\,\delta z} = V''v^2\frac{\delta\Omega}{\delta\tau}\frac{\delta\Omega}{\delta\upsilon};\\
&\frac{\delta^2V}{\delta z^2}\frac{\delta^2V}{\delta x^2} - \left(\frac{\delta^2V}{\delta z\,\delta x}\right)^2 = V''v^2\left(\frac{\delta\Omega}{\delta\tau}\right)^2; \quad \frac{\delta^2V}{\delta y\,\delta z}\frac{\delta^2V}{\delta x\,\delta y} - \frac{\delta^2V}{\delta y^2}\frac{\delta^2V}{\delta z\,\delta x} = V''v^2\frac{\delta\Omega}{\delta\upsilon}\frac{\delta\Omega}{\delta\sigma};\\
&\frac{\delta^2V}{\delta x^2}\frac{\delta^2V}{\delta y^2} - \left(\frac{\delta^2V}{\delta x\,\delta y}\right)^2 = V''v^2\left(\frac{\delta\Omega}{\delta\upsilon}\right)^2; \quad \frac{\delta^2V}{\delta z\,\delta x}\frac{\delta^2V}{\delta y\,\delta z} - \frac{\delta^2V}{\delta z^2}\frac{\delta^2V}{\delta x\,\delta y} = V''v^2\frac{\delta\Omega}{\delta\sigma}\frac{\delta\Omega}{\delta\tau}.
\end{aligned}\right\} \tag{H^4}$$

Having thus obtained expressions (E^4) for the three variations

$$\delta\frac{\delta W}{\delta\sigma},\quad \delta\frac{\delta W}{\delta\tau},\quad \delta\frac{\delta W}{\delta\upsilon},$$

it only remains to substitute these expressions in the four last equations (Y^3), and so to deduce, without any new elimination, the four other variations

$$\delta\frac{\delta W}{\delta x'},\quad \delta\frac{\delta W}{\delta y'},\quad \delta\frac{\delta W}{\delta z'},\quad \delta\frac{\delta W}{\delta\chi};$$

after which, we shall have immediately the twenty-eight coefficients of W, of the second order. The six coefficients, for example, of this order, which are formed by differentiating W with respect to σ, τ, υ, are expressed by the six following equations, deduced from (E^4):

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta\sigma^2} &= -V\,\frac{\delta^2\Omega}{\delta\sigma^2} + \frac{1}{v^2V''}\left(\tau^2\frac{\delta^2V}{\delta z^2} + \upsilon^2\frac{\delta^2V}{\delta y^2} - 2\tau\upsilon\frac{\delta^2V}{\delta y\,\delta z}\right);\\
\frac{\delta^2 W}{\delta\tau^2} &= -V\,\frac{\delta^2\Omega}{\delta\tau^2} + \frac{1}{v^2V''}\left(\upsilon^2\frac{\delta^2V}{\delta x^2} + \sigma^2\frac{\delta^2V}{\delta z^2} - 2\upsilon\sigma\frac{\delta^2V}{\delta z\,\delta x}\right);\\
\frac{\delta^2 W}{\delta\upsilon^2} &= -V\,\frac{\delta^2\Omega}{\delta\upsilon^2} + \frac{1}{v^2V''}\left(\sigma^2\frac{\delta^2V}{\delta y^2} + \tau^2\frac{\delta^2V}{\delta x^2} - 2\sigma\tau\frac{\delta^2V}{\delta x\,\delta y}\right);\\
\frac{\delta^2 W}{\delta\sigma\,\delta\tau} &= -V\,\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau} + \frac{1}{v^2V''}\left(-\upsilon^2\frac{\delta^2V}{\delta x\,\delta y} + \tau\upsilon\frac{\delta^2V}{\delta z\,\delta x} + \upsilon\sigma\frac{\delta^2V}{\delta y\,\delta z} - \sigma\tau\frac{\delta^2V}{\delta z^2}\right);\\
\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} &= -V\,\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon} + \frac{1}{v^2V''}\left(-\sigma^2\frac{\delta^2V}{\delta y\,\delta z} + \upsilon\sigma\frac{\delta^2V}{\delta x\,\delta y} + \sigma\tau\frac{\delta^2V}{\delta z\,\delta x} - \tau\upsilon\frac{\delta^2V}{\delta x^2}\right);\\
\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} &= -V\,\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma} + \frac{1}{v^2V''}\left(-\tau^2\frac{\delta^2V}{\delta z\,\delta x} + \sigma\tau\frac{\delta^2V}{\delta y\,\delta z} + \tau\upsilon\frac{\delta^2V}{\delta x\,\delta y} - \upsilon\sigma\frac{\delta^2V}{\delta y^2}\right):
\end{aligned}\right\}\quad(\mathrm{I}^4)$$

which may be shown to agree with the less simple equations of the same kind in the First Supplement, and may be thus summed up,

$$\begin{aligned}
v^2V''(\delta''^2W + V\,\delta''^2\Omega) = {}& \frac{\delta^2V}{\delta x^2}(\tau\,\delta\upsilon - \upsilon\,\delta\tau)^2 + 2\frac{\delta^2V}{\delta y\,\delta z}(\upsilon\,\delta\sigma - \sigma\,\delta\upsilon)(\sigma\,\delta\tau - \tau\,\delta\sigma)\\
&+ \frac{\delta^2V}{\delta y^2}(\upsilon\,\delta\sigma - \sigma\,\delta\upsilon)^2 + 2\frac{\delta^2V}{\delta z\,\delta x}(\sigma\,\delta\tau - \tau\,\delta\sigma)(\tau\,\delta\upsilon - \upsilon\,\delta\tau)\\
&+ \frac{\delta^2V}{\delta z^2}(\sigma\,\delta\tau - \tau\,\delta\sigma)^2 + 2\frac{\delta^2V}{\delta x\,\delta y}(\tau\,\delta\upsilon - \upsilon\,\delta\tau)(\upsilon\,\delta\sigma - \sigma\,\delta\upsilon),\quad(\mathrm{K}^4)
\end{aligned}$$

the mark of variation δ'' referring only to the variables σ, τ, υ, as δ' referred only to x', y', z', χ.

And the whole system of the twenty-eight expressions for the twenty-eight coefficients of W, of the second order, may be summed up in this one formula:

$$
\begin{aligned}
&\upsilon^2 V''(\delta^2 W + V\,\delta^2\Omega + 2\,\delta' V\,\delta\Omega + \delta'^2 V)\\
&\quad = \frac{\delta^2 V}{\delta x^2}\left\{\tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\}^2\\
&\quad + \frac{\delta^2 V}{\delta y^2}\left\{\upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}^2\\
&\quad + \frac{\delta^2 V}{\delta z^2}\left\{\sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}^2\\
&\quad + 2\frac{\delta^2 V}{\delta x\,\delta y}\left\{\tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\}\left\{\upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\\
&\quad + 2\frac{\delta^2 V}{\delta y\,\delta z}\left\{\upsilon\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right) - \sigma\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right)\right\}\left\{\sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\\
&\quad + 2\frac{\delta^2 V}{\delta z\,\delta x}\left\{\sigma\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right) - \tau\left(\delta\sigma - \delta'\frac{\delta V}{\delta x}\right)\right\}\left\{\tau\left(\delta\upsilon - \delta'\frac{\delta V}{\delta z}\right) - \upsilon\left(\delta\tau - \delta'\frac{\delta V}{\delta y}\right)\right\};
\end{aligned}
\tag{L^4}
$$

in which the symbols δ^2, δ'^2 are easily understood by what precedes, and in which the seven variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, may be treated as independent of each other.

The formula (K^4) has an inverse, deduced from (X^3), namely

$$
\begin{aligned}
\frac{\delta'''^2 V}{\upsilon^2 V''} &= \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta\Omega}{\delta\tau}\,\delta z - \frac{\delta\Omega}{\delta\upsilon}\,\delta y\right)^2\\
&+ \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left(\frac{\delta\Omega}{\delta\upsilon}\,\delta x - \frac{\delta\Omega}{\delta\sigma}\,\delta z\right)^2\\
&+ \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left(\frac{\delta\Omega}{\delta\sigma}\,\delta y - \frac{\delta\Omega}{\delta\tau}\,\delta x\right)^2\\
&+ 2\left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)\left(\frac{\delta\Omega}{\delta\tau}\,\delta z - \frac{\delta\Omega}{\delta\upsilon}\,\delta y\right)\left(\frac{\delta\Omega}{\delta\upsilon}\,\delta x - \frac{\delta\Omega}{\delta\sigma}\,\delta z\right)\\
&+ 2\left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)\left(\frac{\delta\Omega}{\delta\upsilon}\,\delta x - \frac{\delta\Omega}{\delta\sigma}\,\delta z\right)\left(\frac{\delta\Omega}{\delta\sigma}\,\delta y - \frac{\delta\Omega}{\delta\tau}\,\delta x\right)\\
&+ 2\left(\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)\left(\frac{\delta\Omega}{\delta\sigma}\,\delta y - \frac{\delta\Omega}{\delta\tau}\,\delta x\right)\left(\frac{\delta\Omega}{\delta\tau}\,\delta z - \frac{\delta\Omega}{\delta\upsilon}\,\delta y\right),
\end{aligned}
\tag{M^4}
$$

in which δ''' refers to x, y, z, and in which V'' may be deduced from W by the relation

$$
\begin{aligned}
\frac{\sigma^2 + \tau^2 + \upsilon^2}{V''\upsilon^2} &= \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)^2\\
&+ \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right) - \left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)^2\\
&+ \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right) - \left(\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)^2 :
\end{aligned}
\tag{N^4}
$$

and the more extensive formula (L^4) has an inverse also, namely,

$$\begin{aligned}
&\frac{1}{V''\upsilon^2}(\delta^2 V + V\,\delta'^2\Omega + 2\,\delta V\,\delta'\Omega + \delta'^2 W)\\
&= \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}^2\\
&+ \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2\\
&+ \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2\\
&+ 2\left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\\
&+ 2\left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\\
&+ 2\left(\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x - \delta'\frac{\delta W}{\delta\sigma} - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z - \delta'\frac{\delta W}{\delta\upsilon} - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y - \delta'\frac{\delta W}{\delta\tau} - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\},
\end{aligned} \tag{O^4}$$

δ' retaining its recent meaning, so that, as Ω does not contain x', y', z', we have, in the last formula,

$$\left.\begin{array}{l}\delta'\Omega = \dfrac{\delta\Omega}{\delta\chi}\,\delta\chi, \qquad \delta'^2\Omega = \dfrac{\delta^2\Omega}{\delta\chi^2}\,\delta\chi^2,\\ \delta'\dfrac{\delta\Omega}{\delta\sigma} = \dfrac{\delta^2\Omega}{\delta\sigma\,\delta\chi}\,\delta\chi, \quad \delta'\dfrac{\delta\Omega}{\delta\tau} = \dfrac{\delta^2\Omega}{\delta\tau\,\delta\chi}\,\delta\chi, \quad \delta'\dfrac{\delta\Omega}{\delta\upsilon} = \dfrac{\delta^2\Omega}{\delta\upsilon\,\delta\chi}\,\delta\chi.\end{array}\right\} \tag{P^4}$$

If we do not choose to suppose W homogeneous of the first dimension with respect to σ, τ, υ, and if we put for abridgment

$$\sigma\frac{\delta W}{\delta\sigma} + \tau\frac{\delta W}{\delta\tau} + \upsilon\frac{\delta W}{\delta\upsilon} - W = w_1, \tag{Q^4}$$

and denote by δW_1 and $\delta^2 W_1$, the expressions already found on this particular supposition, for the variations of W, of the two first orders, so that, for the first order, by (G') (W^2) (W^3),

$$\delta W_1 = x\,\delta\sigma + y\,\delta\tau + z\,\delta\upsilon + \sigma'\,\delta x' + \tau'\,\delta y' + \upsilon'\,\delta z' - \frac{\delta V}{\delta\chi}\,\delta\chi - V\,\delta\Omega, \tag{R^4}$$

and, for the second order, $\delta^2 W_1 =$ the value of $\delta^2 W$ assigned by the formula (L⁴); we may generalise these particular values δW_1, $\delta^2 W_1$, by the following relations,

$$\left.\begin{aligned}\delta W_1 &= \delta W - w_1\,\delta\Omega,\\ \delta^2 W_1 &= \delta^2 W - w_1\,\delta^2\Omega - 2\,\delta w_1\,\delta\Omega\\ &\qquad + \left(\sigma\frac{\delta w_1}{\delta\sigma} + \tau\frac{\delta w_1}{\delta\tau} + \upsilon\frac{\delta w_1}{\delta\upsilon}\right)\delta\Omega^2,\end{aligned}\right\} \tag{S^4}$$

in which δW, $\delta^2 W$, are general expressions, independent of the condition of homogeneity $w_1 = 0$, and of every other particular supposition respecting the form of W. It is, however, here understood that the final medium is uniform, and that in forming the variations of the function W, the quantities σ, τ, υ, χ, x', y', z', on which it depends, are treated as if they were seven independent variables.

And if we would deduce expressions δW_n, $\delta^2 W_n$, for the variations of W, of the two first orders, on the supposition that W is made, before differentiation, homogeneous of any dimension n, with respect to σ, τ, υ, we may put

$$\sigma\frac{\delta W}{\delta\sigma} + \tau\frac{\delta W}{\delta\tau} + \upsilon\frac{\delta W}{\delta\upsilon} - nW = w_n, \tag{T^4}$$

and we shall have the following relations

$$\left.\begin{aligned}\delta W_n &= \delta W - w_n\,\delta\Omega,\\ \delta^2 W_n &= \delta^2 W - w_n\,\delta^2\Omega - 2\,\delta w_n\,\delta\Omega\\ &\qquad + \left(\sigma\frac{\delta w_n}{\delta\sigma} + \tau\frac{\delta w_n}{\delta\tau} + \upsilon\frac{\delta w_n}{\delta\upsilon} + w_n - nw_n\right)\delta\Omega^2,\end{aligned}\right\} \tag{U^4}$$

which include the relations (S⁴). The general analysis of these homogeneous transformations is interesting, but we cannot dwell upon it here.

Deductions of the Coefficients of T from those of W, and reciprocally.

9. The general principles of investigation, respecting the connexions between the partial differential coefficients of the second order, of the characteristic and auxiliary functions, having been sufficiently explained by the remarks made at the beginning of the seventh number, and by the details into which we have since entered; we shall confine ourselves, in the remaining research of such connexions, for the new auxiliary function T, to the case of extreme uniform media. And having already treated of the mutual connexions between the coefficients of the two functions V and W, it will be sufficient now to connect the coefficients of either of these two, for example, the coefficients of W, with those of T, of the first and second orders: since the connexions between the coefficients of all three functions will thus be sufficiently known. We shall also suppose that W has been made, before differentiation, homogeneous of the first dimension with respect to σ, τ, υ, that our results may be the more easily combined with the symmetric expressions already deduced from this supposition,

expressions which can be generalised in the manner that has been explained: and similarly we shall suppose that T is made homogeneous of the first dimension with respect to σ, τ, υ, and also with respect to σ', τ', υ'. Let us then seek to express the partial differential coefficients of the two first orders, of T, by means of those of W, both functions being thus symmetrically prepared.

In this inquiry, we have, as before, the conditions of homogeneity (U^3) (V^3), relative to the function W, and analogous conditions relative to T, namely, for the first order,

$$\left.\begin{aligned} \sigma\frac{\delta T}{\delta\sigma} + \tau\frac{\delta T}{\delta\tau} + \upsilon\frac{\delta T}{\delta\upsilon} &= T, \\ \sigma'\frac{\delta T}{\delta\sigma'} + \tau'\frac{\delta T}{\delta\tau'} + \upsilon'\frac{\delta T}{\delta\upsilon'} &= T; \end{aligned}\right\} \tag{V^4}$$

and, for the second order,

$$\left.\begin{aligned}
0 &= \sigma\frac{\delta^2 T}{\delta\sigma^2} + \tau\frac{\delta^2 T}{\delta\sigma\,\delta\tau} + \upsilon\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'^2} + \tau'\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\sigma'\,\delta\upsilon'}; \\
0 &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\tau} + \tau\frac{\delta^2 T}{\delta\tau^2} + \upsilon\frac{\delta^2 T}{\delta\tau\,\delta\upsilon}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} + \tau'\frac{\delta^2 T}{\delta\tau'^2} + \upsilon'\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'}; \\
0 &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon} + \tau\frac{\delta^2 T}{\delta\tau\,\delta\upsilon} + \upsilon\frac{\delta^2 T}{\delta\upsilon^2}; & 0 &= \sigma'\frac{\delta^2 T}{\delta\sigma'\,\delta\upsilon'} + \tau'\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'} + \upsilon'\frac{\delta^2 T}{\delta\upsilon'^2}; \\
\frac{\delta T}{\delta\sigma'} &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\sigma'} + \tau\frac{\delta^2 T}{\delta\tau\,\delta\sigma'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\,\delta\sigma'}; & \frac{\delta T}{\delta\sigma} &= \sigma'\frac{\delta^2 T}{\delta\sigma\,\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\sigma\,\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon'}; \\
\frac{\delta T}{\delta\tau'} &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\tau'} + \tau\frac{\delta^2 T}{\delta\tau\,\delta\tau'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\,\delta\tau'}; & \frac{\delta T}{\delta\tau} &= \sigma'\frac{\delta^2 T}{\delta\tau\,\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\tau\,\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\tau\,\delta\upsilon'}; \\
\frac{\delta T}{\delta\upsilon'} &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon'} + \tau\frac{\delta^2 T}{\delta\tau\,\delta\upsilon'} + \upsilon\frac{\delta^2 T}{\delta\upsilon\,\delta\upsilon'}; & \frac{\delta T}{\delta\upsilon} &= \sigma'\frac{\delta^2 T}{\delta\upsilon\,\delta\sigma'} + \tau'\frac{\delta^2 T}{\delta\upsilon\,\delta\tau'} + \upsilon'\frac{\delta^2 T}{\delta\upsilon\,\delta\upsilon'}; \\
\frac{\delta T}{\delta\chi} &= \sigma\frac{\delta^2 T}{\delta\sigma\,\delta\chi} + \tau\frac{\delta^2 T}{\delta\tau\,\delta\chi} + \upsilon\frac{\delta^2 T}{\delta\upsilon\,\delta\chi}; & \frac{\delta T}{\delta\chi} &= \sigma'\frac{\delta^2 T}{\delta\sigma'\,\delta\chi} + \tau'\frac{\delta^2 T}{\delta\tau'\,\delta\chi} + \upsilon'\frac{\delta^2 T}{\delta\upsilon'\,\delta\chi};
\end{aligned}\right\} \tag{W^4}$$

together with the conditions relative to Ω, Ω', namely (B^4), (C^4), and the following,

$$\left.\begin{aligned}
\sigma'\frac{\delta\Omega'}{\delta\sigma'} + \tau'\frac{\delta\Omega'}{\delta\tau'} + \upsilon'\frac{\delta\Omega'}{\delta\upsilon'} &= \Omega' + 1 = 1, \\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'^2} + \tau'\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'} + \upsilon'\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\upsilon'} &= 0, \\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'} + \tau'\frac{\delta^2\Omega'}{\delta\tau'^2} + \upsilon'\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'} &= 0, \\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\upsilon'} + \tau'\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'} + \upsilon'\frac{\delta^2\Omega'}{\delta\upsilon'^2} &= 0, \\
\sigma'\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\chi} + \tau'\frac{\delta^2\Omega'}{\delta\tau'\,\delta\chi} + \upsilon'\frac{\delta^2\Omega'}{\delta\upsilon'\,\delta\chi} &= \frac{\delta\Omega'}{\delta\chi};
\end{aligned}\right\} \tag{X^4}$$

we have also the general equations

$$\frac{\delta W}{\delta x'} = \sigma', \quad \frac{\delta W}{\delta y'} = \tau', \quad \frac{\delta W}{\delta z'} = \upsilon', \tag{G$'$}$$

by combining which with the foregoing conditions and with the partial differential equation (A^2), we find the following, analogous to (A^4),

$$\left.\begin{aligned}
0 &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta x'^2} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta x'\,\delta y'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta x'\,\delta z'},\\
0 &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\,\delta y'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta y'^2} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta y'\,\delta z'},\\
0 &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\,\delta z'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta y'\,\delta z'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta z'^2},\\
\frac{\delta\Omega}{\delta\sigma} &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta\sigma\,\delta x'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta\sigma\,\delta y'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta\sigma\,\delta z'},\\
\frac{\delta\Omega}{\delta\tau} &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta\tau\,\delta x'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta\tau\,\delta y'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta\tau\,\delta z'},\\
\frac{\delta\Omega}{\delta\upsilon} &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta\upsilon\,\delta x'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta\upsilon\,\delta y'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta\upsilon\,\delta z'},\\
\frac{\delta\Omega}{\delta\chi} - \frac{\delta\Omega'}{\delta\chi} &= \frac{\delta\Omega'}{\delta\sigma'}\frac{\delta^2 W}{\delta\chi\,\delta x'} + \frac{\delta\Omega'}{\delta\tau'}\frac{\delta^2 W}{\delta\chi\,\delta y'} + \frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta^2 W}{\delta\chi\,\delta z'};
\end{aligned}\right\} \qquad (\mathrm{Y}^4)$$

and if we combine the conditions of homogeneity of the two functions W, T, with the fundamental relation (E') between these two functions, and with the properties of Ω, Ω', and attend to (G'), we find the following expressions for the partial differential coefficients of T, of the first order,

$$\left.\begin{aligned}
\frac{\delta T}{\delta\sigma} &= \frac{\delta W}{\delta\sigma} + (T - W)\frac{\delta\Omega}{\delta\sigma}; & \frac{\delta T}{\delta\sigma'} &= -x' + W\frac{\delta\Omega'}{\delta\sigma'};\\
\frac{\delta T}{\delta\tau} &= \frac{\delta W}{\delta\tau} + (T - W)\frac{\delta\Omega}{\delta\tau}; & \frac{\delta T}{\delta\tau'} &= -y' + W\frac{\delta\Omega'}{\delta\tau'};\\
\frac{\delta T}{\delta\upsilon} &= \frac{\delta W}{\delta\upsilon} + (T - W)\frac{\delta\Omega}{\delta\upsilon}; & \frac{\delta T}{\delta\upsilon'} &= -z' + W\frac{\delta\Omega'}{\delta\upsilon'};\\
\frac{\delta T}{\delta\chi} &= \frac{\delta W}{\delta\chi} + (T - W)\frac{\delta\Omega}{\delta\chi} + W\frac{\delta\Omega'}{\delta\chi}.
\end{aligned}\right\} \qquad (\mathrm{Z}^4)$$

Differentiating the expressions (Z^4), and eliminating $\delta x'$, $\delta y'$, $\delta z'$, by means of the differentials of the general equations (G'), we obtain, by (Y^4), the following system, analogous to the system (Y^3);

$$
\left.
\begin{aligned}
\lambda_1\,\delta\Omega+\lambda_1'\,\delta\Omega' &= \frac{\delta^2 W}{\delta x'^2}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta x'\,\delta y'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta x'\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)-\delta_{\prime}\frac{\delta W}{\delta x'}+\delta\sigma';\\
\lambda_2\,\delta\Omega+\lambda_2'\,\delta\Omega' &= \frac{\delta^2 W}{\delta x'\,\delta y'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta y'^2}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta y'\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)-\delta_{\prime}\frac{\delta W}{\delta y'}+\delta\tau';\\
\lambda_3\,\delta\Omega+\lambda_3'\,\delta\Omega' &= \frac{\delta^2 W}{\delta x'\,\delta z'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta y'\,\delta z'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta z'^2}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)-\delta_{\prime}\frac{\delta W}{\delta z'}+\delta\upsilon';\\
\lambda_4\,\delta\Omega+\lambda_4'\,\delta\Omega' &= \frac{\delta^2 W}{\delta\sigma\,\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta\sigma\,\delta y'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta\sigma\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad-\delta_{\prime}\frac{\delta W}{\delta\sigma}+\delta\left(\frac{\delta T}{\delta\sigma}-T\frac{\delta\Omega}{\delta\sigma}\right)+W\,\delta\frac{\delta\Omega}{\delta\sigma};\\
\lambda_5\,\delta\Omega+\lambda_5'\,\delta\Omega' &= \frac{\delta^2 W}{\delta\tau\,\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta\tau\,\delta y'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta\tau\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad-\delta_{\prime}\frac{\delta W}{\delta\tau}+\delta\left(\frac{\delta T}{\delta\tau}-T\frac{\delta\Omega}{\delta\tau}\right)+W\,\delta\frac{\delta\Omega}{\delta\tau};\\
\lambda_6\,\delta\Omega+\lambda_6'\,\delta\Omega' &= \frac{\delta^2 W}{\delta\upsilon\,\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta\upsilon\,\delta y'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta\upsilon\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad-\delta_{\prime}\frac{\delta W}{\delta\upsilon}+\delta\left(\frac{\delta T}{\delta\upsilon}-T\frac{\delta\Omega}{\delta\upsilon}\right)+W\,\delta\frac{\delta\Omega}{\delta\upsilon};\\
\lambda_7\,\delta\Omega+\lambda_7'\,\delta\Omega' &= \frac{\delta^2 W}{\delta\chi\,\delta x'}\left(\delta\frac{\delta T}{\delta\sigma'}-W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right)+\frac{\delta^2 W}{\delta\chi\,\delta y'}\left(\delta\frac{\delta T}{\delta\tau'}-W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right)\\
&\quad+\frac{\delta^2 W}{\delta\chi\,\delta z'}\left(\delta\frac{\delta T}{\delta\upsilon'}-W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right)\\
&\quad-\delta_{\prime}\frac{\delta W}{\delta\chi}+\delta\left(\frac{\delta T}{\delta\chi}-T\frac{\delta\Omega}{\delta\chi}\right)+W\,\delta\left(\frac{\delta\Omega}{\delta\chi}-\frac{\delta\Omega'}{\delta\chi}\right):
\end{aligned}
\right\}
\tag{A^5}
$$

in which $\delta_{\prime}$ refers only to the four variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, and in which we may treat the seven variations $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$, $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, as independent, if we assign to the fourteen multipliers $\lambda_1, \ldots \lambda'_7$, the following values;

$$\left.\begin{aligned}
\lambda_1 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'^2} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta x'\,\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta x'\,\delta z'} - \frac{\delta W}{\delta x'};\\
\lambda_2 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\,\delta y'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta y'^2} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta y'\,\delta z'} - \frac{\delta W}{\delta y'};\\
\lambda_3 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta x'\,\delta z'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta y'\,\delta z'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta z'^2} - \frac{\delta W}{\delta z'};\\
\lambda_4 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\sigma\,\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\sigma\,\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\sigma\,\delta z'} - T\frac{\delta\Omega}{\delta\sigma};\\
\lambda_5 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\tau\,\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\tau\,\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\tau\,\delta z'} - T\frac{\delta\Omega}{\delta\tau};\\
\lambda_6 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\upsilon\,\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\upsilon\,\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\upsilon\,\delta z'} - T\frac{\delta\Omega}{\delta\upsilon};\\
\lambda_7 &= \frac{\delta T}{\delta\sigma'}\frac{\delta^2 W}{\delta\chi\,\delta x'} + \frac{\delta T}{\delta\tau'}\frac{\delta^2 W}{\delta\chi\,\delta y'} + \frac{\delta T}{\delta\upsilon'}\frac{\delta^2 W}{\delta\chi\,\delta z'} - T\frac{\delta\Omega}{\delta\chi} + W\frac{\delta\Omega'}{\delta\chi};\\
\lambda'_1 &= \sigma'; \quad \lambda'_2 = \tau'; \quad \lambda'_3 = \upsilon';\\
\lambda'_4 &= \frac{\delta W}{\delta\sigma} - W\frac{\delta\Omega}{\delta\sigma}; \quad \lambda'_5 = \frac{\delta W}{\delta\tau} - W\frac{\delta\Omega}{\delta\tau};\\
\lambda'_6 &= \frac{\delta W}{\delta\upsilon} - W\frac{\delta\Omega}{\delta\upsilon}; \quad \lambda'_7 = \frac{\delta W}{\delta\chi} - W\frac{\delta\Omega}{\delta\chi}:
\end{aligned}\right\} \tag{B^5}$$

the values of $\lambda_1, \ldots \lambda_7$, may also be thus expressed,

$$\left.\begin{aligned}
\lambda_1 &= -\frac{\delta w'}{\delta x'}, & \lambda_4 &= -\frac{\delta w'}{\delta\sigma} + (W - T)\frac{\delta\Omega}{\delta\sigma},\\
\lambda_2 &= -\frac{\delta w'}{\delta y'}, & \lambda_5 &= -\frac{\delta w'}{\delta\tau} + (W - T)\frac{\delta\Omega}{\delta\tau},\\
\lambda_3 &= -\frac{\delta w'}{\delta z'}, & \lambda_6 &= -\frac{\delta w'}{\delta\upsilon} + (W - T)\frac{\delta\Omega}{\delta\upsilon},\\
& & \lambda_7 &= -\frac{\delta w'}{\delta\chi} + (W - T)\frac{\delta\Omega}{\delta\chi},
\end{aligned}\right\} \tag{C^5}$$

if we put for abridgment

$$w' = x'\frac{\delta W}{\delta x'} + y'\frac{\delta W}{\delta y'} + z'\frac{\delta W}{\delta z'}, \tag{D^5}$$

and consider w', like W, as a function of σ, τ, υ, χ, x', y', z', which, relatively to σ, τ, υ, is homogeneous of the first dimension. The four last equations (A^5) give, by addition, after multiplying them respectively by $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\chi$,

$$\left.\begin{aligned}
\delta^2 T = (T - W)\,\delta^2\Omega + W\,\delta^2\Omega' + (W - T)\,\delta\Omega^2 \\
+ (\delta T - \delta_{\prime} w')\,\delta\Omega + (\delta_{\prime} W - W\,\delta\Omega)\,\delta\Omega' + \delta_{\prime}^2 W \\
- \left(\delta_{\prime}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\left(\delta\frac{\delta T}{\delta\sigma'} - W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\right) \\
- \left(\delta_{\prime}\frac{\delta W}{\delta y'} - \delta\tau'\right)\left(\delta\frac{\delta T}{\delta\tau'} - W\,\delta\frac{\delta\Omega'}{\delta\tau'}\right) \\
- \left(\delta_{\prime}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\left(\delta\frac{\delta T}{\delta\upsilon'} - W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\right),
\end{aligned}\right\} \qquad (\mathrm{E}^5)$$

$\delta_{\prime}$ still referring only to the variations of σ, τ, υ, χ; and the three first equations (A^5) give, by elimination,

$$
\left.\begin{aligned}
&\delta\frac{\delta T}{\delta\sigma'} - W\,\delta\frac{\delta\Omega'}{\delta\sigma'}\\
&\quad = \frac{\delta\Omega'}{\delta\sigma'}(\delta_{,}W - W\,\delta\Omega) + \frac{\delta T}{\delta\sigma'}\,\delta\Omega\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\upsilon'\frac{\delta^2 W}{\delta x'\,\delta y'} - \tau'\frac{\delta^2 W}{\delta x'\,\delta z'}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\upsilon'\frac{\delta^2 W}{\delta y'^2} - \tau'\frac{\delta^2 W}{\delta y'\,\delta z'}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\upsilon'\frac{\delta^2 W}{\delta y'\,\delta z'} - \tau'\frac{\delta^2 W}{\delta z'^2}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\};\\
&\delta\frac{\delta T}{\delta\tau'} - W\,\delta\frac{\delta\Omega'}{\delta\tau'}\\
&\quad = \frac{\delta\Omega'}{\delta\tau'}(\delta_{,}W - W\,\delta\Omega) + \frac{\delta T}{\delta\tau'}\,\delta\Omega\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta x'\,\delta z'} - \upsilon'\frac{\delta^2 W}{\delta x'^2}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta y'\,\delta z'} - \upsilon'\frac{\delta^2 W}{\delta x'\,\delta y'}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\sigma'\frac{\delta^2 W}{\delta z'^2} - \upsilon'\frac{\delta^2 W}{\delta x'\,\delta z'}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\};\\
&\delta\frac{\delta T}{\delta\upsilon'} - W\,\delta\frac{\delta\Omega'}{\delta\upsilon'}\\
&\quad = \frac{\delta\Omega'}{\delta\upsilon'}(\delta_{,}W - W\,\delta\Omega) + \frac{\delta T}{\delta\upsilon'}\,\delta\Omega\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'^2} - \sigma'\frac{\delta^2 W}{\delta x'\,\delta y'}\right)\left\{\tau'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right) - \upsilon'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'\,\delta y'} - \sigma'\frac{\delta^2 W}{\delta y'^2}\right)\left\{\upsilon'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right) - \sigma'\left(\delta_{,}\frac{\delta W}{\delta z'} - \delta\upsilon'\right)\right\}\\
&\qquad + \frac{1}{\upsilon'^2 W'''}\left(\tau'\frac{\delta^2 W}{\delta x'\,\delta z'} - \sigma'\frac{\delta^2 W}{\delta y'\,\delta z'}\right)\left\{\sigma'\left(\delta_{,}\frac{\delta W}{\delta y'} - \delta\tau'\right) - \tau'\left(\delta_{,}\frac{\delta W}{\delta x'} - \delta\sigma'\right)\right\};
\end{aligned}\right\}\qquad (\mathrm{F}^5)
$$

in which

$$
W''' = \frac{\delta^2 W}{\delta x'^2}\frac{\delta^2 W}{\delta y'^2} - \left(\frac{\delta^2 W}{\delta x'\,\delta y'}\right)^2 + \frac{\delta^2 W}{\delta y'^2}\frac{\delta^2 W}{\delta z'^2} - \left(\frac{\delta^2 W}{\delta y'\,\delta z'}\right)^2 + \frac{\delta^2 W}{\delta z'^2}\frac{\delta^2 W}{\delta x'^2} - \left(\frac{\delta^2 W}{\delta z'\,\delta x'}\right)^2, \quad (\mathrm{G}^5)
$$

and

$$
\frac{1}{\upsilon'^2} = \left(\frac{\delta\Omega'}{\delta\sigma'}\right)^2 + \left(\frac{\delta\Omega'}{\delta\tau'}\right)^2 + \left(\frac{\delta\Omega'}{\delta\upsilon'}\right)^2, \qquad (\mathrm{H}^5)
$$

υ' having the same meaning as in the second number. In effecting the last elimination, we have attended to the relations (Y^4), which give

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta y'^2}\frac{\delta^2 W}{\delta z'^2} - \left(\frac{\delta^2 W}{\delta y'\,\delta z'}\right)^2 &= W'''\upsilon'^2\left(\frac{\delta\Omega'}{\delta\sigma'}\right)^2;\\
\frac{\delta^2 W}{\delta z'^2}\frac{\delta^2 W}{\delta x'^2} - \left(\frac{\delta^2 W}{\delta z'\,\delta x'}\right)^2 &= W'''\upsilon'^2\left(\frac{\delta\Omega'}{\delta\tau'}\right)^2;\\
\frac{\delta^2 W}{\delta x'^2}\frac{\delta^2 W}{\delta y'^2} - \left(\frac{\delta^2 W}{\delta x'\,\delta y'}\right)^2 &= W'''\upsilon'^2\left(\frac{\delta\Omega'}{\delta\upsilon'}\right)^2;\\
\frac{\delta^2 W}{\delta x'\,\delta y'}\frac{\delta^2 W}{\delta z'\,\delta x'} - \frac{\delta^2 W}{\delta x'^2}\frac{\delta^2 W}{\delta y'\,\delta z'} &= W'''\upsilon'^2\frac{\delta\Omega'}{\delta\tau'}\frac{\delta\Omega'}{\delta\upsilon'};\\
\frac{\delta^2 W}{\delta y'\,\delta z'}\frac{\delta^2 W}{\delta x'\,\delta y'} - \frac{\delta^2 W}{\delta y'^2}\frac{\delta^2 W}{\delta z'\,\delta x'} &= W'''\upsilon'^2\frac{\delta\Omega'}{\delta\upsilon'}\frac{\delta\Omega'}{\delta\sigma'};\\
\frac{\delta^2 W}{\delta z'\,\delta x'}\frac{\delta^2 W}{\delta y'\,\delta z'} - \frac{\delta^2 W}{\delta z'^2}\frac{\delta^2 W}{\delta x'\,\delta y'} &= W'''\upsilon'^2\frac{\delta\Omega'}{\delta\sigma'}\frac{\delta\Omega'}{\delta\tau'}.
\end{aligned}\right\} \tag{I^5}$$

And combining (E^5) (F^5) we obtain the following formula for $\delta^2 T$, analogous to the formula (L^4), which completes the solution of our present problem, because it it equivalent to twenty-eight expressions for the twenty-eight partial differential coefficients of T, of the second order, deduced from the coefficients of W;

$$\begin{aligned}
0 = \upsilon'^2 W'''\{&\delta^2 T + (W - T)\,\delta^2\Omega - W\,\delta^2\Omega' - 2\,\delta_{\prime}W\,.\,\delta\Omega' - \delta_{\prime}^2 W\\
&+ 2(x'\,\delta\sigma' + y'\,\delta\tau' + z'\,\delta\upsilon')\,\delta\Omega\}\\
&+ \frac{\delta^2 W}{\delta x'^2}\left\{\tau'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right)\right\}^2\\
&+ \frac{\delta^2 W}{\delta y'^2}\left\{\upsilon'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right)\right\}^2\\
&+ \frac{\delta^2 W}{\delta z'^2}\left\{\sigma'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right)\right\}^2\\
&+ 2\frac{\delta^2 W}{\delta x'\,\delta y'}\left\{\tau'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right)\right\}\\
&\qquad\times\left\{\upsilon'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right)\right\}\\
&+ 2\frac{\delta^2 W}{\delta y'\,\delta z'}\left\{\upsilon'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right) - \sigma'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right)\right\}\\
&\qquad\times\left\{\sigma'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right)\right\}\\
&+ 2\frac{\delta^2 W}{\delta z'\,\delta x'}\left\{\sigma'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right) - \tau'\left(\delta\sigma' - \delta_{\prime}\frac{\delta W}{\delta x'}\right)\right\}\\
&\qquad\times\left\{\tau'\left(\delta\upsilon' - \delta_{\prime}\frac{\delta W}{\delta z'}\right) - \upsilon'\left(\delta\tau' - \delta_{\prime}\frac{\delta W}{\delta y'}\right)\right\}.
\end{aligned} \tag{K^5}$$

And if we denote by $\delta^2 T_{1,1}$ the value of the second differential $\delta^2 T$ assigned by the formula (K^5), and determined on the supposition that T has been made, before differentiation, homogeneous of the first dimension with respect to σ, τ, υ, and also with respect to σ', τ', υ', and denote by $\delta T_{1,1}$ the corresponding value of δT, determined by the coefficients (Z^4), we may generalise thse values by means of the following relations, analogous to (S^4);

$$\left.\begin{aligned} \delta T_{1,1} &= \delta T - \delta\Omega \,.\, \nabla_1 T - \delta\Omega' \,.\, \nabla'_1 T; \\ \delta^2 T_{1,1} &= \delta^2 T - \delta^2\Omega \,.\, \nabla_1 T - \delta^2\Omega' \,.\, \nabla'_1 T - 2\,\delta\Omega \,.\, \delta\nabla_1 T - 2\,\delta\Omega' \,.\, \delta\nabla'_1 T \\ &\qquad + \delta\Omega^2 \,.\, \nabla_1(\nabla_1 + 1)T + 2\,\delta\Omega \,.\, \delta\Omega' \,.\, \nabla_1\nabla'_1 T + \delta\Omega'^2 \,.\, \nabla'_1(\nabla'_1 + 1)T : \end{aligned}\right\} \qquad (L^5)$$

∇_1, ∇'_1, being here characteristics of operation, defined by the following symbolic equations,

$$\left.\begin{aligned} \nabla_1 &= \sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau} + \upsilon\frac{\delta}{\delta\upsilon} - 1; \\ \nabla'_1 &= \sigma'\frac{\delta}{\delta\sigma'} + \tau'\frac{\delta}{\delta\tau'} + \upsilon'\frac{\delta}{\delta\upsilon'} - 1. \end{aligned}\right\} \qquad (M^5)$$

More generally, if we denote by $T_{n,n'}$ the function deduced from T by the homogeneous preparation mentioned in the sixth number, which coincides with T when the variables σ τ υ σ' τ' υ' χ are connected by the relations $\Omega = 0$, $\Omega' = 0$, and which is, for arbitrary values of those variables, homogeneous of the dimension n with respect to σ, τ, υ, and of the dimension n' with respect to σ', τ', υ', we have the following expressions, analogous to (U^4),

$$\left.\begin{aligned} \delta T_{n,n'} &= \delta T - \delta\Omega \,.\, \nabla_n T - \delta\Omega' \,.\, \nabla'_{n'} T; \\ \delta^2 T_{n,n'} &= \delta^2 T - \delta^2\Omega \,.\, \nabla_n T - \delta^2\Omega' \,.\, \nabla'_{n'} T - 2\,\delta\Omega \,.\, \delta\nabla_n T - 2\,\delta\Omega' \,.\, \delta\nabla'_{n'} T \\ &\qquad + \delta\Omega^2 \,.\, \nabla_n(\nabla_n + 1)T + 2\,\delta\Omega \,.\, \delta\Omega' \,.\, \nabla_n\nabla'_{n'} T + \delta\Omega'^2 \,.\, \nabla'_{n'}(\nabla'_{n'} + 1)T : \end{aligned}\right\} \qquad (N^5)$$

defining the characteristics ∇_n, $\nabla'_{n'}$, as follows,

$$\left.\begin{aligned} \nabla_n &= \sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau} + \upsilon\frac{\delta}{\delta\upsilon} - n; \\ \nabla'_{n'} &= \sigma'\frac{\delta}{\delta\sigma'} + \tau'\frac{\delta}{\delta\tau'} + \upsilon'\frac{\delta}{\delta\upsilon'} - n'. \end{aligned}\right\} \qquad (O^5)$$

Reciprocally to deduce the coefficients of W, of the second order, from those of T, on the same suppositions of homogeneity, and with the same dimensions $n = 1$, $n' = 1$, we are to eliminate $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, between the differentials of (G') and (Z^4), and we find the following system,

$$
\left.
\begin{aligned}
\lambda_1''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'^2} - W\frac{\delta^2\Omega'}{\delta\sigma'^2}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\upsilon'}\right)\delta\frac{\delta W}{\delta z'} + \delta_{,}\frac{\delta T}{\delta\sigma'} - W\,\delta_{,}\frac{\delta\Omega'}{\delta\sigma'} + \delta x' - \frac{\delta\Omega'}{\delta\sigma'}\delta W; \\
\lambda_2''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau'^2} - W\frac{\delta^2\Omega'}{\delta\tau'^2}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'}\right)\delta\frac{\delta W}{\delta z'} + \delta_{,}\frac{\delta T}{\delta\tau'} - W\,\delta_{,}\frac{\delta\Omega'}{\delta\tau'} + \delta y' - \frac{\delta\Omega'}{\delta\tau'}\delta W; \\
\lambda_3''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\upsilon'}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\upsilon'^2} - W\frac{\delta^2\Omega'}{\delta\upsilon'^2}\right)\delta\frac{\delta W}{\delta z'} + \delta_{,}\frac{\delta T}{\delta\upsilon'} - W\,\delta_{,}\frac{\delta\Omega'}{\delta\upsilon'} + \delta z' - \frac{\delta\Omega'}{\delta\upsilon'}\delta W; \\
\lambda_4''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma\,\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\sigma\,\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\sigma\,\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\sigma}\right)\delta\frac{\delta W}{\delta z'} - \delta\frac{\delta W}{\delta\sigma} + \delta_{,}\frac{\delta T}{\delta\sigma} \\
&\quad + (W - T)\,\delta\frac{\delta\Omega}{\delta\sigma} + \frac{\delta\Omega}{\delta\sigma}(\delta W - \delta_{,}T); \\
\lambda_5''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\tau\,\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\tau\,\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\tau\,\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\tau}\right)\delta\frac{\delta W}{\delta z'} - \delta\frac{\delta W}{\delta\tau} + \delta_{,}\frac{\delta T}{\delta\tau} \\
&\quad + (W - T)\,\delta\frac{\delta\Omega}{\delta\tau} + \frac{\delta\Omega}{\delta\tau}(\delta W - \delta_{,}T); \\
\lambda_6''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\upsilon\,\delta\sigma'} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta x'} + \left(\frac{\delta^2 T}{\delta\upsilon\,\delta\tau'} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\upsilon\,\delta\upsilon'} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\upsilon}\right)\delta\frac{\delta W}{\delta z'} - \delta\frac{\delta W}{\delta\upsilon} + \delta_{,}\frac{\delta T}{\delta\upsilon} \\
&\quad + (W - T)\,\delta\frac{\delta\Omega}{\delta\upsilon} + \frac{\delta\Omega}{\delta\upsilon}(\delta W - \delta_{,}T); \\
\lambda_7''\,\delta\Omega &= \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\chi} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\chi} - \frac{\delta T}{\delta\sigma'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta x'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\tau'\,\delta\chi} - W\frac{\delta^2\Omega'}{\delta\tau'\,\delta\chi} - \frac{\delta T}{\delta\tau'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta y'} \\
&\quad + \left(\frac{\delta^2 T}{\delta\upsilon'\,\delta\chi} - W\frac{\delta^2\Omega'}{\delta\upsilon'\,\delta\chi} - \frac{\delta T}{\delta\upsilon'}\frac{\delta\Omega}{\delta\chi}\right)\delta\frac{\delta W}{\delta z'} \\
&\quad - \delta\frac{\delta W}{\delta\chi} + \delta_{,}\frac{\delta T}{\delta\chi} + (W - T)\,\delta\frac{\delta\Omega}{\delta\chi} + \frac{\delta\Omega}{\delta\chi}(\delta W - \delta_{,}T) - W\,\delta_{,}\frac{\delta\Omega'}{\delta\chi} - \frac{\delta\Omega'}{\delta\chi}\delta W;
\end{aligned}
\right\}
\quad (\mathrm{P}^5)
$$

$\delta_{\prime}$ still referring only to variations of σ, τ, υ, χ, and the values of the multipliers being,

$$\left.\begin{aligned}
\lambda''_1 &= -x'; & \lambda''_4 &= \frac{\delta W}{\delta\sigma} - T\frac{\delta\Omega}{\delta\sigma};\\
\lambda''_2 &= -y'; & \lambda''_5 &= \frac{\delta W}{\delta\tau} - T\frac{\delta\Omega}{\delta\tau};\\
\lambda''_3 &= -z'; & \lambda''_6 &= \frac{\delta W}{\delta\upsilon} - T\frac{\delta\Omega}{\delta\upsilon};\\
& & \lambda''_7 &= \frac{\delta W}{\delta\chi} - T\frac{\delta\Omega}{\delta\chi}.
\end{aligned}\right\} \tag{Q^5}$$

Hence may be deduced, by reasonings analogous to those already employed, the following formula for $\delta^2 W$, which is equivalent to twenty-eight separate expressions for the partial differential coefficients of W, of the second order, considered as deduced from the coefficients of T, on the foregoing suppositions of homogeneity:

$$\begin{aligned}
0 = {}& \frac{1}{\upsilon'^2 W'''}\{\delta^2 W - \delta_{\prime}^2 T + (T - W)\,\delta^2\Omega + W\,\delta_{\prime}^2\Omega' + 2\,\delta W\,(\delta_{\prime}\Omega' - \delta\Omega) + 2\,\delta_{\prime}W\,.\,\delta\Omega\}\\
&+ \left(\frac{\delta^2 T}{\delta\sigma'^2} - W\,\frac{\delta^2\Omega'}{\delta\sigma'^2}\right) D^2 + 2\left(\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'}\right) D' D''\\
&+ \left(\frac{\delta^2 T}{\delta\tau'^2} - W\,\frac{\delta^2\Omega'}{\delta\tau'^2}\right) D'^2 + 2\left(\frac{\delta^2 T}{\delta\upsilon'\,\delta\sigma'} - W\frac{\delta^2\Omega'}{\delta\upsilon'\,\delta\sigma'}\right) D'' D\\
&+ \left(\frac{\delta^2 T}{\delta\upsilon'^2} - W\,\frac{\delta^2\Omega'}{\delta\upsilon'^2}\right) D''^2 + 2\left(\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'}\right) D D';
\end{aligned} \tag{R^5}$$

in which we have put for abridgment,

$$\left.\begin{aligned}
D &= \frac{\delta\Omega'}{\delta\tau'}\left(\delta z' + z'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\upsilon'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\upsilon'}\right) - \frac{\delta\Omega'}{\delta\upsilon'}\left(\delta y' + y'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\tau'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\tau'}\right),\\
D' &= \frac{\delta\Omega'}{\delta\upsilon'}\left(\delta x' + x'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\sigma'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\sigma'}\right) - \frac{\delta\Omega'}{\delta\sigma'}\left(\delta z' + z'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\upsilon'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\upsilon'}\right),\\
D'' &= \frac{\delta\Omega'}{\delta\sigma'}\left(\delta y' + y'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\tau'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\tau'}\right) - \frac{\delta\Omega'}{\delta\tau'}\left(\delta x' + x'\,\delta\Omega + \delta_{\prime}\frac{\delta T}{\delta\sigma'} - W\,\delta_{\prime}\frac{\delta\Omega'}{\delta\sigma'}\right),
\end{aligned}\right\} \tag{S^5}$$

and in which W''' can be deduced from T, by the relation

$$\begin{aligned}
\frac{\sigma'^2 + \tau'^2 + \upsilon'^2}{\upsilon'^2 W'''} = {}& \left(\frac{\delta^2 T}{\delta\sigma'^2} - W\,\frac{\delta^2\Omega'}{\delta\sigma'^2}\right)\left(\frac{\delta^2 T}{\delta\tau'^2} - W\,\frac{\delta^2\Omega'}{\delta\tau'^2}\right) - \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} - W\frac{\delta^2\Omega'}{\delta\sigma'\,\delta\tau'}\right)^2\\
&+ \left(\frac{\delta^2 T}{\delta\tau'^2} - W\,\frac{\delta^2\Omega'}{\delta\tau'^2}\right)\left(\frac{\delta^2 T}{\delta\upsilon'^2} - W\,\frac{\delta^2\Omega'}{\delta\upsilon'^2}\right) - \left(\frac{\delta^2 T}{\delta\tau'\,\delta\upsilon'} - W\frac{\delta^2\Omega'}{\delta\tau'\,\delta\upsilon'}\right)^2\\
&+ \left(\frac{\delta^2 T}{\delta\upsilon'^2} - W\,\frac{\delta^2\Omega'}{\delta\upsilon'^2}\right)\left(\frac{\delta^2 T}{\delta\sigma'^2} - W\,\frac{\delta^2\Omega'}{\delta\sigma'^2}\right) - \left(\frac{\delta^2 T}{\delta\upsilon'\,\delta\sigma'} - W\frac{\delta^2\Omega'}{\delta\upsilon'\,\delta\sigma'}\right)^2.
\end{aligned} \tag{T^5}$$

General Remarks and Cautions, with respect to the foregoing deductions. Case of a Single Uniform Medium. Connexions between the Coefficients of the Functions v, Ω, υ, for any Single Medium.

10. We are then able, by combining the formulæ of the three preceding numbers, to deduce the partial differential coefficients of the two first orders, of any one of the three functions V, W, T, from those of either of the other two, when the extreme media are uniform and known: since we have expressed the coefficients of V by those of W, and the coefficients of W by those of T, and reciprocally, for this case of uniform media. And if the extreme media be not uniform, but variable, that is, if they be atmospheres, ordinary or extraordinary, we can still connect the partial differential coefficients of the three functions, by the general method mentioned at the beginning of the seventh number: which method extends to orders higher than the second, without much additional difficulty of elimination, but with the results of greater complexity, and of less interesting application.

This general method consists, as has been said, in differentiating and comparing the equations into which the general expressions (A′) (B′) (C′) for the variations of the three functions resolve themselves: and *in making this preliminary resolution of the general expressions* (A′) (B′) (C′), *it is necessary to attend with care to the relations between the variables* σ, τ, υ, σ', τ', υ', χ, or between σ, τ, υ, x', y', z', χ, when any such relations exist. The investigations into which we have entered in the three last numbers, for the case of extreme uniform media, *suppose that the variables are connected only by the relations* $\Omega = 0$, $\Omega' = 0$, which arise from and express the optical properties of these media; and *other but analogous processes must be deduced from the general method, when any additional relations* $\Omega'' = 0$, $\Omega''' = 0, \ldots$ between the variables of the question, arise from the particular nature of a combination which we wish to study. In the very simple case, for instance, of a single uniform medium, we have the three relations

$$\sigma' = \sigma, \quad \tau' = \tau, \quad \upsilon' = \upsilon, \tag{U^5}$$

which are to be combined with the relation $\Omega = 0$; and with this combination of relations, the general expression (C′) for the variation of T will no longer admit of being resolved in the same way as when more of the quantities on which T depends could vary independently of each other.

In the case last mentioned,of a *single uniform medium*, the characteristic function V involves the co-ordinates x, y, z, x', y', z', only by involving their differences $x - x'$, $y - y'$, $z - z'$, and is, with respect to these differences, homogeneous of the first dimension, being determined by an equation of the form

$$0 = \Psi\left(\frac{x - x'}{V}, \frac{y - y'}{V}, \frac{z - z'}{V}, \chi\right), \tag{V^5}$$

which results from the equation (N) for the medium function v, by first suppressing in that equation the co-ordinates on account of the supposed uniformity, and then making

$$\frac{\alpha}{v} = \frac{x - x'}{V}, \quad \frac{\beta}{v} = \frac{y - y'}{V}, \quad \frac{\gamma}{v} = \frac{z - z'}{V}. \tag{W^5}$$

The relation (V^5) may also be deduced from the relation $\Omega = 0$, by eliminating the ratios of σ, τ, υ, between the three following equations,

$$\frac{x - x'}{V} = \frac{\delta\Omega}{\delta\sigma}, \quad \frac{y - y'}{V} = \frac{\delta\Omega}{\delta\tau}, \quad \frac{z - z'}{V} = \frac{\delta\Omega}{\delta\upsilon}. \tag{X^5}$$

We have also, in this case of a single uniform medium,

$$V = \sigma(x - x') + \tau(y - y') + \upsilon(z - z'), \tag{Y^5}$$

and therefore, by (D′) (E′) (U^5),

$$\left.\begin{aligned} W &= \sigma x' + \tau y' + \upsilon z', \\ T &= 0 : \end{aligned}\right\} \tag{Z^5}$$

the last of which results may be verified by observing that the general expression for the auxiliary function T may be put under the form

$$T = x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y} + z\frac{\delta V}{\delta z} + x'\frac{\delta V}{\delta x'} + y'\frac{\delta V}{\delta y'} + z'\frac{\delta V}{\delta z'} - V, \tag{A^6}$$

so that T vanishes when V is homogeneous of the first dimension with respect to the six extreme co-ordinates. The formulæ of the last number, for the partial differential coefficients of T, all fail in this case of a single uniform medium, for the reason already assigned; but we may consider all these coefficients of T as vanishing, like T itself: we may however give any other values to these coefficients which when combined with the relations between the variables will make the variations of T vanish. The coefficients of W may be obtained by differentiating the expression (Z^5), which is of the homogeneous form that we have already found convenient to adopt; they are, for the first two orders, included in the two following formulæ,

$$\left.\begin{aligned} \delta W &= x'\,\delta\sigma + y'\,\delta\tau + z'\,\delta\upsilon + \sigma\,\delta x' + \tau\,\delta y' + \upsilon\,\delta z', \\ \delta^2 W &= 2\,\delta\sigma\,\delta x' + 2\,\delta\tau\,\delta y' + 2\,\delta\upsilon\,\delta z', \end{aligned}\right\} \tag{B^6}$$

and they vanish for orders higher than the second. And the coefficients of V, of the two first orders, may be deduced from those of W by the formulæ of the eighth number, which are not vitiated by the existence of the relations (U^5), because those relations do not affect the variables that enter into the composition of V and W. The variation of V, of the first order, is

$$\delta V = \sigma(\delta x - \delta x') + \tau(\delta y - \delta y') + \upsilon(\delta z - \delta z') - V\frac{\delta\Omega}{\delta\chi}\,\delta\chi; \tag{C^6}$$

and that of the second order is given by the following equation, deduced from (O^4), (N^4), (B^6),

$$
\begin{aligned}
V\left\{\frac{\delta^2\Omega}{\delta\sigma^2}\frac{\delta^2\Omega}{\delta\tau^2}-\left(\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)^2+\frac{\delta^2\Omega}{\delta\tau^2}\frac{\delta^2\Omega}{\delta\upsilon^2}-\left(\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)^2+\frac{\delta^2\Omega}{\delta\upsilon^2}\frac{\delta^2\Omega}{\delta\sigma^2}-\left(\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)^2\right\}& \\
\times\left(\frac{\delta^2V+V\,\delta'^2\Omega+2\,\delta V\,\delta'\Omega}{\sigma^2+\tau^2+\upsilon^2}\right)& \\
=\frac{\delta\Omega^2}{\delta\sigma^2}\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)-\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}^2& \\
+\frac{\delta\Omega^2}{\delta\tau^2}\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)-\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2& \\
+\frac{\delta\Omega^2}{\delta\upsilon^2}\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)-\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2& \\
+2\frac{\delta\Omega^2}{\delta\sigma\,\delta\tau}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}& \\
+2\frac{\delta\Omega^2}{\delta\tau\,\delta\upsilon}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}& \\
+2\frac{\delta\Omega^2}{\delta\upsilon\,\delta\sigma}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta x-\delta x'-V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta z-\delta z'-V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta y-\delta y'-V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\};\quad (\mathrm{D}^6)&
\end{aligned}
$$

in which the symbol δ' has the same meaning as before, so that as x' y' z' do not enter into the composition of the function Ω, δ' refers here to the variation of colour only. This equation (D^6) may be put under the following simpler form,

$$
\begin{aligned}
\frac{V}{v}(\delta^2 V + V\,\delta'^2\Omega + 2\,\delta V\,\delta'\Omega)&\\
= \frac{\delta^2 v}{\delta\alpha^2}\left(\delta x - \delta x' - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)^2&\\
+ \frac{\delta^2 v}{\delta\beta^2}\left(\delta y - \delta y' - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)^2&\\
+ \frac{\delta^2 v}{\delta\gamma^2}\left(\delta z - \delta z' - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)^2&\\
+ 2\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\delta x - \delta x' - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\left(\delta y - \delta y' - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)&\\
+ 2\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\left(\delta y - \delta y' - V\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\left(\delta z - \delta z' - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)&\\
+ 2\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\left(\delta z - \delta z' - V\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\left(\delta x - \delta x' - V\,\delta'\frac{\delta\Omega}{\delta\sigma}\right),& \qquad (\mathrm{E}^6)
\end{aligned}
$$

if we attend to the equations already established, in the second number,

$$
\begin{aligned}
&\frac{\alpha}{v} = \frac{\delta\Omega}{\delta\sigma}, \quad \frac{\beta}{v} = \frac{\delta\Omega}{\delta\tau}, \quad \frac{\gamma}{v} = \frac{\delta\Omega}{\delta\upsilon}, \quad -\frac{1}{v}\frac{\delta v}{\delta\chi} = \frac{\delta\Omega}{\delta\chi},\\
&\sigma = \frac{\delta v}{\delta\alpha}, \quad \tau = \frac{\delta v}{\delta\beta}, \quad \upsilon = \frac{\delta v}{\delta\gamma},
\end{aligned}
$$

and to the relations which result from these, by differentiation and elimination. For thus we obtain

$$
\left.
\begin{aligned}
\delta\frac{\alpha}{v} - \delta'\frac{\delta\Omega}{\delta\sigma} &= \frac{\delta^2\Omega}{\delta\sigma^2}\,\delta\frac{\delta v}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\,\delta\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}\,\delta\frac{\delta v}{\delta\gamma},\\
\delta\frac{\beta}{v} - \delta'\frac{\delta\Omega}{\delta\tau} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\,\delta\frac{\delta v}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau^2}\,\delta\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\,\delta\frac{\delta v}{\delta\gamma},\\
\delta\frac{\gamma}{v} - \delta'\frac{\delta\Omega}{\delta\upsilon} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}\,\delta\frac{\delta v}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\,\delta\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon^2}\,\delta\frac{\delta v}{\delta\gamma},\\
-\delta\left(\frac{1}{v}\frac{\delta v}{\delta\chi}\right) - \delta'\frac{\delta\Omega}{\delta\chi} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta\chi}\,\delta\frac{\delta v}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau\,\delta\chi}\,\delta\frac{\delta v}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon\,\delta\chi}\,\delta\frac{\delta v}{\delta\gamma},
\end{aligned}
\right\}
\qquad (\mathrm{F}^6)
$$

in which v is considered as a homogeneous function of the first dimension of α, β, γ, involving also the colour χ; and in which, although the three variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, are connected by the relation $\alpha\,\delta\alpha + \beta\,\delta\beta + \gamma\,\delta\gamma = 0$, yet we may treat these variations as independent; because, if we introduced indeterminate multipliers of $\alpha\,\delta\alpha + \beta\,\delta\beta + \gamma\,\delta\gamma$, in ($\mathrm{F}^6$), to allow for the relation, we should find that these multipliers vanish, on account of the homogeneity of v. And if we put for abridgment

$$
\omega'' = \frac{\delta^2\Omega}{\delta\sigma^2}\frac{\delta^2\Omega}{\delta\tau^2} - \left(\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)^2 + \frac{\delta^2\Omega}{\delta\tau^2}\frac{\delta^2\Omega}{\delta\upsilon^2} - \left(\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)^2 + \frac{\delta^2\Omega}{\delta\upsilon^2}\frac{\delta^2\Omega}{\delta\sigma^2} - \left(\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)^2, \qquad (\mathrm{G}^6)
$$

the equations (F^6) give the following formula for $\delta^2 \upsilon$, that is, for the second variation of υ, taken as if α β γ χ, were four independent variables,

$$
\begin{aligned}
&\frac{\upsilon\omega''}{\sigma^2+\tau^2+\upsilon^2}(\delta^2\upsilon + \upsilon\,\delta'^2\Omega + 2\,\delta\upsilon\,\delta'\Omega)\\
&\quad = \frac{\delta^2\Omega}{\delta\sigma^2}\left\{\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right) - \frac{\delta\Omega}{\delta\tau}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\right\}^2\\
&\quad + \frac{\delta^2\Omega}{\delta\tau^2}\left\{\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right) - \frac{\delta\Omega}{\delta\upsilon}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\right\}^2\\
&\quad + \frac{\delta^2\Omega}{\delta\upsilon^2}\left\{\frac{\delta\Omega}{\delta\tau}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right) - \frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\right\}^2\\
&\quad + 2\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\\
&\quad + 2\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\sigma}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\\ -\frac{\delta\Omega}{\delta\upsilon}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\\
&\quad + 2\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\tau}\left(\delta\alpha - \upsilon\,\delta'\frac{\delta\Omega}{\delta\sigma}\right)\\ -\frac{\delta\Omega}{\delta\sigma}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\end{array}\right\}\left\{\begin{array}{l}\frac{\delta\Omega}{\delta\upsilon}\left(\delta\beta - \upsilon\,\delta'\frac{\delta\Omega}{\delta\tau}\right)\\ -\frac{\delta\Omega}{\delta\tau}\left(\delta\gamma - \upsilon\,\delta'\frac{\delta\Omega}{\delta\upsilon}\right)\end{array}\right\};
\end{aligned}
\tag{H^6}
$$

which justifies the passage from (D^6) to (E^6), and expresses the law of dependence of the partial differential coefficients of the second order of the function υ on those of Ω, for the case of a uniform medium.

If the medium be not uniform, and if we would still express the law of this dependence, we have only to change δ', in the four equations (F^6), to a new characteristic $\delta_{\prime\prime}$ referring to the variations of x y z χ, and to combine the four thus altered with the three following,

$$
\left.\begin{aligned}
-\delta\left(\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta x}\right) - \delta_{\prime\prime}\frac{\delta\Omega}{\delta x} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta x}\,\delta\frac{\delta\upsilon}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau\,\delta x}\,\delta\frac{\delta\upsilon}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon\,\delta x}\,\delta\frac{\delta\upsilon}{\delta\gamma},\\
-\delta\left(\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta y}\right) - \delta_{\prime\prime}\frac{\delta\Omega}{\delta y} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta y}\,\delta\frac{\delta\upsilon}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau\,\delta y}\,\delta\frac{\delta\upsilon}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon\,\delta y}\,\delta\frac{\delta\upsilon}{\delta\gamma},\\
-\delta\left(\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta z}\right) - \delta_{\prime\prime}\frac{\delta\Omega}{\delta z} &= \frac{\delta^2\Omega}{\delta\sigma\,\delta z}\,\delta\frac{\delta\upsilon}{\delta\alpha} + \frac{\delta^2\Omega}{\delta\tau\,\delta z}\,\delta\frac{\delta\upsilon}{\delta\beta} + \frac{\delta^2\Omega}{\delta\upsilon\,\delta z}\,\delta\frac{\delta\upsilon}{\delta\gamma},
\end{aligned}\right\}
\tag{I^6}
$$

in which $\delta_{\prime\prime}$ is the same new characteristic, and which are deduced from the equations already established for variable media,

$$
-\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta x} = \frac{\delta\Omega}{\delta x}, \quad -\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta y} = \frac{\delta\Omega}{\delta y}, \quad -\frac{1}{\upsilon}\frac{\delta\upsilon}{\delta z} = \frac{\delta\Omega}{\delta z}:
$$

and we are conducted to a formula for $\delta^2 \upsilon$, which no otherwise differs from (H^6) than by having $\delta_{\prime\prime}$ instead of δ' throughout.

And if, reciprocally, we would express the law of dependence of the coefficients of Ω of the second order, on those of υ, we may do so by the following general formula,

$$
\begin{aligned}
&\upsilon''\upsilon^2(\upsilon\,\delta^2\Omega + \delta_{\prime\prime}^2\upsilon + 2\,\delta_{\prime\prime}\upsilon\,\delta\Omega) \\
&\quad = \frac{\delta^2\upsilon}{\delta\alpha^2}\left\{\upsilon\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right) - \tau\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right)\right\}^2 \\
&\quad + \frac{\delta^2\upsilon}{\delta\beta^2}\left\{\sigma\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right) - \upsilon\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right)\right\}^2 \\
&\quad + \frac{\delta^2\upsilon}{\delta\gamma^2}\left\{\tau\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right) - \sigma\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right)\right\}^2 \\
&\quad + 2\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta}\left\{\upsilon\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right) - \tau\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right)\right\}\left\{\sigma\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right) - \upsilon\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right)\right\} \\
&\quad + 2\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma}\left\{\sigma\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right) - \upsilon\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right)\right\}\left\{\tau\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right) - \sigma\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right)\right\} \\
&\quad + 2\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha}\left\{\tau\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\alpha}\right) - \sigma\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right)\right\}\left\{\upsilon\left(\delta\tau - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\beta}\right) - \tau\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta\upsilon}{\delta\gamma}\right)\right\};
\end{aligned}
\tag{K^6}
$$

in which $\delta_{\prime\prime}$ refers still to the variations x, y, z, χ, and in which υ'' has the same meaning as in the First Supplement, namely

$$\upsilon'' = \frac{\delta^2\upsilon}{\delta\alpha^2}\frac{\delta^2\upsilon}{\delta\beta^2} - \left(\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta}\right)^2 + \frac{\delta^2\upsilon}{\delta\beta^2}\frac{\delta^2\upsilon}{\delta\gamma^2} - \left(\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma}\right)^2 + \frac{\delta^2\upsilon}{\delta\gamma^2}\frac{\delta^2\upsilon}{\delta\alpha^2} - \left(\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha}\right)^2 : \tag{L^6}$$

this quantity υ'' is also connected with the ω'' of (G^6) (H^6), by the relation

$$\upsilon''\omega'' = \frac{\sigma^2 + \tau^2 + \upsilon^2}{\upsilon^4}. \tag{M^6}$$

The formula (K^6) is equivalent to twenty-eight separate expressions for the partial differential coefficients of Ω, of the second order, which extend to variable as well as to uniform media: the formula gives, for example, the six following general expressions, which enable us to introduce the coefficients of the function υ, of the second order, instead of those of Ω, if it be thought desirable so to do, in many of the general equations of the present memoir, as the expressions contained in (H^6) would enable us to introduce Ω instead of υ, in many of those of the First Supplement:

$$\left.\begin{aligned}
\frac{\delta^2\Omega}{\delta\sigma^2} &= \frac{1}{\upsilon''\upsilon^3}\left(\tau^2\frac{\delta^2\upsilon}{\delta\gamma^2} + \upsilon^2\frac{\delta^2\upsilon}{\delta\beta^2} - 2\tau\upsilon\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma}\right);\\
\frac{\delta^2\Omega}{\delta\tau^2} &= \frac{1}{\upsilon''\upsilon^3}\left(\upsilon^2\frac{\delta^2\upsilon}{\delta\alpha^2} + \sigma^2\frac{\delta^2\upsilon}{\delta\gamma^2} - 2\upsilon\sigma\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha}\right);\\
\frac{\delta^2\Omega}{\delta\upsilon^2} &= \frac{1}{\upsilon''\upsilon^3}\left(\sigma^2\frac{\delta^2\upsilon}{\delta\beta^2} + \tau^2\frac{\delta^2\upsilon}{\delta\alpha^2} - 2\sigma\tau\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta}\right);\\
\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau} &= \frac{1}{\upsilon''\upsilon^3}\left(-\upsilon^2\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta} + \tau\upsilon\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha} + \upsilon\sigma\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma} - \sigma\tau\frac{\delta^2\upsilon}{\delta\gamma^2}\right);\\
\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon} &= \frac{1}{\upsilon''\upsilon^3}\left(-\sigma^2\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma} + \upsilon\sigma\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta} + \sigma\tau\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha} - \tau\upsilon\frac{\delta^2\upsilon}{\delta\alpha^2}\right);\\
\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma} &= \frac{1}{\upsilon''\upsilon^3}\left(-\tau^2\frac{\delta^2\upsilon}{\delta\gamma\,\delta\alpha} + \sigma\tau\frac{\delta^2\upsilon}{\delta\beta\,\delta\gamma} + \tau\upsilon\frac{\delta^2\upsilon}{\delta\alpha\,\delta\beta} - \upsilon\sigma\frac{\delta^2\upsilon}{\delta\beta^2}\right).
\end{aligned}\right\} \tag{N^6}$$

To make more complete this theory of the coefficients of the function Ω, which determines the nature of the final uniform or variable medium by the manner of its dependence on the seven variables $\sigma\ \tau\ \upsilon\ x\ y\ z\ \chi$, and is supposed to have been so prepared that $\Omega + 1$ is homogeneous of the first dimension relatively to $\sigma\ \tau\ \upsilon$, let us investigate the connexion of these coefficients of Ω with those of the simpler though less symmetric function υ, considered as depending on the six other variables $\sigma\ \tau\ x\ y\ z\ \chi$ by the relation $\Omega = 0$. For this purpose we are to combine the differentials of that relation with the conditions of homogeneity (B^4) (C^4), and with the following other conditions of the same kind, which are only useful in variable media,

$$\left.\begin{aligned}
\sigma\frac{\delta^2\Omega}{\delta\sigma\,\delta x} + \tau\frac{\delta^2\Omega}{\delta\tau\,\delta x} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\,\delta x} &= \frac{\delta\Omega}{\delta x},\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\,\delta y} + \tau\frac{\delta^2\Omega}{\delta\tau\,\delta y} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\,\delta y} &= \frac{\delta\Omega}{\delta y},\\
\sigma\frac{\delta^2\Omega}{\delta\sigma\,\delta z} + \tau\frac{\delta^2\Omega}{\delta\tau\,\delta z} + \upsilon\frac{\delta^2\Omega}{\delta\upsilon\,\delta z} &= \frac{\delta\Omega}{\delta z}.
\end{aligned}\right\} \tag{O^6}$$

In this manner we find, for the first order,

$$\delta\Omega = \lambda\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \frac{\delta\upsilon}{\delta\tau}\,\delta\tau - \frac{\delta\upsilon}{\delta x}\,x - \frac{\delta\upsilon}{\delta y}\,y - \frac{\delta\upsilon}{\delta z}\,z - \frac{\delta\upsilon}{\delta\chi}\,\delta\chi\right); \tag{P^6}$$

that is

$$\left.\begin{aligned}
&\frac{\delta\Omega}{\delta\sigma} = -\lambda\frac{\delta\upsilon}{\delta\sigma};\quad \frac{\delta\Omega}{\delta\tau} = -\lambda\frac{\delta\upsilon}{\delta\tau};\quad \frac{\delta\Omega}{\delta\upsilon} = \lambda;\\
&\frac{\delta\Omega}{\delta x} = -\lambda\frac{\delta\upsilon}{\delta x};\quad \frac{\delta\Omega}{\delta y} = -\lambda\frac{\delta\upsilon}{\delta y};\quad \frac{\delta\Omega}{\delta z} = -\lambda\frac{\delta\upsilon}{\delta z};\quad \frac{\delta\Omega}{\delta\chi} = -\lambda\frac{\delta\upsilon}{\delta\chi};
\end{aligned}\right\} \tag{Q^6}$$

λ being a multiplier introduced for the purpose of treating the variations of $\sigma\ \tau\ \upsilon\ x\ y\ z\ \chi$ as independent; and to determine the value of this multiplier we have, by the condition of homogeneity (B^4),

$$\lambda\left(\upsilon - \sigma\frac{\delta\upsilon}{\delta\sigma} - \tau\frac{\delta\upsilon}{\delta\tau}\right) = \Omega + 1 = 1: \tag{R^6}$$

the coefficients of Ω of the first order are therefore known, and we have for example,

$$\frac{\delta\Omega}{\delta\upsilon} = \lambda = \frac{1}{\upsilon - \sigma\dfrac{\delta\upsilon}{\delta\sigma} - \tau\dfrac{\delta\upsilon}{\delta\tau}}. \tag{S^6}$$

Again, for the second order,

$$\left.\begin{aligned}
\delta\frac{\delta\Omega}{\delta\sigma} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta\sigma} + \lambda_1\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta\tau} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta\tau} + \lambda_2\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta\upsilon} &= \quad\delta\lambda \qquad + \lambda_3\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta x} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta x} + \lambda_4\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta y} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta y} + \lambda_5\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta z} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta z} + \lambda_6\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);\\
\delta\frac{\delta\Omega}{\delta\chi} &= -\delta\,.\,\lambda\frac{\delta\upsilon}{\delta\chi} + \lambda_7\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \&\text{c.}\right);
\end{aligned}\right\} \tag{T^6}$$

in which, by (C^4) (O^6) (Q^6), the multipliers $\lambda_1, \ldots\, \lambda_7$, have the following values,

$$\left.\begin{aligned}
\lambda_1 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta\sigma};\\
\lambda_2 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta\tau};\\
\lambda_3 &= -\lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right)\lambda;\\
\lambda_4 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta x} - \lambda^2\frac{\delta\upsilon}{\delta x};\\
\lambda_5 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta y} - \lambda^2\frac{\delta\upsilon}{\delta y};\\
\lambda_6 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta z} - \lambda^2\frac{\delta\upsilon}{\delta z};\\
\lambda_7 &= \lambda\left(\sigma\frac{\delta}{\delta\sigma} + \tau\frac{\delta}{\delta\tau}\right).\,\lambda\frac{\delta\upsilon}{\delta\chi} - \lambda^2\frac{\delta\upsilon}{\delta\chi};
\end{aligned}\right\} \tag{U^6}$$

λ, like υ, being here treated as a function of σ, τ, x, y, z, χ: and if we put, as usual,

$$\delta^2\Omega = \delta\sigma\,\delta\frac{\delta\Omega}{\delta\sigma} + \delta\tau\,\delta\frac{\delta\Omega}{\delta\tau} + \delta\upsilon\,\delta\frac{\delta\Omega}{\delta\upsilon} + \delta x\,\delta\frac{\delta\Omega}{\delta x} + \delta y\,\delta\frac{\delta\Omega}{\delta y} + \delta z\,\delta\frac{\delta\Omega}{\delta z} + \delta\chi\,\delta\frac{\delta\Omega}{\delta\chi}, \tag{V^6}$$

and similarly

$$\delta^2 \upsilon = \delta\sigma\,\delta\frac{\delta\upsilon}{\delta\sigma} + \delta\tau\,\delta\frac{\delta\upsilon}{\delta\tau} + \delta x\,\delta\frac{\delta\upsilon}{\delta x} + \delta y\,\delta\frac{\delta\upsilon}{\delta y} + \delta z\,\delta\frac{\delta\upsilon}{\delta z} + \delta\chi\,\delta\frac{\delta\upsilon}{\delta\chi}, \tag{W^6}$$

we find

$$\begin{aligned}\delta^2\Omega = {} & -\lambda\,\delta^2\upsilon \\ & + 2\,\delta\lambda\,.\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \frac{\delta\upsilon}{\delta\tau}\,\delta\tau - \frac{\delta\upsilon}{\delta x}\,\delta x - \frac{\delta\upsilon}{\delta y}\,\delta y - \frac{\delta\upsilon}{\delta z}\,\delta z - \frac{\delta\upsilon}{\delta\chi}\,\delta\chi\right) \\ & - \lambda^3\left(\sigma^2\frac{\delta^2\upsilon}{\delta\sigma^2} + 2\sigma\tau\frac{\delta^2\upsilon}{\delta\sigma\,\delta\tau} + \tau^2\frac{\delta^2\upsilon}{\delta\tau^2}\right)\left(\delta\upsilon - \frac{\delta\upsilon}{\delta\sigma}\,\delta\sigma - \text{\&c.}\right)^2,\end{aligned} \tag{X^6}$$

in which

$$\delta\lambda = \lambda^2\left(\sigma\,\delta\frac{\delta\upsilon}{\delta\sigma} + \tau\,\delta\frac{\delta\upsilon}{\delta\tau} - \frac{\delta\upsilon}{\delta x}\,\delta x - \frac{\delta\upsilon}{\delta y}\,\delta y - \frac{\delta\upsilon}{\delta z}\,\delta z - \frac{\delta\upsilon}{\delta\chi}\,\delta\chi\right), \tag{Y^6}$$

and which is equivalent to twenty-eight expressions for the partial differential coefficients of Ω of the second order: it gives, for example,

$$\frac{\delta^2\Omega}{\delta v^2} = \frac{\sigma^2\dfrac{\delta^2\upsilon}{\delta\sigma^2} + 2\sigma\tau\dfrac{\delta^2\upsilon}{\delta\sigma\,\delta\tau} + \tau^2\dfrac{\delta^2\upsilon}{\delta\tau^2}}{\left(\sigma\dfrac{\delta\upsilon}{\delta\sigma} + \tau\dfrac{\delta\upsilon}{\delta\tau} - \upsilon\right)^3}. \tag{Z^6}$$

And since the forms of the connected functions Ω, υ, v, of which each expresses the optical properties of the final medium, may be deduced, by the method of the second number, from the form of the characteristic function V, it is evident that their partial differential coefficients also, of all orders, are not only related to each other, but may be deduced from the coefficients of that one characteristic function.

General Formula for Reflection or Refraction, Ordinary or Extraordinary. Changes of V, W, T. The Difference ΔV is $= 0$; $\Delta W = \Delta T =$ a Homogeneous Function of the First Dimension of the Differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, depending on the Shape and Position of the Reflecting or Refracting Surface. Theorem of Maxima and Minima, for the Elimination of the Incident Variables. Combinations of Reflectors or Refractors. Compound and Component Combinations.

11. Let us now endeavour to improve our theory of the characteristic and related functions, by applying the methods of the present memoir to improve the determination given in the First Supplement, of the sudden changes produced in these functions and in their coefficients, by reflexion or refraction, ordinary or extraordinary.

The general formula of such changes, which easily results from the nature of the characteristic function V, is

$$0 = \Delta V = V_2 - V_1; \tag{A^7}$$

V_1, V_2, being the two successive forms of the function V, before and after the reflexion or refraction; and the final co-ordinates x, y, z, in these forms, being connected by the equation

$$0 = u(x, y, z) \tag{B^7}$$

of the reflecting or refracting surface. The formula (A^7) may be differentiated any number of times with reference to the final and initial co-ordinates and the colour, attending to the relation (B^7); and such differentiation, combined with the properties of the final uniform or variable media, conducts to the general laws of reflexion and refraction, and to all the conditions necessary for determining the changes of the coefficients of V, and therefore also of the connected coefficients of W and T, as well as to the laws of change of the functions V, W, T, themselves.

Thus, for the first order, we have the general formula

$$\delta V_2 - \delta V_1 = \delta \Delta V = \lambda\, \delta u, \tag{C^7}$$

which, on account of the multiplier λ, and the definitions (E), resolves itself into the seven following,

$$\left.\begin{array}{c} \Delta\sigma = \lambda\dfrac{\delta u}{\delta x}; \quad \Delta\tau = \lambda\dfrac{\delta u}{\delta y}; \quad \Delta\upsilon = \lambda\dfrac{\delta u}{\delta z}; \\ \Delta\sigma' = 0; \quad \Delta\tau' = 0; \quad \Delta\upsilon' = 0; \quad \Delta\dfrac{\delta V}{\delta\chi} = 0: \end{array}\right\} \tag{D^7}$$

the symbol Δ referring, as in (A^7), to the finite changes produced at the surface (B^7), so that $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, denote the differences $\sigma_2 - \sigma_1$, $\tau_2 - \tau_1$, $\upsilon_2 - \upsilon_1$, between the new and the old values of σ, τ, υ, that is, of the partial differential coefficients of the first order, of the characteristic function V, taken with respect to the final co-ordinates. The three first of the equations (D^7) contain the general laws of the sudden reflexion or refraction of a straight or curved ray, ordinary or extraordinary; because, when combined with the equation of the form (F),

$$0 = \Omega_2(\sigma_2, \tau_2, \upsilon_2, x, y, z, \chi), \tag{E^7}$$

which expresses the nature of the final medium, they suffice, in general, when that final medium is known, to determine, or at least to restrict to a finite variety, the new values σ_2, τ_2, υ_2, of the quantities σ, τ, υ, on which the direction of the reflected or refracted ray depends, if we know the old values σ_1 τ_1 υ_1, which depend on the direction of the incident ray and on the properties of the medium containing it, and if we know also χ, x, y, z, and the ratios of $\dfrac{\delta u}{\delta x}$, $\dfrac{\delta u}{\delta y}$, $\dfrac{\delta u}{\delta z}$, that is, the colour, the point of incidence, and the normal to the reflecting or refracting surface at that point. A remarkable case of indeterminateness, however, or rather two such cases, will appear, when we come to treat, in a future number, of external and internal conical refraction.

With respect to the new form V_2 of the characteristic function V, it is to be determined by the two following conditions; first, by the condition of satisfying, at the surface (B^7), the equation in finite differences (A^7), that is, by the condition of becoming equal to the value of the old form V_1, when the final co-ordinates x, y, z, are connected by the relation $u = 0$;

and secondly by the condition of satisfying, when the final co-ordinates are considered as arbitrary, the partial differential equation of the form (C),

$$0 = \Omega_2 \left(\frac{\delta V_2}{\delta x}, \frac{\delta V_2}{\delta y}, \frac{\delta V_2}{\delta z}, x, y, z, \chi \right), \tag{F^7}$$

if the final medium be variable, or the simpler partial differential equation of the form (V′), if that final medium be uniform. And as it has been already shown that the partial differential equations relative to the characteristic function V, may be transformed, and in the case of uniform media integrated, by the help of the auxiliary functions W, T, it is useful to consider here the changes of those auxiliary functions, which are also otherwise interesting.

It easily follows from the definitions of W, T, that the increments of these two functions, acquired in reflexion or refraction, are equal to each other, and may be thus expressed,

$$\Delta W = \Delta T = x\,\Delta\sigma + y\,\Delta\tau + z\,\Delta\upsilon. \tag{G^7}$$

And because the differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are, by the general equations of reflexion or refraction (D^7), proportional to $\dfrac{\delta u}{\delta x}$, $\dfrac{\delta u}{\delta y}$, $\dfrac{\delta u}{\delta z}$, we may consider these differences as equal to the projections, on the rectangular axes of the co-ordinates x, y, z, of a straight line $= \sqrt{(\Delta\sigma^2 + \Delta\tau^2 + \Delta\upsilon^2)}$, perpendicular to the reflecting or refracting surface at the point of incidence, and making with the axes of co-ordinates angles of which the cosines may be called n_x, n_y, n_z; in such a manner that we shall have

$$\left.\begin{aligned}
\Delta\sigma &= n_x\sqrt{(\Delta\sigma^2 + \Delta\tau^2 + \Delta\upsilon^2)};\\
\Delta\tau &= n_y\sqrt{(\Delta\sigma^2 + \Delta\tau^2 + \Delta\upsilon^2)};\\
\Delta\upsilon &= n_z\sqrt{(\Delta\sigma^2 + \Delta\tau^2 + \Delta\upsilon^2)};\\
\Delta W = \Delta T &= (xn_x + yn_y + zn_z)\sqrt{(\Delta\sigma^2 + \Delta\tau^2 + \Delta\upsilon^2)}.
\end{aligned}\right\} \tag{H^7}$$

Now the quantity $xn_x + yn_y + zn_z$ is equal, abstracting from sign, to the perpendicular let fall from the origin of co-ordinates on the plane which touches the reflecting or refracting surface at the point of incidence; it is therefore constant if that surface be plane, and in general it may be considered as a function of the ratios of $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, because when those ratios are given we know the direction of the normal, and therefore, if the surface be curved and given, we know the point of incidence, or at least can in general restrict that point to a finite number of positions: we have therefore in general

$$\Delta W = \Delta T = f(\Delta\sigma, \Delta\tau, \Delta\upsilon), \tag{I^7}$$

the function f being homogeneous of the first dimension, and depending for its form on the shape and position of the reflecting or refracting surface, from the equation (B^7) of which surface it is to be deduced, by eliminating $x\ y\ z\ \lambda$ between the equations (B^7) (G^7) and the three first of those marked (D^7). We have also

$$\left.\begin{aligned}
\frac{f}{\Delta\upsilon} = \phi\left(\frac{\Delta\sigma}{\Delta\upsilon}, \frac{\Delta\tau}{\Delta\upsilon}\right); \quad \frac{\Delta\sigma}{\Delta\upsilon} = -\frac{\delta z}{\delta x}; \quad \frac{\Delta\tau}{\Delta\upsilon} = -\frac{\delta z}{\delta y};\\
z - x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y} = \phi\left(-\frac{\delta z}{\delta x}, -\frac{\delta z}{\delta y}\right);
\end{aligned}\right\} \tag{K^7}$$

the form therefore of the homogeneous function f may easily be deduced from the equation of the surface (B^7), by so preparing that equation as to express $z - x\dfrac{\delta z}{\delta x} - y\dfrac{\delta z}{\delta y}$ as a function ϕ of $-\dfrac{\delta z}{\delta x}$, $-\dfrac{\delta z}{\delta y}$, which function ϕ reduces itself to a constant when the surface is plane: and we have a simple expression for the variation of the homogeneous function f, namely

$$\delta f = x\,\delta\Delta\sigma + y\,\delta\Delta\tau + z\,\delta\Delta\upsilon, \tag{L^7}$$

which, when the reflecting or refracting surface is curved, resolves itself into the following remarkable expressions for the co-ordinates of the point of incidence,

$$x = \frac{\delta f}{\delta\Delta\sigma}, \quad y = \frac{\delta f}{\delta\Delta\tau}, \quad z = \frac{\delta f}{\delta\Delta\upsilon}; \tag{M^7}$$

so that these co-ordinates, which, for a curved surface, we knew before to be functions of the ratios $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are now seen to be, for such a surface, the partial differential coefficients of the homogeneous function f. When the surface (B^7) is plane, the differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, are no longer independent, since their ratios are then given; and although the expression (L^7) for δf still holds, it no longer resolves itself into the three equations (M^7).

Having thus studied some of the chief properties of the common increment f, which the functions W, T, receive, in the act of reflexion or refraction, we are prepared to investigate the new forms W_2, T_2 of these functions W, T, considered as depending on the new quantities σ_2, τ_2, υ_2, instead of the old σ_1, τ_1, υ_1. For this purpose we have first the equations

$$\left.\begin{aligned} W_2 &= W_1 + f(\sigma_2 - \sigma_1, \tau_2 - \tau_1, \upsilon_2 - \upsilon_1),\\ T_2 &= T_1 + f(\sigma_2 - \sigma_1, \tau_2 - \tau_1, \upsilon_2 - \upsilon_1), \end{aligned}\right\} \tag{N^7}$$

by which W_2, T_2, at the reflecting or refracting surface, are expressed as explicit functions of σ_1 τ_1 υ_1 σ_2 τ_2 υ_2; the expression of W_2 involving also x' y' z' χ, and the expression of T_2 involving σ' τ' υ' χ: and to eliminate from these expressions the incident quantities σ_1 τ_1 υ_1 we have, if the surface be curved, the following equations, in which the symbol $\delta_{\sigma_1,\tau_1,\upsilon_1}$ refers to the variation of those incident quantities,

$$\left.\begin{aligned} \delta_{\sigma_1,\tau_1,\upsilon_1} . f = -x\,\delta\sigma_1 - y\,\delta\tau_1 - z\,\delta\upsilon_1 = -\delta_{\sigma_1,\tau_1,\upsilon_1} . W_1 = -\delta_{\sigma_1,\tau_1,\upsilon_1} . T_1;\\ \text{and} \because \quad \delta_{\sigma_1,\tau_1,\upsilon_1} . W_2 = 0; \quad \delta_{\sigma_1,\tau_1,\upsilon_1} . T_2 = 0; \end{aligned}\right\} \tag{O^7}$$

we are therefore to disengage the incident quantities from the expressions for W_2, T_2, by making each of those expressions a maximum or minimum with respect to those quantities, attending to the relation $\Omega_1 = 0$, between them; the phrase *maximum or minimum* being employed with the usual latitude. For the case of a plane surface this method of elimination fails, the form of f becoming indeterminate, on account of the constant ratios which then exist, by (K^7) or (D^7), between $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$; but these very ratios, combined with the relation $\Omega_1 = 0$, between the quantities σ_1 τ_1 υ_1, enable us in this case to eliminate those quantities from W_2, T_2. And when we have thus determined the new forms W_2, T_2, of the functions W, T, for the points of the reflecting or refracting surface, we may extend these forms to the other points of the final medium, if that medium be uniform, because then the final rays are straight, and for any one such ray the quantities σ_2 τ_2 υ_2 W_2 T_2, are constant; but if the final medium be variable, then the final rays are curved, and the general forms of W_2, T_2, for arbitrary points of the medium, are to be determined by combinations of partial differential equations and equations in finite differences, analogous to the combinations of such equations in V_2, and easily deduced from the principles already laid down.

It is easy to extend the foregoing remarks to any combination of reflexions or refractions, and to show, for example, that in the case of any combination of uniform media, producing any system of polygon rays, ordinary or extraordinary, the auxiliary function T is equal to the following expression,

$$T = \Sigma f(\Delta\sigma, \Delta\tau, \Delta\upsilon), \tag{P^7}$$

that is, to the sum of all the homogeneous functions f of the differences of the quantities σ, τ, υ, obtained by considering the successive reflecting or refracting surfaces: from which expression the intermediate quantities of the form σ, τ, υ, are to be eliminated by making the expression a maximum or minimum with respect to those intermediate quantities, attending to the relations between them which result from the properties of the media, and using, for plane surfaces, the other method of elimination, founded on the ratios of $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$. And when the function T is known, we can deduce from it, by the methods of the fourth number, the other auxiliary function W, and the characteristic function V.

In general, for all optical combinations, whether with uniform or with variable media, we have, by the definitions of the functions V, W, T, and by the results of former numbers, the following expressions,

$$\left.\begin{aligned} V = \int_0^s \upsilon\, ds; \quad T = \int_0^s \left(x\frac{\delta\upsilon}{\delta x} + y\frac{\delta\upsilon}{\delta y} + z\frac{\delta\upsilon}{\delta z} \right) ds; \\ W = x'\sigma' + y'\tau' + z'\upsilon' + \int_0^s \left(x\frac{\delta\upsilon}{\delta x} + y\frac{\delta\upsilon}{\delta y} + z\frac{\delta\upsilon}{\delta z} \right) ds : \end{aligned}\right\} \tag{Q^7}$$

ds being, as before, the element of the curved or polygon ray; and hence it follows that if we consider any total combination, of $m+n-1$ media, whether uniform or variable, as resulting from two partial combinations, of m and of n media respectively, combined so that the last medium of the one partial combination (m) is the first of the other partial combination (n), and so that the final rays of the one partial combination are the initial rays of the other, then the functions V, T, (but not in general W,) for the total combination, are the sums of the corresponding functions for the partial combinations: it follows also from the general expressions for the variations of these functions, that the intermediate variables, belonging to the last medium of the first partial combination, or to the first medium of the second, are to be eliminated from the sum, by the condition of making that sum a maximum or minimum with respect to them. Analogous remarks apply to compound combinations, composed of more than two component combinations. These properties of the functions V, T, for total or resultant combinations, will be useful in the theory of double and triple object-glasses, and other compound optical instruments.

Changes of the Coefficients of the Second Order, of V, W, T, produced by Reflexion or Refraction.

12. With respect to the changes produced by reflexion or refraction in the coefficients of the second order, of the characteristic function V, and therefore also of the connected functions W, T, they may be deduced from the following formula, analogous to (C^7),

$$\delta^2 \Delta V = \delta^2 \,.\, \lambda u = \lambda\, \delta^2 u + 2\, \delta\lambda\, \delta u; \tag{R^7}$$

u, λ, having the same meanings as in (B^7) (C^7); and the multiplier λ, which was introduced also in the First Supplement, and was there regarded as a function of the final co-ordinates x, y, z, being now considered as involving also the initial co-ordinates x', y', z', and the chromatic index χ. The seven variations δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, may be treated as independent in (R^7), if we assign a proper value to $\delta\lambda$, as a linear function of these seven variations; so that we may deduce from (R^7) the seven following equations,

$$\left.\begin{aligned}
&\Delta\delta\frac{\delta V}{\delta x} = \lambda\,\delta\frac{\delta u}{\delta x} + \frac{\delta\lambda}{\delta x}\,\delta u + \frac{\delta u}{\delta x}\,\delta\lambda;\\
&\Delta\delta\frac{\delta V}{\delta y} = \lambda\,\delta\frac{\delta u}{\delta y} + \frac{\delta\lambda}{\delta y}\,\delta u + \frac{\delta u}{\delta y}\,\delta\lambda;\\
&\Delta\delta\frac{\delta V}{\delta z} = \lambda\,\delta\frac{\delta u}{\delta z} + \frac{\delta\lambda}{\delta z}\,\delta u + \frac{\delta u}{\delta z}\,\delta\lambda;\\
&\Delta\delta\frac{\delta V}{\delta x'} = \frac{\delta\lambda}{\delta x'}\,\delta u; \quad \Delta\delta\frac{\delta V}{\delta y'} = \frac{\delta\lambda}{\delta y'}\,\delta u; \quad \Delta\delta\frac{\delta V}{\delta z'} = \frac{\delta\lambda}{\delta z'}\,\delta u;\\
&\Delta\delta\frac{\delta V}{\delta \chi} = \frac{\delta\lambda}{\delta \chi}\,\delta u :
\end{aligned}\right\} \tag{S^7}$$

of which each may again be decomposed into seven others. But of the forty-nine expressions thus obtained for the changes of the twenty-eight coefficients of V of the second order, only twenty-eight expressions are distinct; and these involve seven multipliers as yet unknown, namely, the seven partial differential coefficients of λ: however we can determine these seven multipliers, and the twenty-eight coefficients of V_2 of the second order, by introducing the seven additional equations obtained by differentiating the partial differential equation (F^7), with respect to $x\ y\ z\ x'\ y'\ z'\ \chi$.

The differential of the equation (F^7) is

$$0 = \frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\frac{\delta V_2}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\frac{\delta V_2}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\frac{\delta V_2}{\delta z} + \frac{\delta\Omega_2}{\delta x}\,\delta x + \frac{\delta\Omega_2}{\delta y}\,\delta y + \frac{\delta\Omega_2}{\delta z}\,\delta z + \frac{\delta\Omega_2}{\delta\chi}\,\delta\chi; \tag{T^7}$$

and this, when combined with the three first equations (S^7), conducts to the following formula,

$$\begin{aligned}
0 = {}& \frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\frac{\delta V_1}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\frac{\delta V_1}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\frac{\delta V_1}{\delta z} + \frac{\delta\Omega_2}{\delta x}\,\delta x + \frac{\delta\Omega_2}{\delta y}\,\delta y + \frac{\delta\Omega_2}{\delta z}\,\delta z + \frac{\delta\Omega_2}{\delta\chi}\,\delta\chi\\
&+ \lambda\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\delta\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\delta\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\delta\frac{\delta u}{\delta z}\right)\\
&+ \delta u \,.\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta\lambda}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta\lambda}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta\lambda}{\delta z}\right)\\
&+ \delta\lambda \,.\left(\frac{\delta\Omega_2}{\delta\sigma_2}\,\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\,\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\,\frac{\delta u}{\delta z}\right);
\end{aligned} \tag{U^7}$$

which resolves itself into seven separate equations, sufficient to determine the seven multipliers

$$\frac{\delta\lambda}{\delta x}, \quad \frac{\delta\lambda}{\delta y}, \quad \frac{\delta\lambda}{\delta z}, \quad \frac{\delta\lambda}{\delta x'}, \quad \frac{\delta\lambda}{\delta y'}, \quad \frac{\delta\lambda}{\delta z'}, \quad \frac{\delta\lambda}{\delta \chi}.$$

Three of these seven equations into which (U^7) resolves itself, give, by a proper combination, a value for the trinomial

$$\frac{\delta\Omega_2}{\delta\sigma_2}\frac{\delta\lambda}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\frac{\delta\lambda}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\frac{\delta\lambda}{\delta z},$$

which enables us to eliminate that trinomial from (U^7) and so to deduce a value for $\delta\lambda$, which being combined with (R^7) gives,

$$\begin{aligned}
&\left\{\begin{aligned}
&\left(\frac{\delta\Omega_2}{\delta\sigma_2}\right)^2\left(\frac{\delta^2 V_1}{\delta x^2} + \lambda\frac{\delta^2 u}{\delta x^2}\right) + 2\frac{\delta\Omega_2}{\delta\tau_2}\frac{\delta\Omega_2}{\delta\upsilon_2}\left(\frac{\delta^2 V_1}{\delta y\,\delta z} + \lambda\frac{\delta^2 u}{\delta y\,\delta z}\right)\\
&+\left(\frac{\delta\Omega_2}{\delta\tau_2}\right)^2\left(\frac{\delta^2 V_1}{\delta y^2} + \lambda\frac{\delta^2 u}{\delta y^2}\right) + 2\frac{\delta\Omega_2}{\delta\upsilon_2}\frac{\delta\Omega_2}{\delta\sigma_2}\left(\frac{\delta^2 V_1}{\delta z\,\delta x} + \lambda\frac{\delta^2 u}{\delta z\,\delta x}\right)\\
&+\left(\frac{\delta\Omega_2}{\delta\upsilon_2}\right)^2\left(\frac{\delta^2 V_1}{\delta z^2} + \lambda\frac{\delta^2 u}{\delta z^2}\right) + 2\frac{\delta\Omega_2}{\delta\sigma_2}\frac{\delta\Omega_2}{\delta\tau_2}\left(\frac{\delta^2 V_1}{\delta x\,\delta y} + \lambda\frac{\delta^2 u}{\delta x\,\delta y}\right)\\
&+\frac{\delta\Omega_2}{\delta\sigma_2}\frac{\delta\Omega_2}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\frac{\delta\Omega_2}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\frac{\delta\Omega_2}{\delta z}
\end{aligned}\right\} . \,\delta u^2\\
&-2\left(\frac{\delta\Omega_2}{\delta\sigma_2}\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\frac{\delta u}{\delta z}\right)\delta u\\
&\times\left\{\begin{aligned}
&\frac{\delta\Omega_2}{\delta\sigma_2}\left(\delta\frac{\delta V_1}{\delta x} + \lambda\,\delta\frac{\delta u}{\delta x}\right) + \frac{\delta\Omega_2}{\delta\tau_2}\left(\delta\frac{\delta V_1}{\delta y} + \lambda\,\delta\frac{\delta u}{\delta y}\right) + \frac{\delta\Omega_2}{\delta\upsilon_2}\left(\delta\frac{\delta V_1}{\delta z} + \lambda\,\delta\frac{\delta u}{\delta z}\right)\\
&+\frac{\delta\Omega_2}{\delta x}\,\delta x + \frac{\delta\Omega_2}{\delta y}\,\delta y + \frac{\delta\Omega_2}{\delta z}\,\delta z + \frac{\delta\Omega_2}{\delta\chi}\,\delta\chi
\end{aligned}\right\}\\
&= (\delta^2 V_2 - \delta^2 V_1 - \lambda\,\delta^2 u)\left(\frac{\delta\Omega_2}{\delta\sigma_2}\frac{\delta u}{\delta x} + \frac{\delta\Omega_2}{\delta\tau_2}\frac{\delta u}{\delta y} + \frac{\delta\Omega_2}{\delta\upsilon_2}\frac{\delta u}{\delta z}\right)^2 : \qquad (\mathrm{V}^7)
\end{aligned}$$

a formula that is equivalent to twenty-eight separate expressions for the twenty-eight coefficients of V_2, of the second order. This formula supposes the rays to be reflected or refracted into a variable medium; but it can be adapted to the simpler supposition of reflexion or refraction into an uniform medium, by merely making the quantities $\frac{\delta\Omega_2}{\delta x}, \frac{\delta\Omega_2}{\delta y}, \frac{\delta\Omega_2}{\delta z}$, vanish. Whether the last medium be variable or uniform, the formula (V^7) gives,

$$\delta'^2 V_2 = \delta'^2 V_1; \qquad (\mathrm{W}^7)$$

δ' referring, as in former numbers of this Supplement, to the variations x', y', z', χ, alone, that is, to the variations of the initial co-ordinates and of the colour; and the final co-ordinates $x\ y\ z$ being those of any point on the reflecting or refracting surface. Thus the ten differential coefficients, of the second order, of the characteristic function V, like the four of the first order, taken with respect to the initial co-ordinates and the colour, undergo no sudden change by reflexion or refraction; but the differential coefficients of both orders, which involve the final co-ordinates, take suddenly new values which we have shown how to determine: and from these new coefficients of V, we can deduce those of W and T, by the methods of the foregoing numbers. The coefficients thus found, of W_2 and T_2, remain unchanged through the whole extent of the last reflected or refracted portion of the ray, when this last portion is straight, the final medium being uniform; but the coefficients of V_2, of the second order, change gradually in passing from one point to another, even of this straight portion, according to laws deducible from their connexion, already explained, with the constant coefficients of W_2.

The coefficients of W_2 and T_2 of the second and higher orders may also be calculated, whether the last medium be uniform or variable, by differentiating the expressions (N^7), and eliminating the variations of $\sigma_1\ \tau_1\ \upsilon_1$ by the help of the conditions already mentioned, of maximum or minimum.

Another method of calculating the changes produced in the partial differential coefficients of V of the second order, by reflexion or refraction, ordinary or extraordinary, into a medium uniform or variable, is to develope the second differential of the general formula (A^7), considering ΔV as a function of the seven variables x, y, z, x', y', z', χ, and considering x, y, z, as themselves functions of two independent variables; for example, considering z as a function of x, y, of which the form is determined by the equation of the reflecting or refracting surface. In this manner we obtain, besides the formula (W^7), which is equivalent to ten equations, the eleven following;

$$\left.\begin{array}{l}
0 = \dfrac{\delta^2 \Delta V}{\delta x^2} + 2\dfrac{\delta^2 \Delta V}{\delta x\,\delta z}\dfrac{\delta z}{\delta x} + \dfrac{\delta^2 \Delta V}{\delta z^2}\left(\dfrac{\delta z}{\delta x}\right)^2 + \dfrac{\delta \Delta V}{\delta z}\dfrac{\delta^2 z}{\delta x^2};\\
0 = \dfrac{\delta^2 \Delta V}{\delta y^2} + 2\dfrac{\delta^2 \Delta V}{\delta y\,\delta z}\dfrac{\delta z}{\delta y} + \dfrac{\delta^2 \Delta V}{\delta z^2}\left(\dfrac{\delta z}{\delta y}\right)^2 + \dfrac{\delta \Delta V}{\delta z}\dfrac{\delta^2 z}{\delta y^2};\\
0 = \dfrac{\delta^2 \Delta V}{\delta x\,\delta y} + \dfrac{\delta^2 \Delta V}{\delta x\,\delta z}\dfrac{\delta z}{\delta y} + \dfrac{\delta^2 \Delta V}{\delta y\,\delta z}\dfrac{\delta z}{\delta x} + \dfrac{\delta^2 \Delta V}{\delta z^2}\dfrac{\delta z}{\delta x}\dfrac{\delta z}{\delta y} + \dfrac{\delta \Delta V}{\delta z}\dfrac{\delta^2 z}{\delta x\,\delta y};\\
0 = \dfrac{\delta^2 \Delta V}{\delta x\,\delta x'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta x'}\dfrac{\delta z}{\delta x}; \quad 0 = \dfrac{\delta^2 \Delta V}{\delta y\,\delta x'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta x'}\dfrac{\delta z}{\delta y};\\
0 = \dfrac{\delta^2 \Delta V}{\delta x\,\delta y'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta y'}\dfrac{\delta z}{\delta x}; \quad 0 = \dfrac{\delta^2 \Delta V}{\delta y\,\delta y'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta y'}\dfrac{\delta z}{\delta y};\\
0 = \dfrac{\delta^2 \Delta V}{\delta x\,\delta z'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta z'}\dfrac{\delta z}{\delta x}; \quad 0 = \dfrac{\delta^2 \Delta V}{\delta y\,\delta z'} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta z'}\dfrac{\delta z}{\delta y};\\
0 = \dfrac{\delta^2 \Delta V}{\delta x\,\delta \chi} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta \chi}\dfrac{\delta z}{\delta x}; \quad 0 = \dfrac{\delta^2 \Delta V}{\delta y\,\delta \chi} + \dfrac{\delta^2 \Delta V}{\delta z\,\delta \chi}\dfrac{\delta z}{\delta y};
\end{array}\right\} \qquad (\mathrm{X}^7)$$

which may be put under the form

$$\left.\begin{array}{l}
0 = \Delta\left\{\dfrac{\delta^2 V}{\delta x^2} + 2\dfrac{\delta^2 V}{\delta x\,\delta z}\dfrac{\delta z}{\delta x} + \dfrac{\delta^2 V}{\delta z^2}\left(\dfrac{\delta z}{\delta x}\right)^2 + \dfrac{\delta V}{\delta z}\dfrac{\delta^2 z}{\delta x^2};\right\};\\
\quad \&\text{c.};
\end{array}\right\} \qquad (\mathrm{Y}^7)$$

and are deduced by differentiation from the analogous equations of the first order

$$0 = \Delta \left(\frac{\delta V}{\delta x} + \frac{\delta V}{\delta z} \frac{\delta z}{\delta x} \right); \quad 0 = \Delta \left(\frac{\delta V}{\delta y} + \frac{\delta V}{\delta z} \frac{\delta z}{\delta y} \right). \tag{Z^7}$$

And the eleven equations thus deduced, when combined with the ten given by (W^7), and with the seven into which (T^7) resolves itself, suffice, in general, to determine the twenty-eight coefficients of V_2 of the second order.

Changes produced by Transformation of Co-ordinates. Nearly all the foregoing Results may be extended to Oblique Co-ordinates. The Fundamental Formula may be presented so as to extend even to Polar or to any other marks of position, and new Auxiliary Functions may then be found, analogous to, and including, the Functions W, T: together with New and General Differential and Integral Equations for Curved and Polygon Rays, Ordinary or Extraordinary.

13. In all the foregoing investigations, it has been supposed that the final and initial co-ordinates, x, y, z, x', y', z', were referred to one common set of rectangular axes. But since it may be often convenient to change the mode of marking the final and initial positions, let us now express the old rectangular co-ordinates as linear functions of new and more general co-ordinates $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, and $x'_{\prime}$, $y'_{\prime}$, $z'_{\prime}$, which may or may not be rectangular, and may or may not be referred to one common set of final or initial axes. For this purpose we may employ the following formulæ,

$$\left.\begin{aligned} x &= x_0 + x_{x_\prime} x_\prime + x_{y_\prime} y_\prime + x_{z_\prime} z_\prime; \\ y &= y_0 + y_{x_\prime} x_\prime + y_{y_\prime} y_\prime + y_{z_\prime} z_\prime; \\ z &= z_0 + z_{x_\prime} x_\prime + z_{y_\prime} y_\prime + z_{z_\prime} z_\prime; \\ x' &= x'_0 + x'_{x'_\prime} x'_\prime + x'_{y'_\prime} y'_\prime + x'_{z'_\prime} z'_\prime; \\ y' &= y'_0 + y'_{x'_\prime} x'_\prime + y'_{y'_\prime} y'_\prime + y'_{z'_\prime} z'_\prime; \\ z' &= z'_0 + z'_{x'_\prime} x'_\prime + z'_{y'_\prime} y'_\prime + z'_{z'_\prime} z'_\prime; \end{aligned}\right\} \tag{A^8}$$

in which each of the eighteen coefficients of the form $x_{x_\prime}$ is the cosine of the angle between the directions of the two corresponding semiaxes, so that these coefficients are connected by the six following relations, on account of the rectangularity of the old co-ordinates,

$$\left.\begin{aligned} {x_{x_\prime}}^2 + {y_{x_\prime}}^2 + {z_{x_\prime}}^2 = 1; \quad {x'_{x'_\prime}}^2 + {y'_{x'_\prime}}^2 + {z'_{x'_\prime}}^2 = 1; \\ {x_{y_\prime}}^2 + {y_{y_\prime}}^2 + {z_{y_\prime}}^2 = 1; \quad {x'_{y'_\prime}}^2 + {y'_{y'_\prime}}^2 + {z'_{y'_\prime}}^2 = 1; \\ {x_{z_\prime}}^2 + {y_{z_\prime}}^2 + {z_{z_\prime}}^2 = 1; \quad {x'_{z'_\prime}}^2 + {y'_{z'_\prime}}^2 + {z'_{z'_\prime}}^2 = 1. \end{aligned}\right\} \tag{B^8}$$

Let us also establish, according to the analogy of our former notation, the following definitions similar to (P),

$$\left.\begin{aligned} \alpha_\prime = \frac{dx_\prime}{ds}, \quad \beta_\prime = \frac{dy_\prime}{ds}, \quad \gamma_\prime = \frac{dz_\prime}{ds}, \\ \alpha'_\prime = \frac{dx'_\prime}{ds'}, \quad \beta'_\prime = \frac{dy'_\prime}{ds'}, \quad \gamma'_\prime = \frac{dz'_\prime}{ds'}, \end{aligned}\right\} \tag{C^8}$$

and the following, similar to (E),

$$\left.\begin{aligned}\sigma_{\prime} &= \frac{\delta V}{\delta x_{\prime}}, \quad & \tau_{\prime} &= \frac{\delta V}{\delta y_{\prime}}, \quad & \upsilon_{\prime} &= \frac{\delta V}{\delta z_{\prime}},\\ \sigma'_{\prime} &= -\frac{\delta V}{\delta x'_{\prime}}, \quad & \tau'_{\prime} &= -\frac{\delta V}{\delta y'_{\prime}}, \quad & \upsilon'_{\prime} &= -\frac{\delta V}{\delta z'_{\prime}},\end{aligned}\right\}\tag{D^8}$$

we shall then have

$$\left.\begin{aligned}\alpha &= \alpha_{\prime} x_{x_{\prime}} + \beta_{\prime} x_{y_{\prime}} + \gamma_{\prime} x_{z_{\prime}};\\ \beta &= \alpha_{\prime} y_{x_{\prime}} + \beta_{\prime} y_{y_{\prime}} + \gamma_{\prime} y_{z_{\prime}};\\ \gamma &= \alpha_{\prime} z_{x_{\prime}} + \beta_{\prime} z_{y_{\prime}} + \gamma_{\prime} z_{z_{\prime}};\\ \alpha' &= \alpha'_{\prime} x'_{x'_{\prime}} + \beta'_{\prime} x'_{y'_{\prime}} + \gamma'_{\prime} x'_{z'_{\prime}};\\ \beta' &= \alpha'_{\prime} y'_{x'_{\prime}} + \beta'_{\prime} y'_{y'_{\prime}} + \gamma'_{\prime} y'_{z'_{\prime}};\\ \gamma' &= \alpha'_{\prime} z'_{x'_{\prime}} + \beta'_{\prime} z'_{y'_{\prime}} + \gamma'_{\prime} z'_{z'_{\prime}}:\end{aligned}\right\}\tag{E^8}$$

and

$$\left.\begin{aligned}\sigma_{\prime} &= \sigma x_{x_{\prime}} + \tau y_{x_{\prime}} + \upsilon z_{x_{\prime}}; \quad & \sigma'_{\prime} &= \sigma' x'_{x'_{\prime}} + \tau' y'_{x'_{\prime}} + \upsilon' z'_{x'_{\prime}};\\ \tau_{\prime} &= \sigma x_{y_{\prime}} + \tau y_{y_{\prime}} + \upsilon z_{y_{\prime}}; \quad & \tau'_{\prime} &= \sigma' x'_{y'_{\prime}} + \tau' y'_{y'_{\prime}} + \upsilon' z'_{y'_{\prime}};\\ \upsilon_{\prime} &= \sigma x_{z_{\prime}} + \tau y_{z_{\prime}} + \upsilon z_{z_{\prime}}; \quad & \upsilon'_{\prime} &= \sigma' x'_{z'_{\prime}} + \tau' y'_{z'_{\prime}} + \upsilon' z'_{z'_{\prime}}.\end{aligned}\right\}\tag{F^8}$$

And if, by substituting in the former homogeneous medium-functions, v, v', the expressions (E^8) for α, β, γ, α', β', γ', we obtain v under a new form, as a homogeneous function of $\alpha_{\prime}$, $\beta_{\prime}$, $\gamma_{\prime}$, of the first dimension, and v' as a homogeneous function of the same dimension of $\alpha'_{\prime}$, $\beta'_{\prime}$, $\gamma'_{\prime}$, and then differentiate these new forms of v, v', with reference to their new variables, we find, by (E^8), the following relations between the new and the old coefficients,

$$\left.\begin{aligned}\frac{\delta v}{\delta \alpha_{\prime}} &= \frac{\delta v}{\delta \alpha} x_{x_{\prime}} + \frac{\delta v}{\delta \beta} y_{x_{\prime}} + \frac{\delta v}{\delta \gamma} z_{x_{\prime}};\\ \frac{\delta v}{\delta \beta_{\prime}} &= \frac{\delta v}{\delta \alpha} x_{y_{\prime}} + \frac{\delta v}{\delta \beta} y_{y_{\prime}} + \frac{\delta v}{\delta \gamma} z_{y_{\prime}};\\ \frac{\delta v}{\delta \gamma_{\prime}} &= \frac{\delta v}{\delta \alpha} x_{z_{\prime}} + \frac{\delta v}{\delta \beta} y_{z_{\prime}} + \frac{\delta v}{\delta \gamma} z_{z_{\prime}};\\ \frac{\delta v'}{\delta \alpha'_{\prime}} &= \frac{\delta v'}{\delta \alpha'} x'_{x'_{\prime}} + \frac{\delta v'}{\delta \beta'} y'_{x'_{\prime}} + \frac{\delta v'}{\delta \gamma'} z'_{x'_{\prime}};\\ \frac{\delta v'}{\delta \beta'_{\prime}} &= \frac{\delta v'}{\delta \alpha'} x'_{y'_{\prime}} + \frac{\delta v'}{\delta \beta'} y'_{y'_{\prime}} + \frac{\delta v'}{\delta \gamma'} z'_{y'_{\prime}};\\ \frac{\delta v'}{\delta \gamma'_{\prime}} &= \frac{\delta v'}{\delta \alpha'} x'_{z'_{\prime}} + \frac{\delta v'}{\delta \beta'} y'_{z'_{\prime}} + \frac{\delta v'}{\delta \gamma'} z'_{z'_{\prime}};\end{aligned}\right\}\tag{G^8}$$

from which relations, combined with (D^8) (F^8), and with the equations (B) (E), of the second number, we obtain the following generalisations of the equations (B),

$$\left.\begin{aligned}\frac{\delta V}{\delta x_{\prime}} &= \frac{\delta v}{\delta \alpha_{\prime}}; \quad & \frac{\delta V}{\delta y_{\prime}} &= \frac{\delta v}{\delta \beta_{\prime}}; \quad & \frac{\delta V}{\delta z_{\prime}} &= \frac{\delta v}{\delta \gamma_{\prime}};\\ -\frac{\delta V}{\delta x'_{\prime}} &= \frac{\delta v'}{\delta \alpha'_{\prime}}; \quad & -\frac{\delta V}{\delta y'_{\prime}} &= \frac{\delta v'}{\delta \beta'_{\prime}}; \quad & -\frac{\delta V}{\delta z'_{\prime}} &= \frac{\delta v'}{\delta \gamma'_{\prime}}:\end{aligned}\right\}\tag{H^8}$$

and therefore the following important generalisation of the fundamental formula (A),

$$\delta V = \frac{\delta v}{\delta \alpha_\prime} \, \delta x_\prime - \frac{\delta v'}{\delta \alpha'_\prime} \, \delta x'_\prime + \frac{\delta v}{\delta \beta_\prime} \, \delta y_\prime - \frac{\delta v'}{\delta \beta'_\prime} \, \delta y'_\prime + \frac{\delta v}{\delta \gamma_\prime} \, \delta z_\prime - \frac{\delta v'}{\delta \gamma'_\prime} \, \delta z'_\prime, \tag{I^8}$$

which is thus shown to extend to oblique co-ordinates, and not even to require that the initial should coincide with the final axes.

We may adopt nearly all the foregoing reasonings and results, of the present Supplement, to this more general view. We have, for example, partial differential equations of the first order in V, analogous to the equations (C), and of the form

$$\left.\begin{aligned} 0 &= \Omega_\prime \left(\frac{\delta V}{\delta x_\prime}, \frac{\delta V}{\delta y_\prime}, \frac{\delta V}{\delta z_\prime}, x_\prime, y_\prime, z_\prime, \chi \right), \\ 0 &= \Omega'_\prime \left(-\frac{\delta V}{\delta x'_\prime}, -\frac{\delta V}{\delta y'_\prime}, -\frac{\delta V}{\delta z'_\prime}, x'_\prime, y'_\prime, z'_\prime, \chi \right), \end{aligned}\right\} \tag{K^8}$$

which conduct to a partial differential equation of the second order, analogous to (D): and if we put the equations (K^8) under the form

$$\left.\begin{aligned} 0 &= \Omega_\prime \left(\sigma_\prime, \tau_\prime, \upsilon_\prime, x_\prime, y_\prime, z_\prime, \chi \right), \\ 0 &= \Omega'_\prime \left(\sigma'_\prime, \tau'_\prime, \upsilon'_\prime, x'_\prime, y'_\prime, z'_\prime, \chi \right), \end{aligned}\right\} \tag{L^8}$$

and suppose them so prepared, by the method indicated in the second number, that the function $\Omega_\prime + 1$ shall be homogeneous of the first dimension with respect to $\sigma_\prime$, $\tau_\prime$, $\upsilon_\prime$, and that $\Omega'_\prime + 1$ shall be homogeneous of the same dimension with respect to $\sigma'_\prime$, $\tau'_\prime$, $\upsilon'_\prime$, we shall have

$$\left.\begin{aligned} \frac{\alpha_\prime}{v} &= \frac{\delta \Omega_\prime}{\delta \sigma_\prime}, & \frac{\beta_\prime}{v} &= \frac{\delta \Omega_\prime}{\delta \tau_\prime}, & \frac{\gamma_\prime}{v} &= \frac{\delta \Omega_\prime}{\delta \upsilon_\prime}, \\ \frac{\alpha'_\prime}{v'} &= \frac{\delta \Omega'_\prime}{\delta \sigma'_\prime}, & \frac{\beta'_\prime}{v'} &= \frac{\delta \Omega'_\prime}{\delta \tau'_\prime}, & \frac{\gamma'_\prime}{v'} &= \frac{\delta \Omega'_\prime}{\delta \upsilon'_\prime}, \end{aligned}\right\} \tag{M^8}$$

with many other relations, analogous to those of the second number. The differential equations of a curved ray, ordinary or extraordinary, in the third number, may be generalised as follows,

$$\frac{d}{ds} \frac{\delta v}{\delta \alpha_\prime} = \frac{\delta v}{\delta x_\prime}; \quad \frac{d}{ds} \frac{\delta v}{\delta \beta_\prime} = \frac{\delta v}{\delta y_\prime}; \quad \frac{d}{ds} \frac{\delta v}{\delta \gamma_\prime} = \frac{\delta v}{\delta z_\prime}; \tag{N^8}$$

and their integrals may be extended to oblique co-ordinates, under the form,

$$\frac{\delta V}{\delta x'_\prime} = \text{const.}; \quad \frac{\delta V}{\delta y'_\prime} = \text{const.}; \quad \frac{\delta V}{\delta z'_\prime} = \text{const.} : \tag{O^8}$$

while, if the final portion of the ray be straight, we have also, for that final portion,

$$\frac{\delta V}{\delta x_\prime} = \text{const.}; \quad \frac{\delta V}{\delta y_\prime} = \text{const.}; \quad \frac{\delta V}{\delta z_\prime} = \text{const.} \tag{P^8}$$

The formula (A^7) of reflexion or refraction, ordinary or extraordinary, namely

$$\Delta V = 0,$$

extends to oblique co-ordinates; and if we introduce new auxiliary functions $W_{\prime}$, $T_{\prime}$, analogous to W, T, and defined by the new equations

$$\left.\begin{aligned} W_{\prime} &= -V + x_{\prime}\sigma_{\prime} + y_{\prime}\tau_{\prime} + z_{\prime}\upsilon_{\prime}, \\ T_{\prime} &= W_{\prime} - x'_{\prime}\sigma'_{\prime} - y'_{\prime}\tau'_{\prime} - z'_{\prime}\upsilon'_{\prime}, \end{aligned}\right\} \tag{Q^8}$$

analogous to the definitions (D′) (E′), and attend to the meanings and properties of the symbols $\sigma_{\prime}$ $\tau_{\prime}$ $\upsilon_{\prime}$ $\sigma'_{\prime}$ $\tau'_{\prime}$ $\upsilon'_{\prime}$, we shall obtain the following expressions for the variations of V, $W_{\prime}$, $T_{\prime}$,

$$\left.\begin{aligned} \delta V &= \sigma_{\prime}\,\delta x_{\prime} - \sigma'_{\prime}\,\delta x'_{\prime} + \tau_{\prime}\,\delta y_{\prime} - \tau'_{\prime}\,\delta y'_{\prime} + \upsilon_{\prime}\,\delta z_{\prime} - \upsilon'_{\prime}\,\delta z'_{\prime} + \frac{\delta V}{\delta\chi}\,\delta\chi; \\ \delta W_{\prime} &= x_{\prime}\,\delta\sigma_{\prime} + \sigma'_{\prime}\,\delta x'_{\prime} + y_{\prime}\,\delta\tau_{\prime} + \tau'_{\prime}\,\delta y'_{\prime} + z_{\prime}\,\delta\upsilon_{\prime} + \upsilon'_{\prime}\,\delta z'_{\prime} - \frac{\delta V}{\delta\chi}\,\delta\chi; \\ \delta T_{\prime} &= x_{\prime}\,\delta\sigma_{\prime} - x'_{\prime}\,\delta\sigma'_{\prime} + y_{\prime}\,\delta\tau_{\prime} - y'_{\prime}\,\delta\tau'_{\prime} + z_{\prime}\,\delta\upsilon_{\prime} - z'_{\prime}\,\delta\upsilon'_{\prime} - \frac{\delta V}{\delta\chi}\,\delta\chi; \end{aligned}\right\} \tag{R^8}$$

which resemble the expressions (A′) (B′) (C′), and lead to analogous results. Thus, the partial differential coefficients of the new auxiliary functions $W_{\prime}$, $T_{\prime}$, may be deduced, by methods similar to those already employed, from the new coefficients of the characteristic function V, which may themselves be deduced from the old coefficients of that function, by the following general formula,

$$\left.\begin{aligned} &\left(\frac{\delta}{\delta x_{\prime}}\right)^{i}\left(\frac{\delta}{\delta y_{\prime}}\right)^{i'}\left(\frac{\delta}{\delta z_{\prime}}\right)^{i''}\left(\frac{\delta}{\delta x'_{\prime}}\right)^{k}\left(\frac{\delta}{\delta y'_{\prime}}\right)^{k'}\left(\frac{\delta}{\delta z'_{\prime}}\right)^{k''}\left(\frac{\delta}{\delta\chi}\right)^{l} V \\ &\quad= \left(x_{x'}\frac{\delta}{\delta x} + y_{x'}\frac{\delta}{\delta y} + z_{x'}\frac{\delta}{\delta z}\right)^{i}\left(x'_{x'_{\prime}}\frac{\delta}{\delta x'} + y'_{x'_{\prime}}\frac{\delta}{\delta y'} + z'_{x'_{\prime}}\frac{\delta}{\delta z'}\right)^{k} \\ &\qquad\left(x_{y'}\frac{\delta}{\delta x} + y_{y'}\frac{\delta}{\delta y} + z_{y'}\frac{\delta}{\delta z}\right)^{i'}\left(x'_{y'_{\prime}}\frac{\delta}{\delta x'} + y'_{y'_{\prime}}\frac{\delta}{\delta y'} + z'_{y'_{\prime}}\frac{\delta}{\delta z'}\right)^{k'} \\ &\qquad\left(x_{z'}\frac{\delta}{\delta x} + y_{z'}\frac{\delta}{\delta y} + z_{z'}\frac{\delta}{\delta z}\right)^{i''}\left(x'_{z'_{\prime}}\frac{\delta}{\delta x'} + y'_{z'_{\prime}}\frac{\delta}{\delta y'} + z'_{z'_{\prime}}\frac{\delta}{\delta z'}\right)^{k''}\frac{\delta^{l} V}{\delta\chi^{l}}: \end{aligned}\right\} \tag{S^8}$$

and the equations of a straight final ray may be put under the forms,

$$\left.\begin{aligned} \frac{1}{\alpha_{\prime}}\left(x_{\prime} - \frac{\delta W_{\prime}}{\delta\sigma_{\prime}}\right) &= \frac{1}{\beta_{\prime}}\left(y_{\prime} - \frac{\delta W_{\prime}}{\delta\tau_{\prime}}\right) = \frac{1}{\gamma_{\prime}}\left(z_{\prime} - \frac{\delta W_{\prime}}{\delta\upsilon_{\prime}}\right), \\ \frac{1}{\alpha_{\prime}}\left(x_{\prime} - \frac{\delta T_{\prime}}{\delta\sigma_{\prime}}\right) &= \frac{1}{\beta_{\prime}}\left(y_{\prime} - \frac{\delta T_{\prime}}{\delta\tau_{\prime}}\right) = \frac{1}{\gamma_{\prime}}\left(z_{\prime} - \frac{\delta T_{\prime}}{\delta\upsilon_{\prime}}\right), \end{aligned}\right\} \tag{T^8}$$

while those of a straight initial ray may be put under these other forms,

$$\frac{1}{\alpha'_{\prime}}\left(x'_{\prime} + \frac{\delta T_{\prime}}{\delta\sigma'_{\prime}}\right) = \frac{1}{\beta'_{\prime}}\left(y'_{\prime} + \frac{\delta T_{\prime}}{\delta\tau'_{\prime}}\right) = \frac{1}{\gamma'_{\prime}}\left(z'_{\prime} + \frac{\delta T_{\prime}}{\delta\upsilon'_{\prime}}\right); \tag{U^8}$$

these new equations (T^8) (U^8) being analogous to (I^2) and (P^2). It is evident that the arbitrary constants introduced by these transformations of co-ordinates must often assist to simplify the solution of optical problems. In the comparison, for example, of a given polygon ray, ordinary or extraordinary, of any given system, with other near rays of the same system, it will often be found convenient to choose the final portion of the given polygon ray for the axis of $z_{\prime}$, and the initial portion for the axis of $z'_{\prime}$, a choice which will make $\alpha_{\prime}$ $\beta_{\prime}$ $\alpha'_{\prime}$ $\beta'_{\prime}$ and many of the new partial differential coefficients vanish, without producing, by this simplification, any real loss of generality.

We may even carry these transformations farther, and introduce polar co-ordinates, or any other marks of initial and final position, and still obtain results having much analogy to the foregoing. For if we suppose that the final co-ordinates x, y, z are functions of any three quantities ρ, θ, ϕ, and that in like manner the initial co-ordinates x', y', z' are functions of any other three quantities ρ', θ', ϕ', so that

$$\left.\begin{aligned}
\delta x &= \frac{\delta x}{\delta\rho}\,\delta\rho + \frac{\delta x}{\delta\theta}\,\delta\theta + \frac{\delta x}{\delta\phi}\,\delta\phi, & dx &= \frac{\delta x}{\delta\rho}\,d\rho + \frac{\delta x}{\delta\theta}\,d\theta + \frac{\delta x}{\delta\phi}\,d\phi,\\
\delta y &= \frac{\delta y}{\delta\rho}\,\delta\rho + \frac{\delta y}{\delta\theta}\,\delta\theta + \frac{\delta y}{\delta\phi}\,\delta\phi, & dy &= \frac{\delta y}{\delta\rho}\,d\rho + \frac{\delta y}{\delta\theta}\,d\theta + \frac{\delta y}{\delta\phi}\,d\phi,\\
\delta z &= \frac{\delta z}{\delta\rho}\,\delta\rho + \frac{\delta z}{\delta\theta}\,\delta\theta + \frac{\delta z}{\delta\phi}\,\delta\phi, & dz &= \frac{\delta z}{\delta\rho}\,d\rho + \frac{\delta z}{\delta\theta}\,d\theta + \frac{\delta z}{\delta\phi}\,d\phi,\\
\delta x' &= \frac{\delta x'}{\delta\rho'}\,\delta\rho' + \frac{\delta x'}{\delta\theta'}\,\delta\theta' + \frac{\delta x'}{\delta\phi'}\,\delta\phi', & dx' &= \frac{\delta x'}{\delta\rho'}\,d\rho' + \frac{\delta x'}{\delta\theta'}\,d\theta' + \frac{\delta x'}{\delta\phi'}\,d\phi',\\
\delta y' &= \frac{\delta y'}{\delta\rho'}\,\delta\rho' + \frac{\delta y'}{\delta\theta'}\,\delta\theta' + \frac{\delta y'}{\delta\phi'}\,\delta\phi', & dy' &= \frac{\delta y'}{\delta\rho'}\,d\rho' + \frac{\delta y'}{\delta\theta'}\,d\theta' + \frac{\delta y'}{\delta\phi'}\,d\phi',\\
\delta z' &= \frac{\delta z'}{\delta\rho'}\,\delta\rho' + \frac{\delta z'}{\delta\theta'}\,\delta\theta' + \frac{\delta z'}{\delta\phi'}\,\delta\phi', & dz' &= \frac{\delta z'}{\delta\rho'}\,d\rho' + \frac{\delta z'}{\delta\theta'}\,d\theta' + \frac{\delta z'}{\delta\phi'}\,d\phi',
\end{aligned}\right\}\qquad (\mathrm{V}^8)$$

we may consider V as a function of $\rho\ \theta\ \phi\ \rho'\ \theta'\ \phi'\ \chi$, obtained by substituting for $x\ y\ z\ x'\ y'\ z'$ their values; and if we substitute also the values of dx, dy, dz, in the differential dV, or $v\,ds$, which was before a homogeneous function of the first dimension of dx, dy, dz, such that by our fundamental formula

$$\left.\begin{aligned}
\frac{\delta dV}{\delta dx} &= \frac{\delta\,.\,v\,ds}{\delta dx} = \frac{\delta v}{\delta\alpha} = \frac{\delta V}{\delta x},\\
\frac{\delta dV}{\delta dy} &= \frac{\delta\,.\,v\,ds}{\delta dy} = \frac{\delta v}{\delta\beta} = \frac{\delta V}{\delta y},\\
\frac{\delta dV}{\delta dz} &= \frac{\delta\,.\,v\,ds}{\delta dz} = \frac{\delta v}{\delta\gamma} = \frac{\delta V}{\delta z},
\end{aligned}\right\}\qquad (\mathrm{W}^8)$$

we may consider this differential $dv = v\,ds$ as becoming now a homogeneous function of $d\rho$, $d\theta$, $d\phi$, of the first dimension, such that

$$\left.\begin{aligned}
\frac{\delta \,.\, v\, ds}{\delta d\rho} &= \frac{\delta dV}{\delta d\rho} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \rho} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \rho} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \rho} = \frac{\delta V}{\delta \rho},\\
\frac{\delta \,.\, v\, ds}{\delta d\theta} &= \frac{\delta dV}{\delta d\theta} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \theta} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \theta} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \theta} = \frac{\delta V}{\delta \theta},\\
\frac{\delta \,.\, v\, ds}{\delta d\phi} &= \frac{\delta dV}{\delta d\phi} = \frac{\delta V}{\delta x}\frac{\delta x}{\delta \phi} + \frac{\delta V}{\delta y}\frac{\delta y}{\delta \phi} + \frac{\delta V}{\delta z}\frac{\delta z}{\delta \phi} = \frac{\delta V}{\delta \phi},
\end{aligned}\right\} \tag{X^8}$$

the symbol d referring still to motion along a ray. In like manner we may consider the initial differential element of V, namely $v'\, ds'$, as a homogeneous function of the first dimension of $d\rho'$, $d\theta'$, $d\phi'$, and then we shall find that the partial differential coefficients of the first order of this function, are equal respectively to

$$-\frac{\delta V}{\delta \rho'}, \quad -\frac{\delta V}{\delta \theta'}, \quad -\frac{\delta V}{\delta \phi'};$$

we may therefore generalise the fundamental formula (A) as follows

$$\begin{aligned}
\delta V = {} & \frac{\delta \,.\, v\, ds}{\delta d\rho}\,\delta\rho + \frac{\delta \,.\, v\, ds}{\delta d\theta}\,\delta\theta + \frac{\delta \,.\, v\, ds}{\delta d\phi}\,\delta\phi \\
& - \frac{\delta \,.\, v'\, ds'}{\delta d\rho'}\,\delta\rho' - \frac{\delta \,.\, v'\, ds'}{\delta d\theta'}\,\delta\theta' - \frac{\delta \,.\, v'\, ds'}{\delta d\phi'}\,\delta\phi' + \frac{\delta V}{\delta \chi}\,\delta\chi.
\end{aligned} \tag{Y^8}$$

And the auxiliary functions W, T, correspond to the following more general functions,

$$-V + \rho\frac{\delta V}{\delta \rho} + \theta\frac{\delta V}{\delta \theta} + \phi\frac{\delta V}{\delta \phi}, \quad \text{and} \quad -V + \rho\frac{\delta V}{\delta \rho} + \theta\frac{\delta V}{\delta \theta} + \phi\frac{\delta V}{\delta \phi} + \rho'\frac{\delta V}{\delta \rho'} + \theta'\frac{\delta V}{\delta \theta'} + \phi'\frac{\delta V}{\delta \phi'};$$

of which the first may be regarded as a function of

$$\frac{\delta V}{\delta \rho}, \quad \frac{\delta V}{\delta \theta}, \quad \frac{\delta V}{\delta \phi}, \quad \rho', \quad \theta', \quad \phi', \quad \chi,$$

and the second as a function of

$$\frac{\delta V}{\delta \rho}, \quad \frac{\delta V}{\delta \theta}, \quad \frac{\delta V}{\delta \phi}, \quad -\frac{\delta V}{\delta \rho'}, \quad -\frac{\delta V}{\delta \theta'}, \quad -\frac{\delta V}{\delta \phi'}, \quad \chi.$$

It is easy also to establish the following general differential equations of a curved ray, ordinary or extraordinary, and the following general integrals analogous to and including those already assigned for rectangular and oblique co-ordinates,

$$\left.\begin{aligned}
d\frac{\delta dV}{\delta d\rho} = \frac{\delta dV}{\delta \rho}; \quad & d\frac{\delta dV}{\delta d\theta} = \frac{\delta dV}{\delta \theta}; \quad d\frac{\delta dV}{\delta d\phi} = \frac{\delta dV}{\delta \phi} :\\
\frac{\delta V}{\delta \rho'} = \text{const.}; \quad & \frac{\delta V}{\delta \theta'} = \text{const.}; \quad \frac{\delta V}{\delta \phi'} = \text{const.}.
\end{aligned}\right\} \tag{Z^8}$$

General geometrical Relations of infinitely near Rays. Classification of twenty-four independent Coefficients, which enter into the algebraical Expressions of these general Relations. Division of the general Discussion into four principal Problems.

14. It is an important general problem of mathematical optics, included in that fundamental problem which was stated in the second number, to investigate *the general relations of infinitely near rays*, or paths of light; and especially to examine *how the extreme directions change, for any infinitely small changes of the extreme points, and of the colour*: that is, in the notation of this Supplement, to examine the general dependence of the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, on δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. This important case of our fundamental problem is easily resolved by the application of our general methods, and by the partial differential coefficients, of the two first orders, of the characteristic and related functions: it may also be resolved by the partial differentials of the three first orders, of the characteristic function V alone. For from these we can in general deduce six linear expressions for $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in terms of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, involving forty-two coefficients, of which however only twenty-four are independent, because they are connected by fourteen relations included in the formulæ $\alpha\,\delta\alpha + \beta\,\delta\beta + \gamma\,\delta\gamma = 0$, $\alpha'\,\delta\alpha' + \beta'\,\delta\beta' + \gamma'\,\delta\gamma' = 0$, and by four more included in the conditions that the final direction does not change when the initial point takes any new position on the given luminous path, nor the initial direction when the final point is removed to any new point on that given path.

Thus, if we employ the characteristic function V, and the final and initial medium-functions v, v', we have, by (B), the following general relations:

$$\left.\begin{array}{lll} \delta\dfrac{\delta V}{\delta x} = \delta\dfrac{\delta v}{\delta\alpha}; & \delta\dfrac{\delta V}{\delta y} = \delta\dfrac{\delta v}{\delta\beta}; & \delta\dfrac{\delta V}{\delta z} = \delta\dfrac{\delta v}{\delta\gamma}; \\ -\delta\dfrac{\delta V}{\delta x'} = \delta\dfrac{\delta v'}{\delta\alpha'}; & -\delta\dfrac{\delta V}{\delta y'} = \delta\dfrac{\delta v'}{\delta\beta'}; & -\delta\dfrac{\delta V}{\delta z'} = \delta\dfrac{\delta v'}{\delta\gamma'}: \end{array}\right\} \qquad (\mathrm{A}^9)$$

in which, by the last number, we are at liberty to assign different origins and different and oblique directions to the axes of the final and initial co-ordinates, if we assign new and corresponding values to the marks of final and initial direction, α, β, γ, α', β', γ', so as to have still the equations (P),

$$\alpha = \frac{dx}{ds}, \quad \beta = \frac{dy}{ds}, \quad \gamma = \frac{dz}{ds}, \quad \alpha' = \frac{dx'}{ds'}, \quad \beta' = \frac{dy'}{ds'}, \quad \gamma' = \frac{dz'}{ds'},$$

ds being still the final, and ds' the initial element of the curved or polygon path. We may suppose, for example, that both sets of co-ordinates are rectangular, but that the origins of the final and initial co-ordinates are respectively the final and initial points of a given ordinary or extraordinary path, and that the positive semiaxes of z, z', coincide with the final and initial directions, so as to give

$$\left.\begin{array}{l} x = 0, \quad y = 0, \quad z = 0, \quad \alpha = 0, \quad \beta = 0, \quad \gamma = 1, \quad \delta\gamma = 0; \\ x' = 0, \quad y' = 0, \quad z' = 0, \quad \alpha' = 0, \quad \beta' = 0, \quad \gamma' = 1, \quad \delta\gamma' = 0; \end{array}\right\} \qquad (\mathrm{B}^9)$$

and then the six equations (A^9), of which only four are distinct, reduce themselves to the four following,

$$
\left.\begin{aligned}
&\frac{\delta^2 v}{\delta\alpha^2}\,\delta\alpha + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\beta = \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta y' + \left(\frac{\delta^2 V}{\delta x\,\delta\chi} - \frac{\delta^2 v}{\delta\alpha\,\delta\chi}\right)\delta\chi\\
&\quad + \left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right)\delta x + \left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right)\delta y + \left(\frac{\delta v}{\delta x} - \frac{\delta^2 v}{\delta\alpha\,\delta z}\right)\delta z;\\
&\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta^2}\,\delta\beta = \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y' + \left(\frac{\delta^2 V}{\delta y\,\delta\chi} - \frac{\delta^2 v}{\delta\beta\,\delta\chi}\right)\delta\chi\\
&\quad + \left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right)\delta x + \left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right)\delta y + \left(\frac{\delta v}{\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta z}\right)\delta z;\\
-&\frac{\delta^2 v'}{\delta\alpha'^2}\,\delta\alpha' - \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\beta' = \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta y + \left(\frac{\delta^2 V}{\delta x'\,\delta\chi} + \frac{\delta^2 v'}{\delta\alpha'\,\delta\chi}\right)\delta\chi\\
&\quad + \left(\frac{\delta^2 V}{\delta x'^2} + \frac{\delta^2 v'}{\delta\alpha'\,\delta x'}\right)\delta x' + \left(\frac{\delta^2 V}{\delta x'\,\delta y'} + \frac{\delta^2 v'}{\delta\alpha'\,\delta y'}\right)\delta y' - \left(\frac{\delta v'}{\delta x'} - \frac{\delta^2 v'}{\delta\alpha'\,\delta z'}\right)\delta z';\\
-&\frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\alpha' - \frac{\delta^2 v'}{\delta\beta'^2}\,\delta\beta' = \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y + \left(\frac{\delta^2 V}{\delta y'\,\delta\chi} + \frac{\delta^2 v'}{\delta\beta'\,\delta\chi}\right)\delta\chi\\
&\quad + \left(\frac{\delta^2 V}{\delta x'\,\delta y'} + \frac{\delta^2 v'}{\delta\beta'\,\delta x'}\right)\delta x' + \left(\frac{\delta^2 V}{\delta y'^2} + \frac{\delta^2 v'}{\delta\beta'\,\delta y'}\right)\delta y' - \left(\frac{\delta v'}{\delta y'} - \frac{\delta^2 v'}{\delta\beta'\,\delta z'}\right)\delta z' :
\end{aligned}\right\}\tag{C^9}
$$

they give therefore, by easy eliminations, expressions for $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the form

$$
\left.\begin{aligned}
\delta\alpha &= \frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y + \frac{\delta\alpha}{\delta z}\,\delta z + \frac{\delta\alpha}{\delta x'}\,\delta x' + \frac{\delta\alpha}{\delta y'}\,\delta y' + \frac{\delta\alpha}{\delta\chi}\,\delta\chi,\\
\delta\beta &= \frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y + \frac{\delta\beta}{\delta z}\,\delta z + \frac{\delta\beta}{\delta x'}\,\delta x' + \frac{\delta\beta}{\delta y'}\,\delta y' + \frac{\delta\beta}{\delta\chi}\,\delta\chi,\\
\delta\alpha' &= \frac{\delta\alpha'}{\delta x'}\,\delta x' + \frac{\delta\alpha'}{\delta y'}\,\delta y' + \frac{\delta\alpha'}{\delta z'}\,\delta z' + \frac{\delta\alpha'}{\delta x}\,\delta x + \frac{\delta\alpha'}{\delta y}\,\delta y + \frac{\delta\alpha'}{\delta\chi}\,\delta\chi,\\
\delta\beta' &= \frac{\delta\beta'}{\delta x'}\,\delta x' + \frac{\delta\beta'}{\delta y'}\,\delta y' + \frac{\delta\beta'}{\delta z'}\,\delta z' + \frac{\delta\beta'}{\delta x}\,\delta x + \frac{\delta\beta'}{\delta y}\,\delta y + \frac{\delta\beta'}{\delta\chi}\,\delta\chi,
\end{aligned}\right\}\tag{D^9}
$$

which involve twenty-four coefficients, and enable us to determine the general geometrical relations between the final and initial tangents to the near luminous paths.

If the extreme media be ordinary, that is, if the functions v, v', be independent of the directions of the rays, we have

$$
v = \mu\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}, \quad v' = \mu'\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}, \tag{E^9}
$$

μ, μ' being functions of the colour χ, of which μ involves also the final co-ordinates, and μ' the initial co-ordinates, when the exteme media are atmospheres: and then the equations

(C^9) reduce themselves at once to the following expressions of the form (D^9),

$$\left.\begin{aligned}
\delta\alpha &= \frac{1}{\mu}\left(\frac{\delta^2 V}{\delta x^2}\,\delta x + \frac{\delta^2 V}{\delta x\,\delta y}\,\delta y + \frac{\delta\mu}{\delta x}\,\delta z + \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta x\,\delta\chi}\,\delta\chi\right),\\
\delta\beta &= \frac{1}{\mu}\left(\frac{\delta^2 V}{\delta x\,\delta y}\,\delta x + \frac{\delta^2 V}{\delta y^2}\,\delta y + \frac{\delta\mu}{\delta y}\,\delta z + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta x' + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y' + \frac{\delta^2 V}{\delta y\,\delta\chi}\,\delta\chi\right),\\
\delta\alpha' &= -\frac{1}{\mu'}\left(\frac{\delta^2 V}{\delta x'^2}\,\delta x' + \frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta y' - \frac{\delta\mu'}{\delta x'}\,\delta z' + \frac{\delta^2 V}{\delta x\,\delta x'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta x'}\,\delta y + \frac{\delta^2 V}{\delta x'\,\delta\chi}\,\delta\chi\right),\\
\delta\beta' &= -\frac{1}{\mu'}\left(\frac{\delta^2 V}{\delta x'\,\delta y'}\,\delta x' + \frac{\delta^2 V}{\delta y'^2}\,\delta y' - \frac{\delta\mu'}{\delta y'}\,\delta z' + \frac{\delta^2 V}{\delta x\,\delta y'}\,\delta x + \frac{\delta^2 V}{\delta y\,\delta y'}\,\delta y + \frac{\delta^2 V}{\delta y'\,\delta\chi}\,\delta\chi\right).
\end{aligned}\right\} \qquad (\mathrm{F}^9)$$

In general we see that the twenty-four coefficients of the expressions (D^9) can easily be deduced, by (C^9), from the partial differentials of the two first orders of the characteristic function V, and of the extreme medium-functions v, v': we have for example

$$\left.\begin{aligned}
\frac{\delta\alpha}{\delta x} &= \frac{1}{v''}\frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right) - \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right),\\
\frac{\delta\alpha}{\delta y} &= \frac{1}{v''}\frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right) - \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right),\\
\frac{\delta\beta}{\delta x} &= \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right) - \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right),\\
\frac{\delta\beta}{\delta y} &= \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right) - \frac{1}{v''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right),
\end{aligned}\right\} \qquad (\mathrm{G}^9)$$

v'' having the same meaning as in the tenth number. The same twenty-four coefficients of (D^9) may also be deduced (as we have said) from the partial differentials of the two first orders of the other related and auxiliary functions: or even from the partial differentials of the three first orders of the characteristic function V alone. Let us therefore suppose that these twenty-four coefficeints of the expressions (D^9) are known, and let us consider their geometrical meanings and uses: that is, their connexions with questions respecting the infinitely small variations of the extreme directions or tangents of a luminous path, arising from variations of the extreme points and of the colour.

In discussing these connexions, it is evidently permitted, by the linear form of the differential expressions (D^9), to consider separately and successively the influence of the seven variations δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$, of the extreme co-ordinates and the colour, or the influence of any groupes of these seven variations, on the four variations $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the extreme small cosines of direction. Thus, if it be required to compare the extreme directions of a given path of ordinary or extraordinary light of the colour χ, from a given initial point A to a given final point B, which path we shall denote as follows,

$$(A, B)_\chi, \qquad (\mathrm{H}^9)$$

with the extreme directions of an infinitely near path of infinitely near colour $\chi + \delta\chi$ from an infinitely near initial point A' to an infinitely near final point B', which path we shall in like manner denote thus

$$(A', B')_{\chi+\delta\chi}, \qquad (\mathrm{I}^9)$$

we may do so by comparing separately the extreme directions of the given path $(A, B)_\chi$ with those of the three following other infinitely near paths;

$$\text{1st. } (A, B)_{\chi+\delta\chi}; \quad \text{2d. } (A, B')_\chi; \quad \text{3d. } (A', B)_\chi : \tag{K^9}$$

which are obtained by changing, successively and separately, the colour χ, the final point B, and the initial point A. We are therefore led, by this consideration, to examine separately and successively the meanings and uses of the three following groupes, out of the twenty-four coefficients of (D^9):

$$\left.\begin{array}{ll} \text{1st groupe} & \dfrac{\delta\alpha}{\delta\chi}, \quad \dfrac{\delta\beta}{\delta\chi}, \quad \dfrac{\delta\alpha'}{\delta\chi}, \quad \dfrac{\delta\beta'}{\delta\chi}; \\ \text{2d groupe} & \dfrac{\delta\alpha}{\delta x}, \quad \dfrac{\delta\alpha}{\delta y}, \quad \dfrac{\delta\alpha}{\delta z}, \quad \dfrac{\delta\beta}{\delta x}, \quad \dfrac{\delta\beta}{\delta y}, \quad \dfrac{\delta\beta}{\delta z}, \quad \dfrac{\delta\alpha'}{\delta x}, \quad \dfrac{\delta\alpha'}{\delta y}, \quad \dfrac{\delta\beta'}{\delta x}, \quad \dfrac{\delta\beta'}{\delta y}; \\ \text{3d groupe} & \dfrac{\delta\alpha}{\delta x'}, \quad \dfrac{\delta\alpha}{\delta y'}, \quad \dfrac{\delta\beta}{\delta x'}, \quad \dfrac{\delta\beta}{\delta y'}, \quad \dfrac{\delta\alpha'}{\delta x'}, \quad \dfrac{\delta\alpha'}{\delta y'}, \quad \dfrac{\delta\alpha'}{\delta z'}, \quad \dfrac{\delta\beta'}{\delta x'}, \quad \dfrac{\delta\beta'}{\delta y'}, \quad \dfrac{\delta\beta'}{\delta z'}. \end{array}\right\} \tag{L^9}$$

But we may simplify and improve the plan of our investigation, by means of the following considerations.

Of the three comparisons, of the given path (H^9) with the three near paths (K^9), the third is evidently of the same kind with the second, and need not be treated as distinct; because, of the two extreme points of a luminous path, it is indifferent which we consider as initial and which as final. We may therefore omit the third comparison (K^9), and confine ourselves to the first and second, that is, we may omit the consideration of the third groupe (L^9), in forming a theory of the general relations of infinitely near rays. For a similar reason we may omit the consideration of the two last coefficients of the first groupe (L^9), and so may reduce the study of the whole twenty-four to the study of half that number.

On the other hand, the second comparison (K^9) may conveniently be decomposed into two: for instead of the arbitrary infinitesimal line $\overline{BB'}$, connecting the given final point B with the near point B', we may conveniently consider the two projections of this line, on the final element or tangent of the given luminous path, and on the plane perpendicular to this element: that is, we may put

$$\overline{BB'}^2 = \overline{BB'_d}^2 + \overline{BB'_\delta}^2, \tag{M^9}$$

$\overline{BB'_d}$ being the projection on the element, and $\overline{BB'_\delta}$ the projection on the perpendicular plane, and we may consider separately the two near points B_d, B_δ, upon this element and plane, and the two corresponding paths,

$$(A, B_d)_\chi, \quad (A, B_\delta)_\chi, \tag{N^9}$$

instead of considering the more general near point B', and the near path $(A, B')_\chi$. In this manner we are led to consider separately, as one subordinate class or set, suggested by the path $(A, B_d)_\chi$, the system of the two coefficients $\dfrac{\delta\alpha}{\delta z}$, $\dfrac{\delta\beta}{\delta z}$; distinguishing these from the eight other coefficients of the second groupe (L^9), which correspond to the other near path $(A, B_\delta)_\chi$;

and these eight may again be conveniently divided into two distinct classes, according as we consider the changes of final or initial direction.

We are then led to arrange the twelve retained coefficients of the expressions (D^9), in *four new sets* or classes, suggesting *four separate problems*:

$$\left.\begin{array}{llllllllll} \text{First set} & \dfrac{\delta\alpha}{\delta\chi}, & \dfrac{\delta\beta}{\delta\chi}; & & & \text{Second,} & \dfrac{\delta\alpha}{\delta z}, & \dfrac{\delta\beta}{\delta z}; & & \\ \text{Third,} & \dfrac{\delta\alpha}{\delta x}, & \dfrac{\delta\alpha}{\delta y}, & \dfrac{\delta\beta}{\delta x}, & \dfrac{\delta\beta}{\delta y}; & \text{Fourth,} & \dfrac{\delta\alpha'}{\delta x}, & \dfrac{\delta\alpha'}{\delta y}, & \dfrac{\delta\beta'}{\delta x}, & \dfrac{\delta\beta'}{\delta y}. \end{array}\right\} \qquad (\mathrm{O}^9)$$

In each of these four problems, the initial point is considered as given, and may be supposed to be a luminous origin, common to all the infinitely near paths of which we compare the extreme directions. In the first problem, the final point also is given, but the colour χ is variable; and we study the final chromatic dispersion of the different near paths of heterogeneous light, connecting the given final point with the given luminous origin: whereas, in the three remaining problems, the light is considered as homogeneous, but the luminous path varies by the variation of its final point. In the second problem, the new final point B_d is on the original path, or on that path prolonged; and we examine whether and in what manner the final direction varies, on account of the final curvature of that original path. In the third problem, the new final point B_δ is on an infinitely small line

$$\delta l = \overline{BB_\delta}, \qquad (\mathrm{P}^9)$$

which is drawn from the given final point of the original path, perpendicular to the given final element of that path, namely to the element

$$ds = \overline{BB_d}; \qquad (\mathrm{Q}^9)$$

and we inquire into the mutual arrangement and relations of the final system of right lines which coincide with and mark the final directions of the near luminous paths, at the several near points B_δ where they meet the given final plane perpendicular to the given element ds. In the fourth problem, we consider the initial system of right lines, which mark, at the luminous origin, the initial directions of the same near paths of homogeneous light; and we compare these initial directions with the positions of the points B_δ. Let us now consider separately these four principal problems, respecting the geometrical relations of infinitely near rays.

Discussion of the Four Problems. Elements of Arangement of near Luminous Paths. Axis and Constant of Chromatic Dispersion. Axis of Curvature of Ray. Guiding Paraboloid, and Constant of Deviation. Guiding Planes, and Conjugate Guiding Axes.

15. The *first* of these four problems, namely that in which it is required to determine the final chromatic dispersion, by means of the two coefficients $\dfrac{\delta\alpha}{\delta\chi}$, $\dfrac{\delta\beta}{\delta\chi}$, is very easily resolved:

since we have the following equations for the magnitude and plane of this dispersion,

$$\left.\begin{aligned}
&\textit{Final angle of chromatic dispersion} = \xi\,\delta\chi; \quad \xi = \sqrt{\left(\frac{\delta\alpha}{\delta\chi}\right)^2 + \left(\frac{\delta\beta}{\delta\chi}\right)^2}: \\
&\textit{Final plane of dispersion}\ldots\ldots\ldots\ldots \quad y\frac{\delta\alpha}{\delta\chi} = x\frac{\delta\beta}{\delta\chi}.
\end{aligned}\right\} \qquad (\mathrm{R}^9)$$

We may geometrically construct the effect of this dispersion, by making the given final line of direction of the original luminous path revolve through the small angle $\xi\,\delta\chi$, in which ξ may be called the *constant of final chromatic dispersion*, round the following line which may be called *the axis of final chromatic dispersion*,

$$x\frac{\delta\alpha}{\delta\chi} + y\frac{\delta\beta}{\delta\chi} = 0, \quad z = 0. \qquad (\mathrm{S}^9)$$

The *second* problem, which relates to the final curvature of the given luminous path, is resolved by the analogous equations,

$$\left.\begin{aligned}
&\textit{Final curvature of ray} = \sqrt{\left(\frac{\delta\alpha}{\delta z}\right)^2 + \left(\frac{\delta\beta}{\delta z}\right)^2}; \\
&\textit{Plane of curvature}\ldots\ldots \quad y\frac{\delta\alpha}{\delta z} = x\frac{\delta\beta}{\delta z};
\end{aligned}\right\} \qquad (\mathrm{T}^9)$$

we have also the following equations for the axis of curvature, that is, for the axis of the circle of curvature, or of the final osculating circle to the given luminous path,

$$x\frac{\delta\alpha}{\delta z} + y\frac{\delta\beta}{\delta z} = 1, \quad z = 0: \qquad (\mathrm{U}^9)$$

and in all these equations of curvature we may, consistently with the notation of the present Supplement, express the coefficients $\frac{\delta\alpha}{\delta z}$, $\frac{\delta\beta}{\delta z}$ by the symbols $\frac{d\alpha}{dz}$, $\frac{d\beta}{dz}$, because they relate to motion along a given luminous path. It is evident that these coefficients vanish, when the final portion of this path is straight. But when this final portion is curved, we may geometrically construct the effect of this curvature on the final direction, by making the final element ds revolve through an infinitely small angle round the final axis of curvature.

The two remaining problems are more complicated, because each involves two independent variations δx, δy, namely the two rectangular co-ordinates of the near point B_δ on the final plane of xy, which point is considered as the final point of a near luminous path. The equations of the right line, which is the final portion or final tangent of this near path, are

$$\left.\begin{aligned}
x &= \delta x + z\left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y\right), \\
y &= \delta y + z\left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y\right);
\end{aligned}\right\} \qquad (\mathrm{V}^9)$$

and the equations of the right line, which is the initial portion or the initial tangent of the same near path, are

$$\left.\begin{aligned} x' &= z'\left(\frac{\delta\alpha'}{\delta x}\,\delta x + \frac{\delta\alpha'}{\delta y}\,\delta y\right), \\ y' &= z'\left(\frac{\delta\beta'}{\delta x}\,\delta x + \frac{\delta\beta'}{\delta y}\,\delta y\right). \end{aligned}\right\} \tag{W^9}$$

Our *third* problem is to investigate the geometrical relations of the system of right lines (V^9), which we shall call *final ray-lines*, with each other, and with the co-ordinates δx, δy; and our *fourth* problem is to investigate the connexion of the same co-ordinates or variations with the right lines of the system (W^9), which may be called *initial ray-lines.*

The *third* problem may be considered as resolved, if we can assign any surface to which the final ray-lines (V^9) are normals, or with which they are determinately connected by any other known geometrical relation. Let us therefore examine whether the ray-lines of the system (V^9) are normals to any common surface, which passes through the given final point of the original luminous path. If so, this surface may be considered, in our present order of approximation, as perpendicular to the final rays themselves. Now, in general, when rays of a given colour diverge from a given luminous point, and undergo any number of ordinary or extraordinary and gradual or sudden reflexions or refractions, they are, or are not, perpendicular in their final state to a common surface, according as the following differential equation

$$\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z = 0 \tag{X^9}$$

is or is not integrable; and if there be any one surface perpendicular to all the final rays, there is also a series of such surfaces, represented by the integral of this equation. Hence, in the present question, the normal surface sought is such, if it exists at all, as to satisfy the conditions $\delta z = 0$, and

$$\delta^2 z + \delta\alpha\,\delta x + \delta\beta\,\delta y = 0; \tag{Y^9}$$

that is, if it exist, it must touch the given final plane of xy, and must have contact of the second order with the following paraboloid, which may therefore in our present order of approximation be employed instead of it,

$$2z + \frac{\delta\alpha}{\delta x}x^2 + \left(\frac{\delta\alpha}{\delta y} + \frac{\delta\beta}{\delta x}\right)xy + \frac{\delta\beta}{\delta y}y^2 = 0. \tag{Z^9}$$

The normals to this paraboloid, near its summit, that is, near the final point of the given luminous path, or the origin of the final co-ordinates, have for their approximate equations,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y\right) + zn\,\delta y, \\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y\right) - zn\,\delta x, \end{aligned}\right\} \tag{A^{10}}$$

if we put for abridgment

$$n = \tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} - \frac{\delta\alpha}{\delta y}\right): \tag{B^{10}}$$

they coincide therefore with the ray-lines (V⁹) when the following condition is satisfied,

$$\frac{\delta\beta}{\delta x} = \frac{\delta\alpha}{\delta y}, \tag{C^{10}}$$

which is in fact the condition of integrability of the differential equation (X⁹), because we have made $\alpha\ \beta$ vanish by our choice of the axis of z. The condition (C¹⁰) is satisfied, by (F⁹), when the final medium is ordinary; and in fact the final rays whether straight or curved are then perpendicular to the series of surfaces represented by the equation

$$V = \text{const.}: \tag{D^{10}}$$

which is, for ordinary rays, the integral of the equation (X⁹), and gives, as an approximate equation of the normal surface at the origin, the following,

$$0 = \delta V + \tfrac{1}{2}\delta^2 V, \quad \text{or} \quad 0 = \mu z + \tfrac{1}{2}\frac{\delta^2 V}{\delta x^2}x^2 + \frac{\delta^2 V}{\delta x\,\delta y}xy + \tfrac{1}{2}\frac{\delta^2 V}{\delta y^2}y^2; \tag{E^{10}}$$

agreeing, by (F⁹), with the equation of the paraboloid (Z⁹). In general, the condition (C¹⁰) for the existence of a normal surface, may be put, by (G⁹), under the form

$$\begin{aligned}
&\frac{\delta^2 v}{\delta\alpha^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta\,\delta x}\right) - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha\,\delta x}\right) \\
&= \frac{\delta^2 v}{\delta\beta^2}\left(\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta y}\right) - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta\,\delta y}\right):
\end{aligned} \tag{F^{10}}$$

and it is not satisfied by extraordinary rays, except in particular cases. We may however always consider the paraboloid (Z⁹) as an auxiliary surface, with which the final ray-lines of the proposed system (V⁹) are connected by a remarkable and simple relation. For if we take the rectangular planes of curvature of this paraboloid for the co-ordinate planes of xz, yz, and denote the two curvatures corresponding by r, t, so as to have the following form for the equation of the paraboloid

$$z = \tfrac{1}{2}rx^2 + \tfrac{1}{2}ty^2, \tag{G^{10}}$$

we shall satisfy the condition

$$\frac{\delta\alpha}{\delta y} + \frac{\delta\beta}{\delta x} = 0, \tag{H^{10}}$$

and may employ the following expressions for the four coefficients of our problem,

$$\frac{\delta\alpha}{\delta x} = -r, \quad \frac{\delta\alpha}{\delta y} = -n, \quad \frac{\delta\beta}{\delta x} = +n, \quad \frac{\delta\beta}{\delta y} = -t: \tag{I^{10}}$$

the ray-lines of our system (V⁹) may therefore be thus represented

$$\left.\begin{aligned}
x &= \delta x - z(r\,\delta x + n\,\delta y), \\
y &= \delta y - z(t\,\delta y - n\,\delta x),
\end{aligned}\right\} \tag{K^{10}}$$

while the normals to the paraboloid are represented by these equations

$$x = \delta x - zr\,\delta x, \quad y = \delta y - zt\,\delta y; \tag{L^{10}}$$

from which it follows that the angle $\delta\nu$ between a ray-line (K^{10}) and the corresponding normal (L^{10}) may be thus expressed

$$\delta\nu = n\,\delta l, \quad \text{in which} \quad \delta l = \sqrt{\delta x^2 + \delta y^2}, \tag{M^{10}}$$

δl being the same small line $\overline{BB_\delta}$ as before; and that the plane of this angle $\delta\nu$, or in other words, the plane containing the ray-line and the normal, has for equation

$$x\,\delta x + y\,\delta y = \delta l^2 - z(r\,\delta x^2 + t\,\delta y^2) : \tag{N^{10}}$$

this plane therefore contains also the right line having for equations

$$x\,\delta x + y\,\delta y = 0, \quad z = \frac{\delta l^2}{r\,\delta x^2 + t\,\delta y^2}, \tag{O^{10}}$$

that is, the axis of the osculating circle of curvature of the normal or diametral section of the paraboloid, of which the line δl is an element; and *the normal may be brought to coincide with the ray-line by being made to revolve around the element δl, through an angle δv proportional to δl, and equal to that element multiplied by the constant n*: the direction of the rotation depending on the sign of the constant. On account of this simple law of deviation of the final ray-lines from the normals of the paraboloid, we shall call this paraboloid the *guiding surface*: and the constant n, we shall call the *constant of deviation.* And we may consider this theory, of the guiding paraboloid and the constant of deviation, as containing an adequate solution of our third general problem, in the discussion of the geometrical relations of infinitely near rays: since this theory shows adequately the general arrangement of the final system of ray-lines (V^9), and the geometrical meanings of the third set of coefficients (O^9), namely,

$$\frac{\delta\alpha}{\delta x}, \quad \frac{\delta\alpha}{\delta y}, \quad \frac{\delta\beta}{\delta x}, \quad \frac{\delta\beta}{\delta y}.$$

The geometrical construction suggested by this theory may be still farther simplified by observing that the infinitely near normals to the guiding surface all pass through two rectangular lines, namely, the axes of the two principal circles of curvature of the surface; it is therefore sufficient to draw through any proposed point B_δ two planes containing respectively these two given axes of curvature, and then to make the line of intersection of these two planes revolve round the proposed small line δl or $\overline{BB_\delta}$, through the same small angle $n\,\delta l$ as before, in order to obtain the sought final ray-line for the proposed final point.

Finally, to compare, as required in the *fourth* problem, the initial system of ray-lines (W^9) with the corresponding final points B_δ on the given final plane, we may denote these initial ray-lines by the equations

$$x' = z'\,\delta\theta' \,.\, \cos.\,\phi', \quad y' = z'\,\delta\theta' \,.\, \sin.\,\phi', \tag{P^{10}}$$

if we put

$$\delta\alpha' = \delta\theta' \,.\, \cos.\,\phi', \quad \delta\beta' = \delta\theta' \,.\, \sin.\,\phi' : \tag{Q^{10}}$$

and if in like manner we put

$$\delta x = \delta l \,.\, \cos.\,\phi, \quad \delta y = \delta l \,.\, \sin.\,\phi, \tag{R^{10}}$$

we shall have the following relations, between ϕ, ϕ', δl, $\delta\theta'$, and the fourth set of partial differential coefficients (O^9),

$$\left.\begin{aligned} \delta\theta' \,.\, \cos.\,\phi' &= \left(\frac{\delta\alpha'}{\delta x}\cos.\,\phi + \frac{\delta\alpha'}{\delta y}\sin.\,\phi\right)\delta l, \\ \delta\theta' \,.\, \sin.\,\phi' &= \left(\frac{\delta\beta'}{\delta x}\cos.\,\phi + \frac{\delta\beta'}{\delta y}\sin.\,\phi\right)\delta l. \end{aligned}\right\} \tag{S^{10}}$$

These relations give

$$\tan.\,\phi' = \frac{\dfrac{\delta\beta'}{\delta x} + \dfrac{\delta\beta'}{\delta y}\tan.\,\phi}{\dfrac{\delta\alpha'}{\delta x} + \dfrac{\delta\alpha'}{\delta y}\tan.\,\phi}; \tag{T^{10}}$$

they enable us therefore to determine, for any given value of ϕ, that is, for any proposed direction of the small final line δl, or $\overline{BB_\delta}$, the corresponding value of ϕ', that is, the direction of the initial plane of ray-lines, having for equation

$$y' = x' \tan.\,\phi'. \tag{U^{10}}$$

Thus the final line δl and initial plane ϕ' revolve together, but not in general with equal rapidity; and arbitary rectangular directions of the one do not in general give rectangular directions of the other, because the conditions

$$\left.\begin{gathered} \tan.\,\phi_1' = \frac{\dfrac{\delta\beta'}{\delta x} + \dfrac{\delta\beta'}{\delta y}\tan.\,\phi_1}{\dfrac{\delta\alpha'}{\delta x} + \dfrac{\delta\alpha'}{\delta y}\tan.\,\phi_1}, \quad \tan.\,\phi_2' = \frac{\dfrac{\delta\beta'}{\delta x} + \dfrac{\delta\beta'}{\delta y}\tan.\,\phi_2}{\dfrac{\delta\alpha'}{\delta x} + \dfrac{\delta\alpha'}{\delta y}\tan.\,\phi_2}, \\ \phi_2 = \phi_1 + \frac{\pi}{2}, \quad \phi_2' = \phi_1' + \frac{\pi}{2}, \end{gathered}\right\} \tag{V^{10}}$$

(in which π is the semicircumference to the radius unity,) give the following formula for the angle ϕ_1,

$$2\left(\frac{\delta\beta'}{\delta x}\frac{\delta\beta'}{\delta y} + \frac{\delta\alpha'}{\delta x}\frac{\delta\alpha'}{\delta y}\right)\text{cotan.}\,2\phi_1 = \left(\frac{\delta\beta'}{\delta x}\right)^2 - \left(\frac{\delta\beta'}{\delta y}\right)^2 + \left(\frac{\delta\alpha'}{\delta x}\right)^2 - \left(\frac{\delta\alpha'}{\delta y}\right)^2, \tag{W^{10}}$$

which is not in general satisfied by arbitrary values of that angle. There are however in general two rectangular final directions determined by this formula, which correspond to two rectangular initial planes; and if we take these rectangular directions and planes respectively for the directions of x, y, and for the planes of $x'z'$, $y'z'$, we shall have

$$\frac{\delta \alpha'}{\delta y} = 0, \quad \frac{\delta \beta'}{\delta x} = 0. \tag{X^{10}}$$

We may also in general satisfy, at the same time, by a proper choice of the semiaxes of co-ordinates, the following other conditions,

$$\frac{\delta \beta'}{\delta y} > 0, \quad \frac{\delta \alpha'}{\delta x} > \frac{\delta \beta'}{\delta y}. \tag{Y^{10}}$$

By this choice of co-ordinates, the relations (S^{10}) are simplified, and become

$$\left. \begin{aligned} \delta\theta' \,.\, \cos.\,\phi' &= \frac{\delta \alpha'}{\delta x} \,.\, \delta l \,.\, \cos.\,\phi; \\ \delta\theta' \,.\, \sin.\,\phi' &= \frac{\delta \beta'}{\delta y} \,.\, \delta l \,.\, \sin.\,\phi : \end{aligned} \right\} \tag{Z^{10}}$$

while the equations (W^9) of the initial ray-lines reduce themselves to the following,

$$x' = z' \frac{\delta \alpha'}{\delta x} \, \delta x; \quad y' = z' \frac{\delta \beta'}{\delta y} \, \delta y. \tag{A^{11}}$$

If, then, these initial ray-lines form a circular cone having for equation

$$x'^2 + y'^2 = z'^2 \, \delta\theta'^2, \tag{B^{11}}$$

the corresponding locus of the final point B_δ, on the final plane of xy, will not in general be a circle, but an ellipse, having for its equation

$$\left(\frac{\delta \alpha'}{\delta x} \right)^2 \delta x^2 + \left(\frac{\delta \beta'}{\delta y} \right)^2 \delta y^2 = \delta\theta'^2, \tag{C^{11}}$$

of which, by (Y^{10}), the axis of x coincides with the least and the axis of y with the greatest axis; and reciprocally if the final locus be a circle having for equation

$$\delta x^2 + \delta y^2 = \delta l^2, \tag{D^{11}}$$

the initial cone of ray-lines will have for equation

$$x'^2 \left(\frac{\delta \alpha'}{\delta x} \right)^{-2} + y'^2 \left(\frac{\delta \beta'}{\delta y} \right)^{-2} = z'^2 \, \delta l^2, \tag{E^{11}}$$

so that its perpendicular sections are ellipses, having their greater axes in the plane of $x'z'$, and their lesser axes in the plane of $y'z'$. It is evident that a circle equal to the final circle

(D^{11}) may be obtained from the elliptic cone (E^{11}), by cutting that elliptic cone by any one of the four following planes,

$$z' = \pm \left(\frac{\delta\alpha'}{\delta x}\right)^{-1} \pm y' \sqrt{\left(\frac{\delta\alpha'}{\delta x}\right)^{2} \left(\frac{\delta\beta'}{\delta y}\right)^{-2} - 1}; \tag{F^{11}}$$

and in like manner the four elliptic sections of the circular cone (B^{11}), made by the same four planes, are all equal and similar to the final ellipse (C^{11}). In general it is easy to prove by the equations of the initial ray-lines (A^{11}), that whatever final locus we take for the point B_δ, represented by the equation

$$\delta y = f(\delta x), \tag{G^{11}}$$

the corresponding initial cone

$$\frac{y'}{z'} \left(\frac{\delta\beta'}{\delta y}\right)^{-1} = f\left(\frac{x'}{z'} \left(\frac{\delta\alpha'}{\delta x}\right)^{-1}\right) \tag{H^{11}}$$

will have four sections equal and similar to this final locus, namely, the sections by the four planes (F^{11}). We may therefore consider these as *four guiding planes* for the initial ray, since *each contains for any proposed final curve or locus* (G^{11}) *of the final point B_δ, an equal and similar guiding curve or locus, which is a section of the sought initial cone, and by which therefore that cone may be determined.* If, then, we know these four *guiding planes*, or any one of them, and the corresponding system of final and initial rectangular directions, or *conjugate guiding axes*, of which two are determined by a guiding plane, we shall be able to construct the initial ray-line or ray-cone corresponding to any final position or locus of the point B_δ. The fourth and last general problem of those proposed above, may therefore be considered as resolved, by this theory of the guiding planes and guiding axes.

We see then that in order to compare completely the extreme directions of any two near luminous paths

$$(A, B)_\chi, \quad (A', B')_{\chi+\delta\chi},$$

in which A is the initial and B the final point of a given path, and A', B', are any other initial and final points infinitely near to these, the following geometrical *elements of arrangement*, or some data equivalent to them, are necessary and sufficient to be known.

First. The final axis, and the initial axis, of chromatic dispersion; and the corresponding final and initial constants ξ, ξ', with their proper signs, to indicate the directions, as well as the quantities of dispersion.

Second. The final axis, and the initial axis, of curvature of the given path.

Third. The final pair, and the initial pair, of axes of curvature of the guiding paraboloids, at the ends of this given path; and the final and initial constants of deviation n, n'.

Fourth. A guiding plane for the initial ray-lines, and a guiding plane for the final ray-lines; together with the final system and the initial system of rectangular directions, or conjugate guiding axes, connected with these guiding planes.

When these different elements of arrangement of the extreme ray-lines are known, we can deduce from them the dependence of $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, and more generally of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, on δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$; and reciprocally when this latter dependence has been deduced from the partial differential coefficients of the characteristic or related functions, we can deduce it from the geometrical elements above mentioned.

Application of the Elements of Arrangement. Connexion of the two final Vergencies, and Planes of Vergency, and Guiding Lines, with the two principal Curvatures and Planes of Curvature of the Guiding Paraboloid, and with the Constant of Deviation. The Planes of Curvature are the Planes of Extreme Projection of the final Ray-Lines.

16. To give now an example of the application of these geometrical elements of arrangement, let us employ them to determine the *conditions of intersection of two near final ray-lines*, corresponding to a given colour and to a given luminous origin; and let us suppose, for simplicity, that one of these two straight ray-lines being the final portion or final tangent of a given luminous path $(A, B)_\chi$, the other corresponds (as in the third of the foregoing problems) to a final point B_δ on the given final plane perpendicular to this given path at B. Then if the constant n of deviation vanishes, so that the final ray-lines are normals to the guiding paraboloid, the condition of intersection requires evidently that the near point B_δ should be in one of the two principal diametral planes, that is, on one of the two rectangular tangents to the lines of curvature on this surface; and the corresponding point of intersection must be one of the two centres of curvature. But when n does not vanish, the deviation of the ray-lines obliges us to alter this result. The intersection of the near ray-line with the given ray-line will not now take place for the directions of the lines of curvature; but for those other directions, if any, for which the angular deviation $n\,\delta l$ of the ray-line from the normal is equal and contrary to the angular deviation of the normal from the corresponding plane of normal section, that is, from the corresponding diametral plane of the guiding paraboloid. This latter deviation, abstracting from sign, is, by the general properties of normals, equal to the semidifference of curvatures multiplied by the element of the normal section δl, and by the sine of twice the inclination of this element to either of the lines of curvature; it cannot therefore destroy the deviation $n\,\delta l$ of the ray-line from the normal, unless the semidifference of the two principal curvatures of the paraboloid is greater, or at least not less, abstracting from sign, than the constant of deviation n; this then is a necessary condition for the possibility of the intersection sought. But when the semidifference of curvatures is greater (abstracting from sign) than n, then there are two distinct directions P_1, P_2, of the normal or diametral plane of section, symmetrically placed with respect to the two principal planes of curvature, and such that if the element of section δl be contained in either of these two planes P_1, P_2, (but not if the element δl be in any other normal plane,) the corresponding ray-line from the extremity of that element will be contained in the same normal plane P_1 or P_2, and will intersect the given ray-line as required; and the point of intersection of these two near ray-lines will be the centre of curvature of the corresponding normal section. We may therefore call the curvatures of these two diametral sections the *two vergencies* of the final ray-lines; and the two corresponding planes P_1, P_2, we may call the *two planes of vergency.*

The same conclusions may be deduced algebraically from the equations (K^{10}), which give the following conditions of intersection of a near ray-line with the given ray-line or axis of z,

$$0 = (z^{-1} - r)\,\delta x - n\,\delta y; \quad 0 = (z^{-1} - t)\,\delta y + n\,\delta x; \tag{I^{11}}$$

z being the sought ordinate of intersection, and therefore z^{-1} the vergency: for thus we find by elimination the following quadratic to determine the ratio of δx, δy, that is, the direction of δl,

$$(t - r)\,\delta x\,\delta y = n(\delta y^2 + \delta x^2), \tag{K^{11}}$$

which may be put under the form

$$\sin. 2\phi = \frac{2n}{t - r}, \tag{L^{11}}$$

the angle ϕ being, as in (R^{10}), the inclination of δl to the axis of x, that is, to one of the tangents of the lines of curvature, while r, t, are the two curvatures themselves, of the guiding paraboloid; there are therefore two real directions of δl, or one, or none, corresponding to the intersection supposed, according as we have

$$\left(\frac{t - r}{2}\right)^2 >, \text{ or } =, \text{ or } < n^2; \tag{M^{11}}$$

so that we are thus conducted anew to the same conditions of reality, and to the same symmetric directions of the two planes of vergency, which we obtained before by a reasoning of a more geometrical kind. The same conditions may also be obtained by considering the quadratic for the vergency itself, namely,

$$(z^{-1} - r)(z^{-1} - t) + n^2 = 0, \tag{N^{11}}$$

which results from the equations (I^{11}) and shows that the sum and product of the two vergencies may be thus expressed, by means of the curvatures r, t, and the constant of deviation n,

$$z_1^{-1} + z_2^{-1} = r + t; \quad z_1^{-1} z_2^{-1} = rt + n^2. \tag{O^{11}}$$

The equations (I^{11}) give also, by elimination of n,

$$z^{-1} = r \cos. \phi^2 + t \sin. \phi^2; \tag{P^{11}}$$

we see, therefore, as before, that the two vergencies, when real, of the final ray-lines, are the curvatures of the two corresponding sections of the guiding paraboloid. In general the centre of curvature of any section of this surface, made by a normal plane drawn through the given final ray-line, is the common *focus by projection* of all the near ray-lines from the points of that section; that is, the projections of these near ray-lines on this plane, all pass through this centre of curvature. *The two rectangular planes of curvature, or principal diametral planes, of the guiding paraboloid, may therefore be called the planes of extreme projection*; under which view they were considered in the First Supplement, for the case of an uniform medium, and were proposed as *a pair of natural co-ordinate planes* passing through any given straight ray. The two planes of vergency, for the case of straight final rays, were also considered in that First Supplement, in connexion with the two developable pencils or ray-surfaces which pass through a given straight ray, and of which the two tangent planes contain rays infinitely near, and therefore coincide with the two planes of vergency.

When the planes of vergency are real and distinct, then, whether the final rays are straight or curved, there exist *two guiding lines* perpendicular to the given final ray-line, which are both intersected by all the near final ray-lines from the points B_δ on the given final plane of xy, and which therefore suffice to determine the geometrical arrangement and relations of that system of final ray-lines. To prove the existence and determine the positions of these two guiding lines, let us examine what conditions are necessary and sufficient, in order that a right line having for equations

$$y = x \tan.\Phi, \quad z = Z, \tag{Q^{11}}$$

should be intersected by all the near final ray-lines of the system (K^{10}). These conditions are

$$Z^{-1} = r + n \cotan.\Phi = t - n \tan.\Phi; \tag{R^{11}}$$

they give

$$\sin.2\Phi = \frac{2n}{t-r}, \tag{S^{11}}$$

and

$$(Z^{-1} - r)(Z^{-1} - t) + n^2 = 0: \tag{T^{11}}$$

when therefore

$$(t-r)^2 > 4n^2, \tag{U^{11}}$$

that is, when there are two real vergencies there are also two real guiding lines of the kind explained above; and these two guiding lines are contained in the two planes of vergency, and cross the final ray-line in the two corresponding points in which it is crossed by other ray-lines of the same system: the intersection of each guiding line with the given final ray-line being the point of convergence or divergence of the near ray-lines contained in that plane of vergency which contains the other guiding line. When the constant of deviation n vanishes, these guiding lines are necessarily real, and are the axes of the two principal circles of curvature of the guiding paraboloid. And when the final rays are straight, then, whether n vanishes or not, *the two guiding lines* (if real) *are tangents to the two caustic surfaces*; that is, to *the two surfaces which are touched by the final rays, and are the loci of the two points of vergency.* If the guiding lines are imaginary then the points of vergency are so too, and the final rays are not all tangents to any common surface. We shall have occasion to resume hereafter the theory of the caustic and developable surfaces.

If it happen that

$$t - r = \pm 2n, \tag{V^{11}}$$

without $t - r$ and n separately vanishing, then the two planes of vergency close up into one plane, bisecting one pair of the right angles formed by the two principal planes of curvature of the guiding paraboloid; the two vergencies reduce themselves to a single vergency, corresponding to this single plane, and equal to the semisum of the two curvatures of the same surface: and the two guiding lines reduce themselves to a single guiding-line, passing through the corresponding point of convergence or divergence, and having still the property of being intersected by all the near final ray-lines, although this property is not now sufficient to determine this system of ray-lines.

But if the two members of (V^{11}) vanish separately, that is, if the difference of the curvatures and the constant of deviation are separately equal to zero, then the guiding paraboloid is a surface of revolution, having its summit at the given final point B, and all the near final ray-lines are normals to this paraboloid of revolution, and (with the same order of approximation) to the osculating sphere at its summit, and they all pass through the centre of this sphere. Reciprocally, if there be any one point $0, 0, Z$, through which all the final ray-lines pass, the equations (K^{10}) give

$$n = 0, \quad t = r = Z^{-1} : \tag{W^{11}}$$

and the more general equations (V^9), in which the rectangular axes of x and y are arbitrary, give

$$\frac{\delta\alpha}{\delta x} = \frac{\delta\beta}{\delta y} = -Z^{-1}; \quad \frac{\delta\alpha}{\delta y} = 0; \quad \frac{\delta\beta}{\delta x} = 0; \tag{X^{11}}$$

that is, by (G^9), or (C^9),

$$\left.\begin{aligned} \frac{\delta^2 V}{\delta x^2} + Z^{-1}\frac{\delta^2 v}{\delta\alpha^2} &= \frac{\delta^2 v}{\delta\alpha\,\delta x}; \\ \frac{\delta^2 V}{\delta x\,\delta y} + Z^{-1}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} &= \frac{\delta^2 v}{\delta\alpha\,\delta y} = \frac{\delta^2 v}{\delta\beta\,\delta x}; \\ \frac{\delta^2 V}{\delta y^2} + Z^{-1}\frac{\delta^2 v}{\delta\beta^2} &= \frac{\delta^2 v}{\delta\beta\,\delta y}. \end{aligned}\right\} \tag{Y^{11}}$$

When the final rays are straight, and satisfy these last conditions (Y^{11}), which then reduce themselves to the following,

$$\frac{\delta^2 V}{\delta x^2} + Z^{-1}\frac{\delta^2 v}{\delta\alpha^2} = 0, \quad \frac{\delta^2 V}{\delta x\,\delta y} + Z^{-1}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} = 0, \quad \frac{\delta^2 V}{\delta y^2} + Z^{-1}\frac{\delta^2 v}{\delta\beta^2} = 0, \tag{Z^{11}}$$

the given final ray becomes one of those which we have called *principal rays* in former memoirs, and the point of convergence or divergence $0, 0, Z$, is what we have called a *principal focus.*

Second Application of the Elements. Arrangements of the Near Final Ray-lines from an Oblique Plane. Generalisation of the Theory of the Guiding Paraboloid and Constant of Deviation. General Theory of Deflexures of Surfaces. Circles and Axes of Deflexure. Rectangular Planes and Axes of Extreme Deflexure. Deflected Lines passing through these Axes, and having the Centres of Deflexure for their respective Foci by Projection. Conjugate Planes of Deflexure, and Indicating Cylinder of Deflexion.

17. The foregoing theorems respecting the mutual relations of the final ray-lines, suppose that the near final point B_δ is on the given plane which is perpendicular to the given luminous path $(A, B)_\chi$ at its given final point B: but analogous theorems can be found for the more general case where the near final point B' is not in this given perpendicular plane, by combining the solutions of the second and third of the four problems lately discussed; that is, by considering jointly the second and third sets of coefficients (O^9), and therefore by employing the following equations for a final ray-line,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y + \frac{\delta\alpha}{\delta z}\,\delta z\right), \\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y + \frac{\delta\beta}{\delta z}\,\delta z\right). \end{aligned}\right\} \tag{A^{12}}$$

If, in these equations, we establish no relation between δx, δy, δz, then the system of these final ray-lines (A^{12}) is what has been called (in my Theory of Systems of Rays) *a System of the Third Class*, because the equations of a ray-line in this system involve *three* arbitrary elements of position, namely the co-ordinates δx, δy, δz, of the near point B'; but to study more conveniently the properties of this total system of the third class, we may decompose it into partial *systems of the second class*, that is, systems with only two arbitrary elements of position, by assuming some relation, with an arbitrary parameter, between the three co-ordinates δx, δy, δz, or, in other words, by assuming some arbitrary and variable surface, as a locus for the near point B'. For example we may assume, as this locus, an oblique plane passing through the given point B, and having for equation

$$\delta z = p\,\delta x + q\,\delta y, \tag{B^{12}}$$

in which one of the two parameters p, q, is arbitrary, and the other depends on it by some assumed law; and then, for every such assumed plane locus (B^{12}), we shall have to consider a partial system of the second class, deduced from and included in the total system of the third class (A^{12}); namely a system in which the ray lines are as follows,

$$\left.\begin{aligned} x &= \delta x + z\left(\frac{\delta\alpha}{\delta x} + p\frac{\delta\alpha}{\delta z}\right)\delta x + z\left(\frac{\delta\alpha}{\delta y} + q\frac{\delta\alpha}{\delta z}\right)\delta y;\\ y &= \delta y + z\left(\frac{\delta\beta}{\delta x} + p\frac{\delta\beta}{\delta z}\right)\delta x + z\left(\frac{\delta\beta}{\delta y} + q\frac{\delta\beta}{\delta z}\right)\delta y. \end{aligned}\right\} \tag{C^{12}}$$

Let us therefore consider the geometrical arrangement and properties of this system of final ray-lines (C^{12}), corresponding to the oblique plane locus (B^{12}) of the final point B'.

The system (C^{12}), of ray-lines from the arbitrary oblique plane (B^{12}), includes, as a particular case, the system of ray-lines from the plane of no obliquity: that is, the system (V^{9}), considered in a former number. And as the ray-lines of that particular system (V^{9}) were found to have a remarkable connection with the guiding paraboloid (Z^{9}), which touched the given perpendicular plane locus of the near final point B_δ, and which satisfied the differential condition of the second order (Y^{9}): so, the ray-lines of the more general system (C^{12}) may be shown to be connected in an analogous manner with the following more general paraboloid, which satisfies the same differential condition (Y^{9}), and touches the more general oblique plane locus (B^{12}) at the given final point B,

$$z = px + qy + \tfrac{1}{2}rx^2 + sxy + \tfrac{1}{2}ty^2; \tag{D^{12}}$$

in which p, q, retain their recent meanings, and the coefficients r, s, t have the following values,

$$\left.\begin{aligned} r &= -\left(\frac{\delta\alpha}{\delta x} + p\frac{\delta\alpha}{\delta z}\right); \quad t = -\left(\frac{\delta\beta}{\delta y} + q\frac{\delta\beta}{\delta z}\right);\\ s &= -\tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} + \frac{\delta\alpha}{\delta y} + p\frac{\delta\beta}{\delta z} + q\frac{\delta\alpha}{\delta z}\right). \end{aligned}\right\} \tag{E^{12}}$$

But in order to develope this more general connexion, between the ray-lines (C^{12}) and the paraboloid (D^{12}), it will be useful previously to establish some general theorems respecting the deflexure of curved surfaces, which include some of the known theorems respecting their curvatures and planes of curvature.

Let us then consider the paraboloid (D^{12}), or any other curved surface which has, at the origin of co-ordinates, a complete contact of the second order therewith, and which is therefore approximately represented by the same equation: that is, (on account of the arbitrary position of the origin, and arbitrary values of the coefficients p, q, r, s, t,) any surface of continuous curvature, near any assumed point on this surface. The tangent plane at this arbitrary point or origin has for equation

$$z = px + qy; \tag{F^{12}}$$

and the *deflexion* from this tangent plane, measured in the direction of the arbitrary axis of z, which we shall call the *axis of deflexion*, or in any direction infinitely near to this, is, for any point B' infinitely near to the point of contact B,

$$\textit{Deflexion} = \tfrac{1}{2}\delta^2 z = \tfrac{1}{2} r\,\delta x^2 + s\,\delta x\,\delta y + \tfrac{1}{2} t\,\delta y^2. \tag{G^{12}}$$

This deflexion depends therefore on the perpendicular distance δl of the near point B' from the axis of deflexion, and on the direction of the plane containing this point and axis; in such a manner that if we put, as in (R^{10}),

$$\delta x = \delta l \,.\, \cos.\phi, \quad \delta y = \delta l \,.\, \sin.\phi,$$

and give the name of *deflexure* (after the analogy of the known name *curvature*) to the quotient $\dfrac{\delta^2 z}{\delta l^2}$, that is, to the double deflexion divided by the square of the perpendicular distance from the axis of deflexion, we shall have the following law of dependence of this *deflexure*, which we shall denote by f, on the angle ϕ,

$$\textit{Deflexure} = f = \frac{\delta^2 z}{\delta l^2} = r\ \cos.\phi^2 + 2s\ \cos.\phi\ \sin.\phi + t\ \sin.\phi^2. \tag{H^{12}}$$

There are, therefore, *two rectangular planes of extreme deflexure*, corresponding to angles ϕ_1, ϕ_2, determined by the following formula,

$$\tan.2\phi = \frac{2s}{r-t}; \tag{I^{12}}$$

and if we take these for the co-ordinate planes of xz, yz, and denote the *two extreme deflexures* corresponding to f_1, f_2, we have

$$r = f_1, \quad s = 0, \quad t = f_2, \tag{K^{12}}$$

and the general formula for the deflexure becomes

$$f = f_1\ \cos.\phi^2 + f_2\ \sin.\phi^2 : \tag{L^{12}}$$

which is analogous to, and includes, the known formula for the curvature of a normal section. And as it is usual to consider a system of circles of curvature, for any given point of a curved surface, namely, the osculating circles of the normal sections of that surface, so we may now more generally consider a system of *circles of deflexure*: namely, in each plane of deflexure ϕ, a circle passing through the given point of the surface; and having its centre on the given axis of deflexion, and its curvature equal to the deflexure f; so that the radius of this circle, or the ordinate of its centre, which we may call the *radius of deflexure*, is $\dfrac{1}{f}$, and so that the equations of the circle of deflexure are,

$$y = x \tan.\phi, \quad x^2 + y^2 + z^2 = \frac{2z}{f}. \tag{M^{12}}$$

We may also give the name of *axis of deflexure*, to the axis of this circle, that is, to the right line having for equations

$$y = -x \,\mathrm{cotan.}\,\phi, \quad z = \frac{1}{f}: \tag{N^{12}}$$

and we easily see that there are *two principal circles of deflexure*, analogous to the two principal circles of curvature, namely, the two circles having for equations

$$\left.\begin{array}{lll} \text{First} & y = 0, & x^2 + z^2 = \dfrac{2z}{f_1}; \\ \text{Second} & x = 0, & y^2 + z^2 = \dfrac{2z}{f_2}; \end{array}\right\} \tag{O^{12}}$$

and *two principal rectangular axes of deflexure*, namely,

$$\text{First } x = 0, \quad z = \frac{1}{f_1}; \quad \text{Second } y = 0, \quad z = \frac{1}{f_2}. \tag{P^{12}}$$

These principal axes of deflexure are analogous to the principal axes of curvature, that is, to the axes of the two principal osculating circles of the normal sections, in the less general theory of normals. And as, in that theory, the near normals all pass through the two principal axes of curvature, so we may now consider a more general system of right lines, which we shall call the *deflected lines,* all near the arbitrary axis of deflexion, and all passing through the two corresponding principal axes of deflexure, and therefore having for equations,

$$x = \delta x - z f_1 \,\delta x, \quad y = \delta y - z f_2 \,\delta y, \tag{Q^{12}}$$

when the co-ordinates are chosen as before. These deflected lines are normals, in the present order of approximation, to the locus of the circles of deflexure (M^{12}), that is, to the surface of the fourth degree

$$x^2 + y^2 + z^2 = \frac{2z(x^2 + y^2)}{f_1 x^2 + f_2 y^2}; \tag{R^{12}}$$

and they might be defined by this condition, or by the condition that they are normals, in the same order of approximation, to the following paraboloid,

$$z = \tfrac{1}{2}(f_1 x^2 + f_2 y^2), \tag{S^{12}}$$

which osculates to the locus (R^{12}), and has the property that its ordinates measure the deflexions (G^{12}) of the given surface.

A deflected line of the system (Q^{12}) is in the corresponding plane of deflexure

$$y\,\delta x = x\,\delta y, \tag{T^{12}}$$

if that plane coincide with either of those two principal rectangular planes of deflexure, which we have taken for co-ordinate planes; but otherwise the deflected line makes with the plane of deflexure an infinitesimal angle $\delta\psi$, expressed as follows,

$$\delta\psi = \tfrac{1}{2}(f_1 - f_2)\,\delta l\,.\sin.2\phi: \tag{U^{12}}$$

this angle, therefore, is equal to the semidifference of the extreme deflexures multiplied by the infinitesimal perpendicular distance from the axis of deflexion, and by the sine of twice the inclination ϕ of this perpendicular (or of the plane of deflexure containing it) to one of the two rectangular planes of extreme deflexure. In this general case, the deflected line (Q^{12}) does not intersect the given axis of deflexion, which we have made the axis of z; but the deflected line (Q^{12}) always intersects its own axis of deflexure (N^{12}), in a point of which the co-ordinates may be thus expressed

$$x = -\frac{\delta\psi}{f}\,.\sin.\phi, \quad y = \frac{\delta\psi}{f}\,.\cos.\phi, \quad z = \frac{1}{f}, \tag{V^{12}}$$

the symbols f, ϕ, and $\delta\psi$, retaining their recent meanings. It is easy to see that if a near deflected line be projected on the corresponding plane of deflexure, the projection will cross the axis of deflexion in the centre of the circle of deflexure; and therefore that this centre of deflexure may be considered as a *focus by projection*, and that *the planes of extreme deflexure are planes of extreme projection.*

The foregoing results respecting the deflexures and deflected lines of a curved surface, near any given point upon that surface, and for any given axis of deflexion, may easily be expressed by general formulæ extending to an arbitrary origin and arbitrary axes of co-ordinates. If, for simplicity, we still suppose the co-ordinates rectangular, and still take the given point upon the surface for origin, and the given axis of deflexion for axis of z, but leave the rectangular co-ordinate planes of xz and yz arbitrary, so that the coefficient s in the equation of the surface shall not in general vanish, then the equations of a deflected line become

$$x = \delta x - z(r\,\delta x + s\,\delta y), \quad y = \delta y - z(s\,\delta x + t\,\delta y); \tag{W^{12}}$$

since the equation of the paraboloid (S^{12}), to which they are nearly normals, and of which the ordinates measure the deflexions (G^{12}) of the given surface, becomes

$$z = \tfrac{1}{2}r\,\delta x^2 + s\,\delta x\,\delta y + \tfrac{1}{2}t\,\delta y^2. \tag{X^{12}}$$

The deflexure for any plane ϕ is expressed by the general formula (H^{12}); and in like manner the general formulæ (M^{12}) (N^{12}) determine still the circle and axis of deflexure. The two principal planes of deflexure ϕ_1, ϕ_2, are still determined by the formula (I^{12}), while the corresponding extreme deflexures f_1, f_2, are the roots of the following quadratic

$$f^2 - f(r+t) + rt - s^2 = 0: \qquad (\mathrm{Y}^{12})$$

and the angular deviation $\delta\psi$ of a deflected line from the corresponding plane of deflexure, is thus expressed,

$$\delta\psi = \tfrac{1}{2}(f_1 - f_2) \,.\, \sin.(2\phi - 2\phi_1) \,.\, \delta l = \left(\frac{r-t}{2} \,.\, \sin. 2\phi - s \,.\, \cos. 2\phi\right) \delta l. \qquad (\mathrm{Z}^{12})$$

Before we proceed to apply these general remarks on the deflexures of surfaces to the optical question proposed in the present number, that is, to the study of the connexion of the ray-lines (C^{12}) with the paraboloid (D^{12}), we may remark that the theory which M. Dupin has given, in his excellent *Développements de Géométrie*, of the *indicating curves* and *conjugate tangents* of a surface, may be extended from curvatures to deflexures. For if we consider the deflexion ($\frac{1}{2}\delta^2 z = \frac{1}{2} f\, \delta l^2$) in the given arbitrary direction of z as equal to any given infinitesimal quantity of the second order, that is, if we cut the given surface by a plane

$$z - px - qy = \tfrac{1}{2}\delta^2 z = \mathit{deflexion} = \text{const.}, \qquad (\mathrm{A}^{13})$$

parallel and infinitely near to the given tangent plane (F^{12}), we obtain in general a plane curve of section which may be considered as of the second degree, namely, the *indicating curve* considered by M. Dupin, of which the axes by their directions and values indicate the shape of the given surface near the given point, by indicating its curvatures and planes of curvature. This indicating curve is on the following *cylinder of the second degree*, which has for its indefinite axis the axis of deflexion, and which we shall call the *indicating cylinder of deflexion*,

$$rx^2 + 2sxy + ty^2 = \delta^2 z = \text{const.}; \qquad (\mathrm{B}^{13})$$

and it is easy to see that the two principal planes of deflexure, ϕ_1, ϕ_2, are the principal diametral planes of this indicating cylinder, and that the two principal deflexures f_1, f_2, positive or negative, are equal respectively to the given double deflexion $\delta^2 z$ divided by the squares of the real or imaginary principal semidiameters or seimiaxes of the cylinder, perpendicular to its indefinite axis. In general, the positive or negative deflexure f, corresponding to any plane of deflexure ϕ, is equal to the given double deflexion $\delta^2 z$ divided by the square of the real or imaginary semidiameter of the cylinder, contained in this plane of deflexure, and perpendicular to the axis of deflexion, that is, to the indefinite axis of the cylinder. Hence it follows, that if we consider any two conjugate diametral planes ϕ, $\phi_{\prime}$, which we shall call *conjugate planes of deflexure*, and which are connected by the relation

$$0 = r + s(\tan.\phi + \tan.\phi_{\prime}) + t \,.\, \tan.\phi \,\tan.\phi_{\prime}, \qquad (\mathrm{C}^{13})$$

the sum of the two corresponding conjugate radii of deflexure $\dfrac{1}{f} + \dfrac{1}{f_{\prime}}$, *is constant, and equal to the sum of the two extreme or principal radii*: that is, we have

$$\frac{1}{f} + \frac{1}{f_{\prime}} = \frac{1}{f_1} + \frac{1}{f_2}, \qquad (\mathrm{D}^{13})$$

a relation which might also have been deduced from the general expression for the deflexure, without its being necessary to employ the indicating cylinder. We may remark that any two conjugate planes of deflexure, connected by the relation (C^{13}), intersect the tangent plane of the surface in two conjugate tangents of the kind considered by M. Dupin.

Let us now resume the system of ray-lines (C^{12}), of which the equations may be put by (E^{12}) under the form

$$\left.\begin{aligned} x &= \delta x - z(r\,\delta x + s\,\delta y) - zn\,\delta y, \\ y &= \delta y - z(s\,\delta x + t\,\delta y) + zn\,\delta x, \end{aligned}\right\} \tag{E^{13}}$$

if we make

$$n = \tfrac{1}{2}\left(\frac{\delta\beta}{\delta x} - \frac{\delta\alpha}{\delta y} + p\frac{\delta\beta}{\delta z} - q\frac{\delta\alpha}{\delta z}\right): \tag{F^{13}}$$

and let us compare these ray-lines with the deflected lines from the auxiliary paraboloid (D^{12}), which have for equations

$$x = \delta x - z(r\,\delta x + s\,\delta y), \quad y = \delta y - z(s\,\delta x + t\,\delta y). \tag{W^{12}}$$

We easily see, by this comparison, that the infinitesimal angle of deviation $\delta\nu$ of a ray-line (E^{13}) from the corresponding deflected line (W^{12}), is still determined by the same formula (M^{10})

$$\delta\nu = n\,\delta l,$$

as in the simpler theory of the guiding paraboloid explained in the fifteenth number; that is, this angular deviation $\delta\nu$ is still equal to the perpendicular distance δl of the near final point from the given final ray-line, multiplied by a constant of deviation n. The plane of this angle $\delta\nu$, that is, the plane containing the ray-line (E^{13}) and the deflected line (W^{12}), has for equation

$$x\,\delta x + y\,\delta y = \delta l^2 - z(r\,\delta x^2 + 2s\,\delta x\,\delta y + t\,\delta y^2), \tag{G^{13}}$$

and therefore contains the right line having for equations

$$x\,\delta x + y\,\delta y = 0, \quad z = \frac{\delta x^2 + \delta y^2}{r\,\delta x^2 + 2s\,\delta x\,\delta y + t\delta y^2}, \tag{H^{13}}$$

that is, the axis of deflexure (N^{12}): results which are analogous to those of the fifteenth number, expressed by the equations (N^{10}) (O^{10}). And we may construct the final ray-line (E^{13}) by a process of rotation analogous to that already employed, namely, by making the deflected line (W^{12}), which passes through the two rectangular axes of deflexure of the auxiliary paraboloid (D^{12}), revolve round the perpendicular δl, through the infinitesimal angle $\delta\nu$, proportional to that perpendicular. The theory, therefore, of the guiding paraboloid and constant of deviation, which was given in the fifteenth number, for the ray-lines from the near points B_δ on the final perpendicular plane, extends with little modification to the ray-lines from the points B' on any final oblique plane locus passing through the given final point: namely, by employing a more general auxiliary paraboloid, and by considering deflexures and deflected lines, instead of curvatures and normals. And we may transfer to this more general auxiliary paraboloid, and to its connected constant of deviation, the reasonings of the sixteenth number, respecting the system of final ray-lines; for example, the reasoning respecting the foci by projection, and those respecting the condition of intersection of such ray-lines. And since for any given values of p, q, that is, for any given position of the oblique plane (B^{12}), we can construct the new auxiliary paraboloid (D^{12}), and its new constant of deviation (F^{13}), by the coefficients

$$\frac{\delta\alpha}{\delta x}, \quad \frac{\delta\beta}{\delta x}, \quad \frac{\delta\alpha}{\delta y}, \quad \frac{\delta\beta}{\delta y}, \quad \frac{\delta\alpha}{\delta z}, \quad \frac{\delta\beta}{\delta z},$$

that is, by means of the former guiding paraboloid (Z^{9}) and the former constant of deviation (B^{10}), and by the magnitude and plane of curvature (T^{9}) of the final ray, we may be considered as having reduced the theory of the geometrical arrangement and relations of the system of final ray-lines (C^{12}), from an oblique plane (B^{12}), to the theory of the *elements of arrangement*, which was given in the fifteenth number.

Construction of the New Auxiliary Paraboloid, (or of an Osculating Hyperboloid,) and of the New Constant of Deviation, for Ray-lines from an Oblique Plane, by the former Elements of Arrangement.

18. To construct the new auxiliary paraboloid (D^{12}) by the former elements of arrangement, we may observe that this new paraboloid not only touches the given oblique plane (B^{12}) at the given final point B of the original luminous path, but osculates in all directions at that given point to a certain hyperboloid, represented by the following equation,

$$z = px + qy + \tfrac{1}{2}r_0x^2 + s_0xy + \tfrac{1}{2}t_0y^2 - \tfrac{1}{2}z\left(x\frac{\delta\alpha}{\delta z} + y\frac{\delta\beta}{\delta z}\right); \qquad (\mathrm{I}^{13})$$

in which r_0 s_0 t_0 are the particular values

$$r_0 = -\frac{\delta\alpha}{\delta x}, \quad s_0 = -\tfrac{1}{2}\left(\frac{\delta\alpha}{\delta y} + \frac{\delta\beta}{\delta x}\right), \quad t_0 = -\frac{\delta\beta}{\delta y}, \qquad (\mathrm{K}^{13})$$

of the coefficients r s t, deduced from the general expressions (E^{12}) by making

$$p = 0, \quad q = 0, \qquad (\mathrm{L}^{13})$$

that is, by passing to the case of no obliquity; so that the equation (Z^{9}) of the guiding paraboloid may be put under the form

$$z = \tfrac{1}{2}r_0x^2 + s_0xy + \tfrac{1}{2}t_0y^2, \qquad (\mathrm{M}^{13})$$

which includes the form (G^{10}). Reciprocally, the sought paraboloid (D^{12}) is the only paraboloid which has its indefinite axis parallel to the given final ray-line, and osculates in all directions at the given final point to the hyperboloid (I^{13}): it is therefore sufficient to construct this osculating hyperboloid, in order to deduce the sought paraboloid (D^{12}). We might even employ the hyperboloid as a new guiding surface for the ray-lines from the oblique plane, instead of employing the paraboloid, since these two osculating surfaces have the same deflexures and deflected lines, near their given point of osculation.

Now to construct the osculating hyperboloid (I^{13}), by the oblique plane (B^{12}) or (F^{12}), and by the former elements of arrangement, that is, by the guiding paraboloid (M^{13}), and by the coefficients $\dfrac{\delta\alpha}{\delta z}$, $\dfrac{\delta\beta}{\delta z}$, which determine the magnitude and plane of curvature of the final ray, we may compare the sought hyperboloid (I^{13}) with the following new paraboloid

$$z = px + qy + \tfrac{1}{2} r_0 x^2 + s_0 xy + \tfrac{1}{2} t_0 y^2, \tag{N^{13}}$$

which may be called the *guiding paraboloid removed*, since it is equal and similar to the guiding paraboloid (M^{13}), and may be obtained by transporting that guiding paraboloid without rotation to a new position such that it touches the given oblique plane at the given point. The intersection of the hyperboloid (I^{13}) and paraboloid (N^{13}) consists in general of an ellipse or hyperbola in the given plane

$$z = 0, \tag{O^{13}}$$

perpendicular to the given final ray, and of a parabola in the plane

$$x\frac{\delta\alpha}{\delta z} + y\frac{\delta\beta}{\delta z} = 0, \tag{P^{13}}$$

which contains the given final ray-line or ray-tangent, and is perpendicular to the final plane of curvature of the ray. If then, we make this final plane of curvature the plane of xz, so that its equation shall be

$$y = 0, \tag{Q^{13}}$$

and so that, by (T^9),

$$\frac{\delta\beta}{\delta z} = 0, \tag{R^{13}}$$

we shall have the following equations for the two curves of intersection; first, for the ellipse or hyperbola,

$$z = 0, \quad px + qy + \tfrac{1}{2} r_0 x^2 + s_0 xy + \tfrac{1}{2} t_0 y^2 = 0; \tag{S^{13}}$$

and secondly, for the parabola,

$$x = 0, \quad z = qy + \tfrac{1}{2} t_0 y^2 : \tag{T^{13}}$$

and these two curves may be considered as known, since they are the intersections of two known planes with the known guiding paraboloid removed to a known position. To examine now how far a surface of the second degree is restricted by the condition of containing these two known curves, and what other conditions are necessary, in order to oblige this surface to be the hyperboloid sought, let us employ the following general form for the equation of a surface of the second degree,

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + K = 0, \tag{U^{13}}$$

and let us seek the relations which restrict the coefficients of this equation when the surface is obliged to contain the two known curves. The condition of containing the parabola (T^{13}) gives

$$K = 0, \quad H = -Iq, \quad E = 0, \quad C = 0, \quad B = -\tfrac{1}{2}It_0; \tag{V^{13}}$$

so that, by this condition alone, the general equation (U^{13}) is reduced to the following form,

$$z = qy + \tfrac{1}{2}t_0 y^2 - \frac{x}{I}(G + Fz + Dy + Ax). \tag{W^{13}}$$

In order that this less general surface of the second degree (W^{13}), should contain the ellipse or hyperbola (S^{13}), it is necessary and sufficient that we should have the relations,

$$G = -Ip, \quad D = -Is_0, \quad A = -\tfrac{1}{2}Ir_0 : \tag{X^{13}}$$

the general equation, therefore, of those surfaces of the second degree which contain at once the two known curves (S^{13}) (T^{13}), involves only one arbitrary coefficient, and may be put under the form

$$z = px + qy + \tfrac{1}{2}r_0x^2 + s_0xy + \tfrac{1}{2}t_0y^2 + \lambda xz. \tag{Y^{13}}$$

This general equation, with the arbitrary coefficient λ, belongs to the guiding paraboloid removed, that is, to the surface (N^{13}), when we suppose

$$\lambda = 0; \tag{Z^{13}}$$

and the same general equation belongs by (R^{13}) to the sought hyperboloid (I^{13}), when

$$\lambda = -\tfrac{1}{2}\frac{\delta\alpha}{\delta z}. \tag{A^{14}}$$

To put this last condition under a geometrical form, let us, as we have already considered the intersections of the hyperboloid with the two rectangular co-ordinate planes of xy and yz, consider now its intersection with the third co-ordinate plane of xz, that is, with the plane of curvature (Q^{13}) of the given final ray. This intersection is the following hyperbola,

$$y = 0, \quad z = px + \tfrac{1}{2}r_0x^2 - \tfrac{1}{2}\frac{\delta\alpha}{\delta z}xz, \tag{B^{14}}$$

and the corresponding intersection for the surface (Y^{13}) is

$$y = 0, \quad z = px + \tfrac{1}{2}r_0x^2 + \lambda xz; \tag{C^{14}}$$

the condition (A^{14}) is therefore equivalent to an expression of the coincidence of these two intersections; and if we oblige the surface of the second degree (U^{13}) to contain the three curves (S^{13}) (T^{13}) (B^{14}), in the three rectangular co-ordinate planes, we shall thereby oblige it to become the sought hyperboloid (I^{13}). It is not necessary, however, though it is sufficient, to assign the hyperbola (B^{14}), as a third curve upon this hyperboloid. For, in general, if we know the intersections of a surface of the second degree with two known planes, there remains only one unknown quantity in the equation of that surface, and the intersection with a third known plane is more than sufficient to determine it. Thus, in the present question, if the intersection (C^{14}) be distinct from the following parabola

$$y = 0, \quad z = px + \tfrac{1}{2} r_0 x^2, \tag{D^{14}}$$

that is, if the surface (Y^{13}), containing the two known curves (S^{13}) (T^{13}), be distinct from the known guiding paraboloid removed, which also contains the same two curves, the intersection (C^{14}) with the plane of curvature of the ray is in general a hyperbola, which touches the known parabola (D^{14}) at the known origin of co-ordinates, and meets this parabola again in another known point on the axis of x, that is, on the radius of curvature of the known final ray, namely, in the point

$$x = -\frac{2p}{r_0}, \quad y = 0, \quad z = 0; \tag{E^{14}}$$

the hyperbola (C^{14}) has also one asymptote parallel to the known final ray-line or axis of z, namely, the asymptote having for equations

$$x = \frac{1}{\lambda}, \quad y = 0, \tag{F^{14}}$$

and it will be entirely determined, if, in addition to the foregoing properties, we know also a line parallel to its other asymptote, namely, to that which has for equations,

$$x = -2\left(\frac{\lambda}{r_0}\right) z - \frac{1}{\lambda} - \frac{2p}{r_0}, \quad y = 0: \tag{G^{14}}$$

it will therefore be obliged to coincide with the hyperbola (B^{14}), if only we oblige its second asymptote (G^{14}) to be parallel to the following known right line,

$$x = \frac{z}{r_0} \frac{\delta\alpha}{\delta z}, \quad y = 0, \tag{H^{14}}$$

in which the coefficient

$$\frac{1}{r_0} \frac{\delta\alpha}{\delta z} = \frac{\textit{curvature of the final ray}}{\textit{deflexure of the guiding paraboloid}}, \tag{I^{14}}$$

the plane of the deflexure r_0 being the plane of curvature of the ray. We see, then, that this last condition, respecting the direction of the second asymptote (G^{14}) of the hyperbolic section (C^{14}), is sufficient, when combined with the conditions of containing the two known curves (S^{13}) (T^{13}), to determine completely the sought hyperboloid (I^{13}). Even the conditions of containing the two curves (S^{13}) (T^{13}) are not perfectly distinct and independent; nor would their coexistence be possible, in the determination of a surface of the second degree, if the two points in which the parabola (T^{13}) is intersected by the axis of y, that is, by the intersection-line of the planes of the two curves, namely, the origin and the point

$$x = 0, \quad y = -\frac{2q}{t_0}, \quad z = 0, \tag{K^{14}}$$

were not also contained on the ellipse or hyperbola (S^{13}). But we may confine ourselves to the last chosen conditions, of having these two known curves as the intersections of the hyperboloid with two known planes, and of having known directions for the asymptotes of its hyperbolic curve of intersection with a third known plane, as adequate and sufficiently simple conditions for the construction of the sought hyperboloid, and thereby of the auxiliary paraboloid (D^{12}), to which that hyperboloid osculates. And with respect to the new constant of deviation n, connected with this auxiliary paraboloid, we may put its general value (F^{13}) under the form

$$n = n_0 + \tfrac{1}{2} p \frac{\delta \beta}{\delta z} - \tfrac{1}{2} q \frac{\delta \alpha}{\delta z}, \tag{L^{14}}$$

n_0 being the particular value

$$n_0 = \tfrac{1}{2} \left(\frac{\delta \beta}{\delta x} - \frac{\delta \alpha}{\delta y} \right) \tag{M^{14}}$$

for the plane of no obliquity, that is, the value (B^{10}) connected with the guiding paraboloid (Z^9) in the theory of the elements of arrangement which was given in a former number: we may therefore construct the new constant n as the ordinate z of a plane

$$z = px + qy + n_0, \tag{N^{14}}$$

which is parallel to the given oblique plane (B^{12}), and contains the point

$$x = 0, \quad y = 0, \quad z = n_0, \tag{O^{14}}$$

so that it intersects the axis of z at a distance from the origin = the old constant of deviation n_0. The other co-ordinates x, y, to which the ordinate $z = n$ corresponds, are

$$x = \tfrac{1}{2} \frac{\delta \beta}{\delta z}, \quad y = -\tfrac{1}{2} \frac{\delta \alpha}{\delta z}, \tag{P^{14}}$$

so that the corresponding line $\sqrt{x^2 + y^2}$ is equal to half the curvature of the ray, and is perpendicular to the radius of that curvature.

The details of the present number have been given, in order to illustrate the subject, by combining it more closely with geometrical conceptions; but the new auxiliary paraboloid, and the new constant of deviation, might have been considered as sufficiently defined by their former algebraical expressions.

Condition of Intersection of Two Near Final Ray-lines. Conical Locus of the Near Final Points, in a variable medium, which satisfy this condition. Investigations of MALUS. *Illustration of the Condition of Intersection, by the Theory of the Auxiliary Paraboloid, for Ray-lines from an Oblique Plane.*

19. Returning now to the system of final ray-lines (C^{12}) from an oblique plane (B^{12}), let us consider the condition necessary in order that one of these near ray-lines (C^{12}) may intersect the given final ray-line or axis of z. This condition may be at once obtained by

making x and y vanish in the equations (C^{12}), and then eliminating z; it may therefore be thus expressed,

$$\delta x\,.\left\{\left(\frac{\delta\beta}{\delta x}+p\frac{\delta\beta}{\delta z}\right)\delta x+\left(\frac{\delta\beta}{\delta y}+q\frac{\delta\beta}{\delta z}\right)\delta y\right\}=\delta y\,.\left\{\left(\frac{\delta\alpha}{\delta x}+p\frac{\delta\alpha}{\delta z}\right)\delta x+\left(\frac{\delta\alpha}{\delta y}+q\frac{\delta\alpha}{\delta z}\right)\delta y\right\}, \tag{Q^{14}}$$

or more concisely thus, on account of the equation of the oblique plane (B^{12}),

$$\delta x\,.\left(\frac{\delta\beta}{\delta x}\,\delta x+\frac{\delta\beta}{\delta y}\,\delta y+\frac{\delta\beta}{\delta z}\,\delta z\right)=\delta y\,.\left(\frac{\delta\alpha}{\delta x}\,\delta x+\frac{\delta\alpha}{\delta y}\,\delta y+\frac{\delta\alpha}{\delta z}\,\delta z\right), \tag{R^{14}}$$

that is,

$$\delta x\,\delta\beta=\delta y\,\delta\alpha; \tag{S^{14}}$$

it is therefore necessary and sufficient, for the intersection sought, that the near final point B' should be on a certain conical locus of the second degree, determined by the equation (R^{14}), between the co-ordinates δx, δy, δz. A conical locus of this kind appears to have been first discovered by MALUS. That excellent mathematician and observer had occasion, in his *Traité D'Optique*, to make some remarks on the general properties of a system of right-lines in space, represented by equations of the form

$$\frac{x-x'}{m}=\frac{y-y'}{n}=\frac{z-z'}{o},$$

in which m, n, o, are any given functions of the co-ordinates x', y', z', of a point through which the line is supposed to pass, and by which it is supposed to be determined; and he remarked that the condition of intersection of a line thus determined, with the corresponding near line from a point infinitely near, was expressed by an equation of the second degree between the differentials of the co-ordinates x', y', z', which might be considered as the equation of a conical locus of the second degree for the infinitely near point. The theory of systems of rays which was given by MALUS, differs much, in form and in extent, from that proposed in the present Supplement; especially because, in the former theory, the coefficients which mark the direction of a ray were left as independent and unconnected functions, whereas, in the latter, they are shown to be connected with each other, and to be deducible by uniform methods from one characteristic function. But the mere consideration of the existence of some functional laws, whether connected or arbitrary, of dependence of the coefficients $m\ n\ o$ on the co-ordinates $x'\ y'\ z'$, or of $\alpha\ \beta\ \gamma$ on $x\ y\ z$, conducts easily, as we have seen, to a conical locus of the kind (R^{14}). This result may however be illustrated by the theory which we have given of the geometrical relations of the near final ray-lines from an oblique plane with the deflected lines of a certain auxiliary paraboloid, and with a certain law and constant of deviation.

For, according to the theory of these relations, the ray-line from a near final point B' on a given oblique plane drawn through the given point B, will or will not intersect the given final ray-line from B, according as its deviation $\delta\nu$ from its own deflected line does or does not compensate for the deviation $\delta\psi$ of that deflected line from the corresponding plane of deflexure, by these two deviations being equal in magnitude but opposite in direction; the condition of intersection may therefore be thus expressed,

$$\delta\nu + \delta\psi = 0; \tag{T^{14}}$$

or, by the values of the deviations $\delta\nu$, $\delta\psi$, established in the seventeenth number,

$$n = \frac{t-r}{2} \,.\, \sin.\, 2\phi + s \,.\, \cos.\, 2\phi, \tag{U^{14}}$$

that is,

$$n(\delta x^2 + \delta y^2) = (t-r)\, \delta x \, \delta y + s(\delta x^2 - \delta y^2) : \tag{V^{14}}$$

and the condition of intersection thus obtained, by the consideration of two equal and opposite deviations, is, on account of the meanings (E^{12}) (F^{13}) of n, r, s, t, equivalent to (Q^{14}), and therefore to the equation (R^{14}) of the cone of the second degree. In this manner, then, as well as by the former less geometrical process, we might perceive that the two planes of vergency for the ray-lines from an oblique plane, (determined by (U^{14}) or (V^{14}), and analogous to the two less general planes of vergency considered in the sixteenth number,) intersect the oblique plane in the same two lines in which that plane intersects a certain cone of the second degree, through the centre of which cone it passes; and that the planes of vergency are imaginary when the oblique plane does not intersect this cone. We may remark that the intersection of the oblique plane with the cone, or of a near final ray-line from the oblique plane with the given final ray-line, is impossible, when the constant of deviation corresponding to the oblique plane is greater (abstracting from its sign) than the semidifference of the extreme deflexures of the auxiliary paraboloid: for then the compensation of the two deviations $\delta\nu$, $\delta\psi$, is impossible, the near ray-line always deviating more from the corresponding deflected line of the auxiliary paraboloid, than this deflected line from the corresponding plane of deflexure. And when the compensation and therefore the intersection becomes possible, by the constant of deviation being less than the semidifference of the two extreme deflexures, then the two real planes of vergency of the near final ray-lines from the oblique plane are symmetrically situated with respect to the two rectangular planes of extreme deflexure: which latter planes may also, for a reason already alluded to, be called the planes of extreme projection of the final ray-lines.

Other Geometrical Illustrations of the Condition of Intersection, and of the Elements of Arrangement. Composition of Partial Deviations. Rotation round the Axis of Curvature of a Final Ray.

20. The condition of intersection of two near final ray-lines may also be illustrated, and might have obtained, by other geometrical considerations, on which we shall dwell a little, because they will help to illustrate and improve the theory of the elements of arrangement.

It was remarked, in the fourteenth number, that the general comparison of a given luminous path $(A, B)_\chi$ with a near path $(A', B')_{\chi+\delta\chi}$ might be decomposed into several particular comparisons, such as the comparisons with the less general near paths $(A, B_d)_\chi$, $(A, B_\delta)_\chi$, and others, on account of the linear form of the expressions (D^9) for the variations $\delta\alpha$, $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, of the extreme small cosines of direction, which form permits us to consider separately and successively the influence of the variations of the extreme co-ordinates and colour, or the influence of any groupes of these variations. Accordingly, by an *Analysis* founded on this remark, we decomposed the general discussion of the geometrical relations of infinitely near rays into four less general problems, which were treated of, in the fifteenth number. The applications, in the sixteenth number, to questions respecting the mutual intersections of the final ray-lines from the final perpendicular plane, may be considered as only illustrations and corollaries of the third of those four problems: but the questions since discussed, respecting the ray-lines from an oblique plane, require a combination of the solutions of the second and third of the four problems, and furnish, therefore, an example of the *Synthesis* of those elements of arrangement of near rays, to which the former *Analysis* had conducted. This synthesis, however, has in the foregoing numbers been itself *algebraically* performed, (namely, by the algebraical addition of certain partial variations,) although many of the results were enunciated geometrically, and combined with geometrical conceptions: but a *geometrical* idea and method, of the Synthesis of the Elements of Arrangement, may be obtained by considering, in a general manner, the geometrical composition of partial deviations.

To understand more fully the occasion of such composition, let us remember that our theory of the Elements of Arrangement enables us to pass from the extreme directions of a given luminous path $(A, B)_\chi$, to the four following sets of near extreme directions, by the solution of the four problems considered in the fifteenth number.

First. The extreme directions of the near path $(A, B)_{\chi+\delta\chi}$, which has the same extreme points A, B, but differs by chromatic dispersion.

Second. The final direction of $(A, B_d)_\chi$, that is, of the original path prolonged at the end, and the initial direction of $(A_d, B)_\chi$, that is, of the same path prolonged at the beginning; these near extreme directions being in general affected by curvature.

Third. The final direction of $(A, B_\delta)_\chi$, and the initial direction of $(A_\delta, B)_\chi$; the small lines $\overline{AA_\delta}$, $\overline{BB_\delta}$, being perpendicular to the given path at its extremities.

Fourth. The initial direction of $(A, B_\delta)_\chi$, and the final direction of $(A_\delta, B)_\chi$.

We saw also that the initial direction of $(A, B_d)_\chi$ and the final direction of $(A_d, B)_\chi$ do not differ from the corresponding extreme directions of the original luminous path.

If then we would apply this theory to determine the final direction of an arbitrary near path $(A', B')_{\chi+\delta\chi}$, we would have to consider and compound, algebraically or geometrically, the following partial deviations from the given final direction of the given path $(A, B)_\chi$: first, the chromatic deviation of the final direction of the near path $(A, B)_{\chi+\delta\chi}$ from that given final direction; second, the deviation of curvature of the final direction of $(A, B_d)_\chi$; third, the final deviation of the path $(A, B_\delta)_\chi$, to be determined by the theory of the final guiding paraboloid; and fourth, the deviation of the final direction of $(A_\delta, B)_\chi$, to be found by the theory of the guiding planes and conjugate guiding axes. A similar composition of four partial deviations is required for the determination of the initial direction of the same arbitrary near path $(A', B')_{\chi+\delta\chi}$.

Now to compound in a geometric manner the four preceding partial deviations of the final ray-line, we may proceed as follows. We may construct each partial deviation, by drawing the deviated final ray-line corresponding, or a line parallel thereto, through the given final point B; the line thus drawn will differ little in direction from the given final ray-line or axis of z, and if we take its length equal to unity, then its small projection on the given final plane of xy, to which it is nearly perpendicular, will measure the magnitude and will indicate the direction of the deviation: and if we compound all these projections according to the usual geometrical rule of composition of forces, the result will be the projection of the equal line which represents in direction the resultant or total deviation. And similarly we may compound the four partial deviations of a near initial ray-line.

The geometrical synthesis of the partial deviations may also be performed in other ways. For example, we may consider each partial deviation as arising from a partial or component rotation, and we may compound these several rotations by the geometrical methods proper for such composition.

In particular, we may compound the final deviation of curvature with any of the other partial deviations, by making the deviated ray-line, obtained without considering the final curvature of the ray, revolve through an infinitely small angle round the axis of final curvature, that is, round the axis of the final osculating circle of the given final ray. By this rotation, the projection B_δ of a near final point B' on the final perpendicular plane, will be brought into the position B'; and, by the same rotation, the near final ray-line, which had been obtained by abstracting from the final curvature, and by considering B_δ as the final point, will be brought, at the same time, into the position of the sought ray-line, which corresponds to a final point at B'.

Applying now these general principles to the particular question respecting the condition of intersection of two near final ray-lines, from two near final points B, B', (the colour χ and the initial point A being considered as common and given,) we see that if the projection B_δ of B' be given, the small projecting perpendicular $\overline{B'B_\delta}$ or δz and therefore also the near point B' itself may in general be determined so as to satisfy the condition of intersection: for the final ray-line from B_δ may in general be brought to intersect the given final ray-line, by revolving through an infinitesimal angle round the axis of curvature of the given final ray. We see also that the angular quantity of rotation and therefore the length $\delta z = \overline{B_\delta B'}$ depends on the position of the projection B_δ, that is, on the co-ordinates δx, δy; and therefore that there must be some determined surface as the locus of the near final point B', when the final ray-line from that point is supposed to intersect the given final ray-line.

To investigate the form of this locus, by the help of the foregoing geometrical conceptions, we may observe that the only point, on the near ray-line from B_δ, which is brought by the supposed rotation to meet the given final ray-line, is the point contained in the final plane of curvature of the given final ray; and that if we call this point, where the ray-line from B_δ intersects the given plane of curvature, the point P, the angle of rotation required is the angle between the line $\overline{BP}$ and the given final ray-line; because the same infinitesimal rotation which brings the near ray-line from B_δ, that is, the line $\overline{B_\delta P}$, into a new position in which it intersects the given final ray-line, brings also the line $\overline{BP}$ into the position of the given final ray-line itself. Translating now these geometrical results into algebraical language, and taking the given final plane of curvature for the plane of xz, so as to satisfy the condition (R^{13}), we find the following co-ordinates of the point P of intersection of this plane of curvature with the ray-line (V^9) from B_δ,

$$x = \delta x - \frac{\delta y \,.\, \left(\dfrac{\delta\alpha}{\delta x}\,\delta x + \dfrac{\delta\alpha}{\delta y}\,\delta y\right)}{\dfrac{\delta\beta}{\delta x}\,\delta x + \dfrac{\delta\beta}{\delta y}\,\delta y}; \quad y = 0; \quad z = \frac{-\delta y}{\dfrac{\delta\beta}{\delta x}\,\delta x + \dfrac{\delta\beta}{\delta y}\,\delta y} : \tag{W^{14}}$$

so that the angle between the line $\overline{BP}$ which connects this point with the origin of co-ordinates, and the given final ray-line or axis of z, is

$$\delta\theta = \left(-\frac{x}{z} =\right) \frac{1}{\delta y} \,.\, \delta x \,.\, \left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y\right) - \left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y\right); \tag{X^{14}}$$

and this being equal to the infinitesimal angle of rotation, that is, to the small line δz or $\overline{B_\delta B'}$ multiplied by $\dfrac{\delta\alpha}{\delta z}$ or by the final curvature of the given ray taken with its proper sign, we have the following equation for the locus of the near point B', when the condition of intersection is to be satisfied,

$$\frac{\delta\alpha}{\delta z}\,\delta z = \frac{1}{\delta y}\,\delta x \left(\frac{\delta\beta}{\delta x}\,\delta x + \frac{\delta\beta}{\delta y}\,\delta y\right) - \left(\frac{\delta\alpha}{\delta x}\,\delta x + \frac{\delta\alpha}{\delta y}\,\delta y\right) : \tag{Y^{14}}$$

which is, accordingly, the equation of the former conical locus (R^{14}), only simplified by the condition (R^{13}), arising from a choice of co-ordinates. Without making that choice, we might easily have deduced in a similar manner the equation (R^{14}), under the form

$$\delta z = \frac{\delta x \left(\dfrac{\delta\beta}{\delta x}\,\delta x + \dfrac{\delta\beta}{\delta y}\,\delta y\right) - \delta y \left(\dfrac{\delta\alpha}{\delta x}\,\delta x + \dfrac{\delta\alpha}{\delta y}\,\delta y\right)}{\dfrac{\delta\alpha}{\delta z}\,\delta y - \dfrac{\delta\beta}{\delta z}\,\delta x}, \tag{Z^{14}}$$

in which each member is an expression for the infinitesimal angle of rotation divided by the curvature of the ray.

Another way of applying the foregoing geometrical principles to investigate the condition of intersection of two near final ray-lines, is to consider the infinitesimal angle by which the ray-line from B_δ deviates from the plane containing the given final ray-line and the near point B_δ. This angular deviation is expressed by the numerator of the fraction (Z^{14}), divided by δl, that is, divided by the small line $\overline{BB_\delta}$; and the denominator of the same fraction (Z^{14}), divided also by δl, is equal to the final curvature of the ray multiplied by the sine of the inclination of the line δl to the radius of this final curvature: and hence it is easy to see, by geometrical considerations, that the fraction in the second member of (Z^{14}) is equal to the infinitesimal angle of rotation required for destroying the last mentioned deviation, divided by the curvature of the ray, and therefore equal to the ordinate δz of the sought locus of the near point B', as expressed by the first member. We might therefore easily have obtained, by calculations founded on this other geometrical view, the same condition of intersection as before, and the same conical locus.

Relations between the Elements of Arrangement, depending only on the Extreme Points, Directions, and Colour, of a Given Luminous Path, and on the Extreme Media. In a Final Uniform Medium, Ordinary or Extraordinary, the two Planes of Vergencey are Conjugate Planes of Deflexure of any Surface of a certain class, determined by the Final Medium; and also of a certain Analogous Surface determined by the whole combination. Relations between the Visible Magnitudes and Distortions of any two small objects viewed from each other through any Optical Combination. Interchangable Eye-axes and Object-axes of Distortion. Planes of No Distortion.

21. It was shown in the fourteenth number, and the result has since been developed in detail, that the general geometrical relations between the extreme directions of infinitely near rays are determined by the coefficients of the linear variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, of the six marks of extreme direction, considered as functions of the six extreme co-ordinates and of the colour; and that, between the forty-two general coefficients of these six linear variations, there exists eighteen general relations, leaving only twenty-four coefficients arbitrary, if we suppose for simplicity that the final and initial co-ordinates are referred to rectangular axes. But besides these eighteen general relations which are common to all optical combinations, there arise certain other relations between the coefficients, when the extreme media are considered as given, and when the extreme points, directions, and colour, of any one luminous path, are also supposed to be known. For, if we then employ the general equations (A^9), we may consider the extreme medium functions v, v', and their partial differentials, as known, and may deduce general expressions for the coefficients before mentioned of the linear variations of the extreme cosines of direction, involving only, as unknown quantities, twenty-seven partial differentials of the second order of the characteristic function V, namely, all of this order, which are not relative to the variation of colour only; but these twenty-seven are connected by the fourteen general relations (Q) (U) (X) (Y), deduced in the third number, of which however only thirteen are distinct, because the two systems (U) (Y) conduct both to one common equation (D); there remain, therefore, as independent quantities, only fourteen of the partial differentials of V, in the general expressions of those twenty-four coefficients of the linear variations of the extreme direction-cosines, which had before been considered as independent, when the extreme medium-functions v, v' were supposed unknown and arbitrary: and if we eliminate the fourteen independent differentials of V between the expressions of these twenty-four coefficients, we shall obtain *ten general relations, between the elements of arrangement of infinitely near rays, involving only the extreme points, directions, and colour, of the given luminous path, and the properties of the extreme media.*

The simplest manner of obtaining these ten general relations is to eliminate the fourteen differentials of V which enter into the twenty-four expressions, deducible from (C^9), from the twenty-four coefficients (D^9). The ten relations thus obtained, may be arranged in three different groupes: the first groupe containing the two following

$$\left.\begin{aligned}
\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta z} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\beta}{\delta z} + \frac{\delta^2 v}{\delta\alpha\,\delta z} &= \frac{\delta v}{\delta x},\\
\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\alpha}{\delta z} + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta z} + \frac{\delta^2 v}{\delta\beta\,\delta z} &= \frac{\delta v}{\delta y},
\end{aligned}\right\} \tag{A^{15}}$$

and two others similar to these, but with accented or initial symbols; the second groupe containing the final relation

$$\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta y} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\beta}{\delta y} + \frac{\delta^2 v}{\delta\alpha\,\delta y} = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\alpha}{\delta x} + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta x} + \frac{\delta^2 v}{\delta\beta\,\delta x}, \tag{B^{15}}$$

and a similar initial relation; and the third groupe comprising the four following,

$$\left.\begin{aligned}
\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta x'} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\beta}{\delta x'} + \frac{\delta^2 v'}{\delta\alpha'^2}\frac{\delta\alpha'}{\delta x} + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\frac{\delta\beta'}{\delta x} &= 0,\\
\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta y'} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\beta}{\delta y'} + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\frac{\delta\alpha'}{\delta x} + \frac{\delta^2 v'}{\delta\beta'^2}\frac{\delta\beta'}{\delta x} &= 0,\\
\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\alpha}{\delta x'} + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta x'} + \frac{\delta^2 v'}{\delta\alpha'^2}\frac{\delta\alpha'}{\delta y} + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\frac{\delta\beta'}{\delta y} &= 0,\\
\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta\alpha}{\delta y'} + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta y'} + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\frac{\delta\alpha'}{\delta y} + \frac{\delta^2 v'}{\delta\beta'^2}\frac{\delta\beta'}{\delta y} &= 0.
\end{aligned}\right\} \tag{C^{15}}$$

The two first relations of the first groupe, namely, the equations (A^{15}), are equivalent to the two first differential equations (O) of a curved ray, and express that the magnitude and plane of final curvature of a luminous path, in a final variable medium, are determined, in general, by the properties of that medium, the colour of the light, the position of the final point, and the direction of the final tangent. And the two other relations of the same groupe express, in like manner, a dependence of the initial magnitude and plane of curvature of a luminous path, on the initial medium, colour, point, and tangent.

The equation (B^{15}), belonging to the second groupe, is a relation between the four coefficients $\frac{\delta\alpha}{\delta x}, \frac{\delta\alpha}{\delta y}, \frac{\delta\beta}{\delta x}, \frac{\delta\beta}{\delta y}$, and therefore a relation between the guiding paraboloid and constant of deviation for the final ray-lines, depending on the final medium, colour, point and tangent. And similarly the other equation of the second groupe expresses an analogous relation for the initial medium.

In the extensive case of a final uniform medium, the equation (B^{15}) reduces itself to the following,

$$0 = \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta\alpha}{\delta y} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta\beta}{\delta y} - \frac{\delta\alpha}{\delta x}\right) - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta\beta}{\delta x}; \tag{D^{15}}$$

and, in the same case, the general conical locus of the second degree (R^{14}), connected with the condition of intersection of the final ray-lines, reduces itself to two real or imaginary planes of vergency, represented by the quadratic

$$0 = \frac{\delta\alpha}{\delta y}\,\delta y^2 + \left(\frac{\delta\alpha}{\delta x} - \frac{\delta\beta}{\delta y}\right)\delta x\,\delta y - \frac{\delta\beta}{\delta x}\,\delta x^2, \tag{E^{15}}$$

and coinciding with the two planes of vergency considered in the sixteenth number: attending therefore to (C^{13}), the relation (D^{15}) may be geometrically enunciated by saying, that *in a final uniform medium the two planes of vergency are conjugate planes of deflexure of any* *surface of a certain class determined by the nature of the medium*, namely, that class for which, at the origin of co-ordinates,

$$\frac{\delta^2 z}{\delta x^2} = \lambda \frac{\delta^2 v}{\delta \alpha^2}, \quad \frac{\delta^2 z}{\delta x \, \delta y} = \lambda \frac{\delta^2 v}{\delta \alpha \, \delta \beta}, \quad \frac{\delta^2 z}{\delta y^2} = \lambda \frac{\delta^2 v}{\delta \beta^2}, \tag{F^{15}}$$

and therefore nearly, for points near to this origin,

$$z = px + qy + \frac{\lambda}{2} \left(x^2 \frac{\delta^2 v}{\delta \alpha^2} + 2xy \frac{\delta^2 v}{\delta \alpha \, \delta \beta} + y^2 \frac{\delta^2 v}{\delta \beta^2}, \right) \tag{G^{15}}$$

the given final ray or axis of z being taken as the axis of deflexion, and the constants p, q, λ, being arbitrary. This relation may be still farther simplified, by choosing the arbitrary constants as follows,

$$p = -\frac{1}{v} \frac{\delta v}{\delta \alpha}, \quad q = -\frac{1}{v} \frac{\delta v}{\delta \beta}, \quad \lambda = \frac{1}{vZ}, \tag{H^{15}}$$

Z being any constant ordinate; for then, (by the theory of the characteristic function V for a single uniform medium, which was given in the tenth number,) the surface (G^{15}) acquires a simple optical property, and becomes, in the final uniform medium, the approximate locus of the points x, y, z, for which

$$V_{\prime} = \int v \, ds = v\rho = \text{const.}, \tag{I^{15}}$$

the integral $V_{\prime} = \int v \, ds$ being taken here, in the positive direction, along the variable line ρ, from the fixed point $0, 0, Z$, to the variable point x, y, z, or from the latter to the former, according as Z is negative or positive. And though the equation (G^{15}) is only an approximate representation of the medium-surface (I^{15}), which was called in the First Supplement a *spheroid of constant action*, and which is in the undulatory theory a curved *wave* propagated from or to a point in the final medium, yet since the equation (G^{15}) gives a correct development of the ordinate z of this surface as far as terms of the second dimension inclusive, when the constants are determined by (H^{15}), the conclusion respecting the deflexures applies rigorously to the surface (I^{15}); and *the two planes of vergency* (E^{15}), *in a final uniform medium, are conjugate planes of deflexure of the spheroid or wave* (I^{15}). We shall soon resume this result, and endeavour to illustrate and extend it. In the mean time we may remark that the same planes of vergency (E^{15}) are also conjugate planes of deflexure of a certain analogous surface, determined by the whole combination, and not merely by the final uniform medium, namely, the surface (D^{10}), for which

$$\int v \, ds \, (= V) \; = \text{const.}, \tag{K^{15}}$$

the integral being here extended to the whole luminous path, and being therefore equal to the characteristic function V of the whole optical combination; an additional property of the planes of vergency, which is proved by the following relation, analogous to (D^{15}), and deducible from (C^9) or (G^9),

$$0 = \frac{\delta^2 V}{\delta x^2}\frac{\delta\alpha}{\delta y} + \frac{\delta^2 V}{\delta x\,\delta y}\left(\frac{\delta\beta}{\delta y} - \frac{\delta\alpha}{\delta x}\right) - \frac{\delta^2 V}{\delta y^2}\frac{\delta\beta}{\delta x}; \tag{L^{15}}$$

Finally, with respect to the four remaining equations, of the third groupe (C^{15}), it is evident that they express certain general relations depending on the extreme media, between the coefficients which determine the guiding planes and conjugate guiding axes, for the final and inital ray-lines. In the extensive case of extreme ordinary media, they reduce themselves to the four following, which may also be deduced from (F^9),

$$\left.\begin{aligned} \mu\frac{\delta\alpha}{\delta x'} + \mu'\frac{\delta\alpha'}{\delta x} = 0, \quad \mu\frac{\delta\alpha}{\delta y'} + \mu'\frac{\delta\beta'}{\delta x} = 0, \\ \mu\frac{\delta\beta}{\delta x'} + \mu'\frac{\delta\alpha'}{\delta y} = 0, \quad \mu\frac{\delta\beta}{\delta y'} + \mu'\frac{\delta\beta'}{\delta y} = 0, \end{aligned}\right\} \tag{M^{15}}$$

μ, μ' being the *indices* of the media; and they conduct to some simple conclusions, respecting the general relations between the visible magnitudes and distortions of a small plane object, placed alternately at each end of any given luminous path, and viewed from the other end, through any ordinary or extraordinary combination: at least so far as we suppose these distortions and magnitudes to be measured by the shape and size of the initial and final ray-cones. For then the conjugate guiding axes, initial and final, perpendicular to the given path at its extremities, and determined in the fifteenth number, may be called the *eye-axes and object-axes of distortion*, for a small object placed in the final perpendicular plane, and viewed from the initial point; and if we take these for the axes of initial and final co-ordinates, so as to have, by (X^{10}) (Y^{10}),

$$\frac{\delta\alpha'}{\delta y} = 0, \quad \frac{\delta\beta'}{\delta x} = 0, \quad \frac{\delta\beta'}{\delta y} > 0, \quad \frac{\delta\alpha'}{\delta x} > \frac{\delta\beta'}{\delta y},$$

we shall then have also, by (M^{15}), (the extreme media being supposed ordinary, and their indices μ, μ' positive,)

$$\frac{\delta\alpha}{\delta y'} = 0, \quad \frac{\delta\beta}{\delta x'} = 0, \quad -\frac{\delta\beta}{\delta y'} > 0, \quad -\frac{\delta\alpha}{\delta x'} > -\frac{\delta\beta}{\delta y'}; \tag{N^{15}}$$

that is, in this case, *the guiding axes for the initial ray-lines are also the guiding axes of the same kind for the final ray-lines measured backward*; which is already a remarkable relation, and may be enunciated by saying that *the eye-axes and object-axes of distortion are interchangeable, when the extreme media are ordinary*: that is, for such extreme media, *the eye-axes of distortion become object-axes, and the object-axes become eye-axes, when the object is removed from the final to the initial perpendicular plane, and is viewed from the final instead of the initial point.* And while the equations of the fifteenth number,

$$x' = z'\frac{\delta\alpha'}{\delta x}\,\delta x, \quad y' = z'\frac{\delta\beta'}{\delta y}\,\delta y, \tag{A^{11}}$$

represent the initial visual ray-line corresponding to a final visible point B' which has for co-ordinates δx, δy, δz, the following other equations,

$$x = -z \,.\, \frac{\mu'}{\mu} \frac{\delta\alpha'}{\delta x}\, \delta x', \quad y = -z \,.\, \frac{\mu'}{\mu} \frac{\delta\beta'}{\delta y}\, \delta y', \tag{O^{15}}$$

will represent by (M^{15}) the final visual ray-line corresponding to an initial visible point A' which has for co-ordinates $\delta x'$, $\delta y'$, $\delta z'$; the initial visual ray-cone corresponding to any small object

$$\delta y = f(\delta x) \tag{G^{11}}$$

in the final perpendicular plane is therefore represented by the equation

$$\frac{y'}{z'} \left(\frac{\delta\beta'}{\delta y} \right)^{-1} = f \left(\frac{x'}{z'} \left(\frac{\delta\alpha'}{\delta x} \right)^{-1} \right), \tag{H^{11}}$$

and the final visual ray-cone corresponding to any small object

$$\delta y' = f'(\delta x') \tag{P^{15}}$$

in the initial perpendicular plane is represented by the following analogous equation

$$-\frac{y}{z} \frac{\mu}{\mu'} \left(\frac{\delta\beta'}{\delta y} \right)^{-1} = f' \left(-\frac{x}{z} \frac{\mu}{\mu'} \left(\frac{\delta\alpha'}{\delta x} \right)^{-1} \right); \tag{Q^{15}}$$

if therefore these two small objects (G^{11}) (P^{15}), at the ends of a given luminous path, be equal and similar and similarly placed with respect to the conjugate axes of distortion, that is, if the final and initial functions f, f' be the same, and if we cut the two ray-cones (H^{11}) (Q^{15}) respectively by perpendicular planes having for equations

$$z' = \mu' R, \quad z = -\mu R, \tag{R^{15}}$$

in which R is any constant length, while μ, μ' are the same constant indices as before of the extreme ordinary media, the two perpendicular sections thus obtained will be equal and similar to each other; and if, besides, we put, by (Y^{10}),

$$\frac{\delta\beta'}{\delta y} = \frac{\delta\alpha'}{\delta x} \cos.G, \tag{S^{15}}$$

(G being by (F^{11}) the inclination of an initial guiding plane to the plane perpendicular to the given initial ray-line,) and determine also the arbitrary quantity R as follows,

$$R = \frac{1}{\mu'} \left(\frac{\delta\alpha'}{\delta x} \right)^{-1} = -\frac{1}{\mu} \left(\frac{\delta\alpha}{\delta x'} \right)^{-1}, \tag{T^{15}}$$

the perpendicular sections of the initial and final ray-cones may then be represented as follows,

$$y' = \cos. G \,.\, f(x'), \quad z' = \left(\frac{\delta\alpha'}{\delta x}\right)^{-1}, \tag{U^{15}}$$

and

$$y = \cos. G \,.\, f(x), \quad z = \left(\frac{\delta\alpha}{\delta x'}\right)^{-1}: \tag{V^{15}}$$

the visible distortions therefore, depending on the inclination G, *are the same for any two small equal objects, thus perpendicularly and similarly placed at the ends of any given luminous path, and viewed from each other along that path, through any optical combination.*

The distortion here considered will in general change, if the object at either end of the given luminous path be made to revolve in the perpendicular plane at that end, so as to change its position with respect to the axes of distortion. For example, if the object be a small right-angled triangle in the final perpendicular plane, having the summit of the right angle at the given final point B of the path, we know, by the theory given in the fifteenth number, that the right angle will appear right to an eye placed at the initial point A, when the rectangular directions of its sides ϕ'_1, ϕ'_2, coincide with those of the final guiding axes, or object-axes of distortion; but that otherwise the right angle $\phi'_2 - \phi'_1$ will appear acute or obtuse, its apparent magnitude $\phi_2 - \phi_1$ being determined by the formula

$$-\tan.\left(\phi_2 - \phi_1 - \frac{\pi}{2}\right) = \frac{\left(\frac{\delta\alpha'}{\delta x}\right)^2 - \left(\frac{\delta\beta'}{\delta y}\right)^2}{2\frac{\delta\alpha'}{\delta x}\frac{\delta\beta'}{\delta y}} \,.\, \sin. 2\phi'_1, \tag{W^{15}}$$

which may, by (S^{15}), be reduced to the following,

$$-\tan.\left(\phi_2 - \phi_1 - \frac{\pi}{2}\right) = \tfrac{1}{2}\sin. G \,.\, \tan. G \,.\, \sin. 2\phi'_1. \tag{X^{15}}$$

The law of change of the distortion, corresponding to a rotation in the final perpendicular plane, may also be deduced from the theory of the guiding planes, explained in the fifteenth number.

The distortion will also change if the small plane object be removed into an oblique instead of a perpendicular plane. In this case we may still employ the equations (A^{11}) (O^{15}) for the initial and final ray-lines, and may still represent the initial and final ray-cones by the equations (H^{11}) (Q^{15}); but we are now to consider the equations (G^{11}) (P^{15}), for the final and initial objects, as representing the projections of those objects on the extreme perpendicular planes; or rather the *projecting cylinders*, which contain the objects, and which determine their visible magnitudes and distortions, by determining the connected ray-cones. For example, the equation (C^{11}) may be considered as representing a final elliptic cylinder, of which any section near the final point B of the given luminous path will correspond to an initial circular ray-cone (B^{11}), and will therefore appear a circle to an eye placed at the initial point A; while on the other hand we may regard the equation (D^{11}) as representing a final circular cylinder, such that any section of this cylinder, near the final point B, will give an initial elliptic ray-cone (E^{11}), and will appear an ellipse at A. And as the elliptic ray-cone (E^{11}) conducted, by its circular sections, to the guiding planes (F^{11}) for the initial ray-lines, so, for small plane final objects, the planes

$$z = \pm x \tan. G, \tag{Y^{15}}$$

namely, by (S^{15}), *the planes of circular section of the elliptic cylinder* (C^{11}), *are planes of no distortion*; in such a manner that not only, by what has been said, the circular sections themselves in these two planes appear each circular, but every other small final object in either of the same two planes appears with its proper shape to an eye placed at the initial point A of the given luminous path; the angular magnitude of the final object thus placed, being the same as if it were viewed perpendicularly by straight rays, without any refracting or reflecting surface or medium interposed, from a final distance $= \left(\dfrac{\delta\beta'}{\delta y}\right)^{-1}$. In like manner, the planes

$$z' = \pm x' \tan. G, \tag{Z^{15}}$$

which are the planes of circular section of an analogous initial elliptic cylinder, are *initial planes of no distortion*, of the same kind as the final planes (Y^{15}); since any small initial object, placed in either of these two initial planes (Z^{15}), and viewed from the final point B of the given luminous path, will appear with its proper shape, and with the same angular magnitude as if it were viewed directly from an initial distance $= -\left(\dfrac{\delta\beta}{\delta y'}\right)^{-1} = \dfrac{\mu}{\mu'}\left(\dfrac{\delta\beta'}{\delta y}\right)^{-1}$.

This theory of the *planes of no distortion* gives a simple determination of the visible shape and size of any small object placed in any manner near either end of a given luminous path; since we have only to project the object on one of the two planes of no distortion at that end, by lines parallel to the corresponding extreme direction of the path, and then to suppose this projection viewed directly from a final or initial distance determined as above. We might, for example, deduce from this theory the property of the guiding planes, the circular and elliptic appearances (B^{11}) (E^{11}) of the ellipse and circle (C^{11}) (D^{11}), and the acute or obtuse appearance (X^{15}) of a right angle in the final perpendicular plane, when the directions of the sides of this angle are different from those of the object-axes of distortion. And the relations (M^{15}) for extreme ordinary media may be expressed by the following theorems: first, that *the angle* ($2G$) *between the final pair of planes of no distortion* (Y^{15}), *is equal to that between the initial pair* (Z^{15}); second, *the visible angular magnitudes of any small and equal linear objects in final and initial planes of no distortion, are proportional to the indices of the final and initial media*, when the objects are viewed along a given luminous path, from the initial and final points; and third, *the two intersection-lines of the two pairs of planes of no distortion coincide each with the visible direction of the other*, when viewed along the path.

Calculation of the Elements of Arrangement, for Arbitrary Axes of Co-ordinates.

22. In the foregoing formulæ for the elements of arrangement of near rays, we have chosen for simplicity the final and initial points of a given luminous path, as the respective origins of

two sets of rectangular co-ordinates, final and initial, and we have made the final and initial ray-lines, or tangents to the given path, the axes of z and z'; a choice of co-ordinates which had the convenience of reducing to zero eighteen of the forty-two general coefficients in the expressions of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, as linear functions of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. The twenty-four remaining coefficients (D^9) may however be easily deduced, by the methods already established, and by the partial differential coefficients of the characteristic and related functions, from other systems of final and initial co-ordinates, for example, from any other rectangular sets of final and initial axes.

In effecting this deduction, it will be useful to distinguish by lower accents the particular co-ordinates and cosines of direction, which enter into the expressions (D^9), and are referred to particular axes of the kind already described; and then we may connect these particular co-ordinates and cosines with the more general analogous quantities $x\ y\ z\ x'\ y'\ z'\ \alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, by the formulæ of transformation given in the thirteenth number, which may easily be shown to extend to the case of two distinct rectangular sets of given or unaccented co-ordinates. In this manner the axes of $z_{\prime}$ and $z'_{\prime}$, considered in the thirteenth number, become the final and initial ray-lines, and we have, by (A^8),

$$\left.\begin{aligned}
\delta x &= x_{x_\prime}\,\delta x_\prime + x_{y_\prime}\,\delta y_\prime + \alpha\,\delta z_\prime,\\
\delta y &= y_{x_\prime}\,\delta x_\prime + y_{y_\prime}\,\delta y_\prime + \beta\,\delta z_\prime,\\
\delta z &= z_{x_\prime}\,\delta x_\prime + z_{y_\prime}\,\delta y_\prime + \gamma\,\delta z_\prime,\\
\delta x' &= x'_{x'_\prime}\,\delta x'_\prime + x'_{y'_\prime}\,\delta y'_\prime + \alpha'\,\delta z'_\prime,\\
\delta y' &= y'_{x'_\prime}\,\delta x'_\prime + y'_{y'_\prime}\,\delta y'_\prime + \beta'\,\delta z'_\prime,\\
\delta z' &= z'_{x'_\prime}\,\delta x'_\prime + z'_{y'_\prime}\,\delta y'_\prime + \gamma'\,\delta z'_\prime,
\end{aligned}\right\} \tag{A^{16}}$$

because

$$\left.\begin{aligned}
x_{z_\prime} &= \alpha, & y_{z_\prime} &= \beta, & z_{z_\prime} &= \gamma,\\
x'_{z'_\prime} &= \alpha', & y'_{z'_\prime} &= \beta', & z'_{z'_\prime} &= \gamma';
\end{aligned}\right\} \tag{B^{16}}$$

we have also

$$\left.\begin{aligned}
\alpha_\prime &= 0, & \beta_\prime &= 0, & \gamma_\prime &= 1, & \delta\gamma_\prime &= 0,\\
\alpha'_\prime &= 0, & \beta'_\prime &= 0, & \gamma'_\prime &= 1, & \delta\gamma'_\prime &= 0,
\end{aligned}\right\} \tag{C^{16}}$$

and therefore, by (E^8),

$$\left.\begin{aligned}
\delta\alpha &= x_{x_\prime}\,\delta\alpha_\prime + x_{y_\prime}\,\delta\beta_\prime; & \delta\alpha' &= x'_{x'_\prime}\,\delta\alpha'_\prime + x'_{y'_\prime}\,\delta\beta'_\prime;\\
\delta\beta &= y_{x_\prime}\,\delta\alpha_\prime + y_{y_\prime}\,\delta\beta_\prime; & \delta\beta' &= y'_{x'_\prime}\,\delta\alpha'_\prime + y'_{y'_\prime}\,\delta\beta'_\prime;\\
\delta\gamma &= z_{x_\prime}\,\delta\alpha_\prime + z_{y_\prime}\,\delta\beta_\prime; & \delta\gamma' &= z'_{x'_\prime}\,\delta\alpha'_\prime + z'_{y'_\prime}\,\delta\beta'_\prime;
\end{aligned}\right\} \tag{D^{16}}$$

and substituting these values (A^{16}) (D^{16}) for the twelve variations δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in the general linear relations (A^9) between these twelve variations and the variation of colour $\delta\chi$, or in any other linear relations of the same kind, deduced from the characteristic and related functions, and referred to arbitrary rectangular co-ordinates, we shall easily discover the particular dependence, of the form (D^9), of $\delta\alpha_{\prime}$, $\delta\beta_{\prime}$, on $\delta x_{\prime}$, $\delta y_{\prime}$, $\delta z_{\prime}$, $\delta x'_{\prime}$, $\delta y'_{\prime}$, $\delta\chi$, and of $\delta\alpha'_{\prime}$, $\delta\beta'_{\prime}$, on $\delta x_{\prime}$, $\delta y_{\prime}$, $\delta x'_{\prime}$, $\delta y'_{\prime}$, $\delta z'_{\prime}$, $\delta\chi$.

We seem by this transformation, to introduce twelve arbitrary cosines or coefficients, namely,

$$x_{x_{\prime}}, \quad y_{x_{\prime}}, \quad z_{x_{\prime}}, \quad x_{y_{\prime}}, \quad y_{y_{\prime}}, \quad z_{y_{\prime}}, \quad x'_{x'_{\prime}}, \quad y'_{x'_{\prime}}, \quad z'_{x'_{\prime}}, \quad x'_{y'_{\prime}}, \quad y'_{y'_{\prime}}, \quad z'_{y'_{\prime}};$$

but these twelve coefficients are connected by ten relations, arising from the rectangularity of each of the four sets of co-ordinates, and from the given directions of the semiaxes of $z_{\prime}$ and $z'_{\prime}$; so that there remain only two arbitrary quantities, corresponding to the arbitrary planes of $x_{\prime}z_{\prime}$, $x'_{\prime}z'_{\prime}$, of which planes we often, lately, disposed at pleasure, so as to make them coincide with certain given planes of curvature, or otherwise to simplify the recent geometrical discussions. Thus, although we may assign to the semiaxis of $x_{\prime}$ any position in the given final plane perpendicular to the luminous path, and therefore may assign to its cosines of direction $x_{x_{\prime}}$, $y_{x_{\prime}}$, $z_{x_{\prime}}$, any values consistent with the first equation (B^8), namely

$$x_{x_{\prime}}^2 + y_{x_{\prime}}^2 + z_{x_{\prime}}^2 = 1,$$

and with the following

$$\alpha x_{x_{\prime}} + \beta y_{x_{\prime}} + \gamma z_{x_{\prime}} = 0, \tag{E^{16}}$$

yet when the axis of $x_{\prime}$ has been so assumed, the perpendicular axis of $y_{\prime}$ in the final perpendicular plane is determined, and we have

$$\left.\begin{aligned} x_{y_{\prime}} &= \pm(\beta z_{x_{\prime}} - \gamma y_{x_{\prime}}), \\ y_{y_{\prime}} &= \pm(\gamma x_{x_{\prime}} - \alpha z_{x_{\prime}}), \\ z_{y_{\prime}} &= \pm(\alpha y_{x_{\prime}} - \beta x_{x_{\prime}}), \end{aligned}\right\} \tag{F^{16}}$$

the upper or lower signs being here obliged to accompany each other: and similarly for the initial axes of $x'_{\prime}$ and $y'_{\prime}$.

The characteristic and related functions give immediately, by their partial differentials of the first order, the dependence of the quantities which we have denoted by σ, τ, υ, σ', τ', υ', rather than that of α, β, γ, α', β', γ', on the extreme co-ordinates and the colour; and therefore the same functions give immediately, by their partial differentials of the second order, the variations of $\delta\sigma$, $\delta\tau$, $\delta\upsilon$, $\delta\sigma'$, $\delta\tau'$, $\delta\upsilon'$, rather than $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta\alpha'$, $\delta\beta'$, $\delta\gamma'$, in terms of δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, $\delta\chi$. But we can easily deduce the variations of $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$ from those of $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'$ and of $x\ y\ z\ x'\ y'\ z'\ \chi$, by differentiating the relations

$$\begin{aligned} \sigma &= \frac{\delta v}{\delta\alpha}, & \tau &= \frac{\delta v}{\delta\beta}, & \upsilon &= \frac{\delta v}{\delta\gamma}, \\ \sigma' &= \frac{\delta v'}{\delta\alpha'}, & \tau' &= \frac{\delta v'}{\delta\beta'}, & \upsilon' &= \frac{\delta v'}{\delta\gamma'}, \end{aligned}$$

which have often been employed already in the present Supplement; for thus we obtain

$$\left.\begin{aligned}
\frac{\delta^2 v}{\delta\alpha^2}\,\delta\alpha + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\beta + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\gamma &= \delta\sigma - \delta_{\prime\prime}\frac{\delta v}{\delta\alpha},\\
\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta^2}\,\delta\beta + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\gamma &= \delta\tau - \delta_{\prime\prime}\frac{\delta v}{\delta\beta},\\
\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\beta + \frac{\delta^2 v}{\delta\gamma^2}\,\delta\gamma &= \delta\upsilon - \delta_{\prime\prime}\frac{\delta v}{\delta\gamma},\\
\frac{\delta^2 v'}{\delta\alpha'^2}\,\delta\alpha' + \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\beta' + \frac{\delta^2 v'}{\delta\alpha'\,\delta\gamma'}\,\delta\gamma' &= \delta\sigma' - \delta'\frac{\delta v'}{\delta\alpha'},\\
\frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\,\delta\alpha' + \frac{\delta^2 v'}{\delta\beta'^2}\,\delta\beta' + \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\,\delta\gamma' &= \delta\tau' - \delta'\frac{\delta v'}{\delta\beta'},\\
\frac{\delta^2 v'}{\delta\alpha'\,\delta\gamma'}\,\delta\alpha' + \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\,\delta\beta' + \frac{\delta^2 v'}{\delta\gamma'^2}\,\delta\gamma' &= \delta\upsilon' - \delta'\frac{\delta v'}{\delta\gamma'},
\end{aligned}\right\} \qquad (\mathrm{G}^{16})$$

$\delta_{\prime\prime}$ referring, as in former numbers, to the variations of x, y, z, χ, and δ' to those of x', y', z', χ: and hence we have, by some symmetric eliminations,

$$\left.\begin{aligned}
&v''\,\delta\alpha\\
&\quad= \left(\frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right) - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\delta\tau - \delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right) - \frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right),\\
&v''\,\delta\beta\\
&\quad= \left(\frac{\delta^2 v}{\delta\gamma^2} + \frac{\delta^2 v}{\delta\alpha^2}\right)\left(\delta\tau - \delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right) - \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right) - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right),\\
&v''\,\delta\gamma\\
&\quad= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2}\right)\left(\delta\upsilon - \delta_{\prime\prime}\frac{\delta v}{\delta\gamma}\right) - \frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\left(\delta\sigma - \delta_{\prime\prime}\frac{\delta v}{\delta\alpha}\right) - \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\left(\delta\tau - \delta_{\prime\prime}\frac{\delta v}{\delta\beta}\right),\\
&v'''\,\delta\alpha'\\
&\quad= \left(\frac{\delta^2 v'}{\delta\beta'^2} + \frac{\delta^2 v'}{\delta\gamma'^2}\right)\left(\delta\sigma' - \delta'\frac{\delta v'}{\delta\alpha'}\right) - \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\left(\delta\tau' - \delta'\frac{\delta v'}{\delta\beta'}\right) - \frac{\delta^2 v'}{\delta\gamma'\,\delta\alpha'}\left(\delta\upsilon' - \delta'\frac{\delta v'}{\delta\gamma'}\right),\\
&v'''\,\delta\beta'\\
&\quad= \left(\frac{\delta^2 v'}{\delta\gamma'^2} + \frac{\delta^2 v'}{\delta\alpha'^2}\right)\left(\delta\tau' - \delta'\frac{\delta v'}{\delta\beta'}\right) - \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\left(\delta\upsilon' - \delta'\frac{\delta v'}{\delta\gamma'}\right) - \frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\left(\delta\sigma' - \delta'\frac{\delta v'}{\delta\alpha'}\right),\\
&v'''\,\delta\gamma'\\
&\quad= \left(\frac{\delta^2 v'}{\delta\alpha'^2} + \frac{\delta^2 v'}{\delta\beta'^2}\right)\left(\delta\upsilon' - \delta'\frac{\delta v'}{\delta\gamma'}\right) - \frac{\delta^2 v'}{\delta\gamma'\,\delta\alpha'}\left(\delta\sigma' - \delta'\frac{\delta v'}{\delta\alpha'}\right) - \frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\left(\delta\tau' - \delta'\frac{\delta v'}{\delta\beta'}\right),
\end{aligned}\right\} \qquad (\mathrm{H}^{16})$$

v'' having the meaning (L^6), and v''' the analogous meaning

$$v''' = \frac{\delta^2 v'}{\delta\alpha'^2}\frac{\delta^2 v'}{\delta\beta'^2} - \left(\frac{\delta^2 v'}{\delta\alpha'\,\delta\beta'}\right)^2 + \frac{\delta^2 v'}{\delta\beta'^2}\frac{\delta^2 v'}{\delta\gamma'^2} - \left(\frac{\delta^2 v'}{\delta\beta'\,\delta\gamma'}\right)^2 + \frac{\delta^2 v'}{\delta\gamma'^2}\frac{\delta^2 v'}{\delta\alpha'^2} - \left(\frac{\delta^2 v'}{\delta\gamma'\,\delta\alpha'}\right)^2. \qquad (\mathrm{I}^{16})$$

We might also deduce the variations of $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$ from those of $\sigma\ \tau\ \upsilon\ \sigma'\ \tau'\ \upsilon'\ x\ y\ z\ x'\ y'\ z'\ \chi$, by differentiating the equations (I) of the second number, and by applying the functions Ω, Ω', instead of v, v'.

The general Linear Expressions for the Arrangement of Near Rays, fail at a Point of Vergency. Determination of these Points, and of their Loci, the Caustic Surfaces, in a Straight or Curved System, by the Methods of the present Supplement.

23. We have hitherto supposed that the infinitesimal or limiting expressions of the variations of the extreme cosines of direction of a luminous path, are linear functions of the variations of the extreme co-ordinates and colour. But although this supposition is in general true, it admits of an important and extensive exception; for the linear form becomes inapplicable when the given luminous path $(A, B)_\chi$, with which other near paths are to be compared, is intersected in its initial and final points A, B, by another path infinitely near, and having the same colour χ: since then the extreme directions may undergo certain infinitesimal variations, while the extreme positions A, B, and the colour χ, remain unaltered. It is therefore an important general problem of mathematical optics, to determine, for any proposed optical combination, *the relations between the extreme co-ordinates and the colour of a luminous path which is intersected in its extreme points by another infinitely near path of the same colour.* This general problem, of which the solution includes the general theory of the caustic surfaces touched by the straight or curved rays of any proposed optical system, may easily be resolved by the methods of the present Supplement.

In applying these methods to the present question, we are to differentiate the general equations which connect the extreme directions with the extreme positions and colour, by the partial differential coefficients of the first order of the characteristic and related functions, and then to suppress the variations of $x\ y\ z\ x'\ y'\ z'\ \chi$. And of the partial differential coefficients of the second order, introduced by such differentiation, it is easy to see by (A^9) that those of the characteristic function V, or at least some of them, are infinite in the present research: it is therefore advantageous here to employ one of the auxiliary functions W, T, combined if necessary with the functions v, v', or Ω, Ω', which express by their form the properties of the extreme media.

Thus, when the final medium is uniform, and therefore the final rays straight, we may conveniently employ the following equations, which involve the coefficients of the functions W, Ω, and were established in the sixth number,

$$x = \frac{\delta W}{\delta\sigma} + V\frac{\delta\Omega}{\delta\sigma}, \quad y = \frac{\delta W}{\delta\tau} + V\frac{\delta\Omega}{\delta\tau}, \quad z = \frac{\delta W}{\delta\upsilon} + V\frac{\delta\Omega}{\delta\upsilon}. \tag{W^2}$$

Differentiating these equations with respect to $\sigma\ \tau\ \upsilon$ as the only variables, and suppressing the variation of the first order of V, as well as those of $x\ y\ z\ x'\ y'\ z'\ \chi$, we obtain

$$\left.\begin{aligned}
0 &= \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)\delta\tau + \left(\frac{\delta^2 W}{\delta\sigma\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}\right)\delta\upsilon,\\
0 &= \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\delta\tau + \left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)\delta\upsilon,\\
0 &= \left(\frac{\delta^2 W}{\delta\sigma\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}\right)\delta\sigma + \left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)\delta\tau + \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\delta\upsilon,
\end{aligned}\right\} \tag{K^{16}}$$

and hence, by a symmetric elimination, and by the forms of W, Ω,

$$\begin{aligned}0 = \ & \left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)^2 \\ & + \left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right)\left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right) - \left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}\right)^2 \\ & + \left(\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2}\right)\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right) - \left(\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma}\right)^2 : \qquad (\mathrm{L}^{16})\end{aligned}$$

which is a form of the condition required, for the final and initial intersections of two near luminous paths, of any common colour, the final medium being uniform. The condition (L^{16}) is quadratic with respect to V, and determines, for any final system of straight rays, corresponding to any given luminous or initial point A, and to any given colour χ, two real or imaginary points of vergency B_1, B_2, on any one straight final ray, that is, two points in which this ray is intersected by infinitely near rays of the same final system; and the joint equation in $x\ y\ z$, (involving also $x'\ y'\ z'\ \chi$ as parameters,) of the *two caustic surfaces* which are touched by all the final rays and are the loci of the points of vergency, may be obtained by eliminating $\sigma\ \tau\ \upsilon$ between the equations (W^2) and the quadratic (L^{16}): which quadratic, by the homogeneity of the functions W and $\Omega + 1$, may be put under the following simpler form,

$$\left(\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2}\right)\left(\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2}\right) - \left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau}\right)^2 = 0, \qquad (\mathrm{M}^{16})$$

and admits of several other transformations. When V has either of the two values determined by this quadratic, that is, when the final point B of the luminous path has any position B_1 or B_2 on either of the two caustic surfaces, then the equations deduced from (W^2) by differentiating with respect to $x\ y\ z$ as well as $\sigma\ \tau\ \upsilon$, namely,

$$\left.\begin{aligned}\delta x - \frac{\delta\Omega}{\delta\sigma}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z) &= \delta\frac{\delta W}{\delta\sigma} + V\,\delta\frac{\delta\Omega}{\delta\sigma}, \\ \delta y - \frac{\delta\Omega}{\delta\tau}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z) &= \delta\frac{\delta W}{\delta\tau} + V\,\delta\frac{\delta\Omega}{\delta\tau}, \\ \delta z - \frac{\delta\Omega}{\delta\upsilon}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z) &= \delta\frac{\delta W}{\delta\upsilon} + V\,\delta\frac{\delta\Omega}{\delta\upsilon},\end{aligned}\right\} \qquad (\mathrm{N}^{16})$$

conduct to a linear relation between δx, δy, δz, which may be put under several forms, for example under the following,

$$\begin{aligned}\frac{1}{\lambda}\left\{\delta x - \frac{\delta\Omega}{\delta\sigma}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z)\right\} &= \frac{1}{\lambda'}\left\{\delta y - \frac{\delta\Omega}{\delta\tau}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z)\right\} \\ &= \frac{1}{\lambda''}\left\{\delta z - \frac{\delta\Omega}{\delta\upsilon}(\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z)\right\},\end{aligned} \qquad (\mathrm{O}^{16})$$

in which we may assign to $\lambda\ \lambda'\ \lambda''$ any of the following systems of values,

$$\left.\begin{array}{llll}
\text{First} & \lambda = \dfrac{\delta^2 W}{\delta\sigma^2} + V\dfrac{\delta^2\Omega}{\delta\sigma^2}, & \lambda' = \dfrac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\dfrac{\delta^2\Omega}{\delta\sigma\,\delta\tau} & \lambda'' = \dfrac{\delta^2 W}{\delta\sigma\,\delta\upsilon} + V\dfrac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}; \\
\text{Second} & \lambda = \dfrac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\dfrac{\delta^2\Omega}{\delta\sigma\,\delta\tau}, & \lambda' = \dfrac{\delta^2 W}{\delta\tau^2} + V\dfrac{\delta^2\Omega}{\delta\tau^2} & \lambda'' = \dfrac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\dfrac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}; \\
\text{Third} & \lambda = \dfrac{\delta^2 W}{\delta\sigma\,\delta\upsilon} + V\dfrac{\delta^2\Omega}{\delta\sigma\,\delta\upsilon}, & \lambda' = \dfrac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\dfrac{\delta^2\Omega}{\delta\tau\,\delta\upsilon}, & \lambda'' = \dfrac{\delta^2 W}{\delta\upsilon^2} + V\dfrac{\delta^2\Omega}{\delta\upsilon^2}:
\end{array}\right\} \quad (\mathrm{P}^{16})$$

and it is easy to see that the linear relation thus deduced, between δx, δy, δz, is the differential equation, or equation of the tangent plane, of the caustic surface at the point of vergency $x\ y\ z$. The same linear equation represents also the plane of vergency, or the tangent plane to the developable pencil of straight rays, corresponding to the other or conjugate point of vergency on the given final ray.

When the final medium is variable, the three first equations (H′), namely,

$$x = \frac{\delta W}{\delta\sigma}, \quad y = \frac{\delta W}{\delta\tau}, \quad z = \frac{\delta W}{\delta\upsilon},$$

are to be differentiated with respect to σ, τ, υ; and thus we obtain

$$\left.\begin{array}{l}
\dfrac{\delta^2 W}{\delta\sigma^2}\,\delta\sigma + \dfrac{\delta^2 W}{\delta\sigma\,\delta\tau}\,\delta\tau + \dfrac{\delta^2 W}{\delta\sigma\,\delta\upsilon}\,\delta\upsilon = 0, \\
\dfrac{\delta^2 W}{\delta\sigma\,\delta\tau}\,\delta\sigma + \dfrac{\delta^2 W}{\delta\tau^2}\,\delta\tau + \dfrac{\delta^2 W}{\delta\tau\,\delta\upsilon}\,\delta\upsilon = 0, \\
\dfrac{\delta^2 W}{\delta\sigma\,\delta\upsilon}\,\delta\sigma + \dfrac{\delta^2 W}{\delta\tau\,\delta\upsilon}\,\delta\tau + \dfrac{\delta^2 W}{\delta\upsilon^2}\,\delta\upsilon = 0,
\end{array}\right\} \quad (\mathrm{Q}^{16})$$

and consequently, by elimination,

$$\frac{\delta^2 W}{\delta\sigma^2}\frac{\delta^2 W}{\delta\tau^2}\frac{\delta^2 W}{\delta\upsilon^2} + 2\frac{\delta^2 W}{\delta\sigma\,\delta\tau}\frac{\delta^2 W}{\delta\tau\,\delta\upsilon}\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} = \frac{\delta^2 W}{\delta\sigma^2}\left(\frac{\delta^2 W}{\delta\tau\,\delta\upsilon}\right)^2 + \frac{\delta^2 W}{\delta\tau^2}\left(\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma}\right)^2 + \frac{\delta^2 W}{\delta\upsilon^2}\left(\frac{\delta^2 W}{\delta\sigma\,\delta\tau}\right)^2: \quad (\mathrm{R}^{16})$$

this equation, therefore, (which may be put under other forms,) takes the place, when the final medium is variable, of the quadratic (L^{16}) for a final uniform medium; and if we eliminate from it $\sigma\ \tau\ \upsilon$ by (H′), it will give, for any proposed initial point and colour, the equation of the *single or multiple caustic surface, touched by the curved rays* of the corresponding final system.

The auxiliary function T may also be employed for the case of curved rays, but it is chiefly useful when both the extreme media are uniform. In that case the extreme portions of a luminous path are straight, and we may employ for these extreme straight portions the equations (S^2) under the form

$$x = \frac{\delta S}{\delta\sigma}, \quad y = \frac{\delta S}{\delta\tau}, \quad x' = -\frac{\delta S}{\delta\sigma'}, \quad y' = -\frac{\delta S}{\delta\tau'}, \quad (\mathrm{S}^{16})$$

in which we have put, for abridgement,

$$S = T - zv + z'v', \tag{T^{16}}$$

and in which we consider v as a function of σ, τ, χ; v' as a function of σ', τ', χ; T as a function of σ, τ, σ', τ', χ; and S as a function of z, z', σ, τ, σ', τ', χ. Differentiating these equations (S^{16}) with respect to σ, τ, σ', τ', we find that if the extreme straight portions, ordinary or extraordinary, of two infinitely near paths of light of the same colour, intersect in an initial point x' y' z', and in a final point x y z, the final and initial variations $\delta\sigma$, $\delta\tau$, $\delta\sigma'$, $\delta\tau'$, and the final and initial ordinates of intersection z, z', must satisfy the four following conditions,

$$\left.\begin{aligned}
0 &= \frac{\delta^2 S}{\delta\sigma^2}\,\delta\sigma + \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\delta\tau + \frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\delta\sigma' + \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\delta\tau',\\
0 &= \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\,\delta\sigma + \frac{\delta^2 S}{\delta\tau^2}\,\delta\tau + \frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\,\delta\sigma' + \frac{\delta^2 S}{\delta\tau\,\delta\tau'}\,\delta\tau',\\
0 &= \frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\,\delta\sigma + \frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\,\delta\tau + \frac{\delta^2 S}{\delta\sigma'^2}\,\delta\sigma' + \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\,\delta\tau',\\
0 &= \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\,\delta\sigma + \frac{\delta^2 S}{\delta\tau\,\delta\tau'}\,\delta\tau + \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\,\delta\sigma' + \frac{\delta^2 S}{\delta\tau'^2}\,\delta\tau';
\end{aligned}\right\} \tag{U^{16}}$$

which give, by eliminating between the two first,

$$\left.\begin{aligned}
\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\delta\sigma' &= \left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right)\delta\sigma\\
&\quad + \left(\frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right)\delta\tau;\\
\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\delta\tau' &= \left(\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right)\delta\sigma\\
&\quad + \left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right)\delta\tau;
\end{aligned}\right\} \tag{V^{16}}$$

and therefore, by substituting these values of $\delta\sigma'$, $\delta\tau'$, in the two last,

$$0 = \delta\sigma\left\{ \frac{\delta^2 S}{\delta\sigma'^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right) + \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right) \right.$$
$$\left. + \frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right\}$$
$$+ \delta\tau\left\{ \frac{\delta^2 S}{\delta\sigma'^2}\left(\frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right) + \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right) \right.$$
$$\left. + \frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right\};$$
$$0 = \delta\sigma\left\{ \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right) + \frac{\delta^2 S}{\delta\tau'^2}\left(\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right) \right.$$
$$\left. + \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right\}$$
$$+ \delta\tau\left\{ \frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right) + \frac{\delta^2 S}{\delta\tau'^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} - \frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right) \right.$$
$$\left. + \frac{\delta^2 S}{\delta\tau\,\delta\tau'}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right\}; \tag{W^{16}}$$

so that by a new elimination we obtain, between the final and initial ordinates z, z', the following equation, which, by the form of S, is quadratic with respect to each ordinate separately, and involves the product of their squares:

$$0 = \left\{\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau^2} - \left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\right)^2\right\}\left\{\frac{\delta^2 S}{\delta\sigma'^2}\frac{\delta^2 S}{\delta\tau'^2} - \left(\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\right)^2\right\}$$
$$+ \left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)^2$$
$$- \frac{\delta^2 S}{\delta\sigma'^2}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right)^2 - 2\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} + \frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\right)^2\right\}$$
$$+ 2\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\left\{\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} + \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right.$$
$$\left. + \frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\right\}$$
$$- \frac{\delta^2 S}{\delta\tau'^2}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)^2 - 2\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} + \frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right)^2\right\}. \tag{X^{16}}$$

When the point of intersection of the infinitely near initial rays removes to an infinite distance, this equation reduces itself to the following,

$$0 = \frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau^2} - \left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\right)^2$$
$$= \left(\frac{\delta^2 T}{\delta\sigma^2} - z\frac{\delta^2 \upsilon}{\delta\sigma^2}\right)\left(\frac{\delta^2 T}{\delta\tau^2} - z\frac{\delta^2 \upsilon}{\delta\tau^2}\right) - \left(\frac{\delta^2 T}{\delta\sigma\,\delta\tau} - z\frac{\delta^2 \upsilon}{\delta\sigma\,\delta\tau}\right)^2 : \tag{Y^{16}}$$

and when in like manner the two infinitely near final rays become parallel, it gives the following quadratic to determine the two corresponding positions of the point of initial intersection,

$$0 = \frac{\delta^2 S}{\delta\sigma'^2}\frac{\delta^2 S}{\delta\tau'^2} - \left(\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\right)^2$$
$$= \left(\frac{\delta^2 T}{\delta\sigma'^2} + z'\frac{\delta^2 \upsilon}{\delta\sigma'^2}\right)\left(\frac{\delta^2 T}{\delta\tau'^2} + z'\frac{\delta^2 \upsilon}{\delta\tau'^2}\right) - \left(\frac{\delta^2 T}{\delta\sigma'\,\delta\tau'} + z'\frac{\delta^2 \upsilon}{\delta\sigma'\,\delta\tau'}\right)^2 . \tag{Z^{16}}$$

The caustic surfaces of straight systems, ordinary or extraordinary, were determined in the First Supplement: but it seemed useful to resume the subject in a more general manner here, and to treat it by the new methods of the present memoir.

Connexion of the Conditions of Initial and Final Intersection of two Near Paths of Light, Polygon or Curved, with the Maxima or Minima of the Time or Action-Function $V + V_{\prime} = \Sigma \int v\, ds$. Separating Planes, Transition Planes, and Transition Points, suggested by these Maxima and Minima. The Separating Planes divide the Near Points of less from those of greater Action, and they contain the Directions of Osculation or Intersection of the Surfaces for which V and $V_{\prime}$ are constant; the Transition Planes touch the Caustic Pencils, and the Transition Points are on the Caustic Curves. Extreme Osculating Waves, or Action-Surfaces: Law of Osculation. Analogous Theorems for Sudden Reflexion or Refraction.

24. The conditions of initial and final intersection of two near luminous paths, have a remarkable connexion with the maxima and minima of the integral in the law of least action, that is, with those of the characteristic function V, or rather with those of the sum of two such integrals or functions, which may be investigated in the following manner.

Let A, B, C, be three successive points, at finite intervals, on one common luminous path. Let the rectangular co-ordinates of these three points be x', y', z' for A; x, y, z for B; and $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ for C. Let $V(A, B)$ denote the integral $\int v\, ds$ taken from the first point A to the second point B; let $V(B, C)$ denote the same integral, taken from the second point B to the third point C; and similarly, let $V(A, C)$ be the integral from A to C, which is evidently equal to the sum of the two former,

$$V(A, C) = V(A, B) + V(B, C), \tag{A^{17}}$$

so that, if we put for abridgment

$$V(A, B) = V, \quad V(B, C) = V_{\prime}, \tag{B^{17}}$$

we shall have, by the continuity of the integral,

$$V(A,C) = V + V_{\prime}. \tag{C^{17}}$$

If we do not suppose that the intermediate point B is a point of sudden reflexion or refraction, the final direction of the part (A,B) will coincide with the initial direction of the part (B,C), and the final direction-cosines $\alpha\ \beta\ \gamma$ of the one part will be equal to the initial direction-cosines of the other; considering V therefore, as usual, as a function of $x\ y\ z\ x'\ y'\ z'\ \chi$, and $V_{\prime}$ as a function of $x_{\prime}\ y_{\prime}\ z_{\prime}\ x\ y\ z\ \chi$, we have, by our fundamental formula (A),

$$\frac{\delta V}{\delta x} = \frac{\delta v}{\delta \alpha} = -\frac{\delta V_{\prime}}{\delta x}, \quad \frac{\delta V}{\delta y} = \frac{\delta v}{\delta \beta} = -\frac{\delta V_{\prime}}{\delta y}, \quad \frac{\delta V}{\delta z} = \frac{\delta v}{\delta \gamma} = -\frac{\delta V_{\prime}}{\delta z}; \tag{D^{17}}$$

that is, we have

$$\delta V + \delta V_{\prime} = 0, \tag{E^{17}}$$

for any infinitesimal variations of the co-ordinates $x\ y\ z$, and therefore, to the accuracy of the first order,

$$V(A,B') + V(B',C) = V(A,B) + V(B,C) = V(A,C), \tag{F^{17}}$$

B' being any new intermediate point infinitely near to B, and the path (B',C) being not in general a continuation of the path (A,B'). If therefore we regard the extreme points A, C, as fixed, but consider the intermediate point B as variable and as not necessarily situated on the path (A,C), the function $V + V_{\prime}$, or $\Sigma \int v\,ds$, composed of the two partial and now not necessarily continuous integrals (B^{17}), will acquire what may be called a *stationary value* when the paths (A,B) (B,C) become continuous, that is, when the intermediate point B takes any position on the path (A,C) from one given extreme point to the other: since then the change of this function will be infinitely small of the the second order, for any infinitely small alteration $\overline{BB'}$, of the first order, in the position of the point B. The stationary value thus determined, namely $V(A,C)$, might be called, by that customary latitude of expression which leads to the received phrase of *least action*, a *maximum* or *minimum* of the function $V + V_{\prime}$: but in order that this value should really be greater than all the neighbouring values, or less than all, a new condition is necessary. To find this new condition, we may observe that the relations

$$\left.\begin{aligned}
\alpha\frac{\delta^2 V}{\delta x^2} + \beta\frac{\delta^2 V}{\delta x\,\delta y} + \gamma\frac{\delta^2 V}{\delta x\,\delta z} = \frac{\delta v}{\delta x} &= -\left(\alpha\frac{\delta^2 V_{\prime}}{\delta x^2} + \beta\frac{\delta^2 V_{\prime}}{\delta x\,\delta y} + \gamma\frac{\delta^2 V_{\prime}}{\delta x\,\delta z}\right),\\
\alpha\frac{\delta^2 V}{\delta x\,\delta y} + \beta\frac{\delta^2 V}{\delta y^2} + \gamma\frac{\delta^2 V}{\delta y\,\delta z} = \frac{\delta v}{\delta y} &= -\left(\alpha\frac{\delta^2 V_{\prime}}{\delta x\,\delta y} + \beta\frac{\delta^2 V_{\prime}}{\delta y^2} + \gamma\frac{\delta^2 V_{\prime}}{\delta y\,\delta z}\right),\\
\alpha\frac{\delta^2 V}{\delta x\,\delta z} + \beta\frac{\delta^2 V}{\delta y\,\delta z} + \gamma\frac{\delta^2 V}{\delta z^2} = \frac{\delta v}{\delta y} &= -\left(\alpha\frac{\delta^2 V_{\prime}}{\delta x\,\delta z} + \beta\frac{\delta^2 V_{\prime}}{\delta y\,\delta z} + \gamma\frac{\delta^2 V_{\prime}}{\delta z^2}\right),
\end{aligned}\right\} \tag{G^{17}}$$

which result from the third number, give

$$\begin{aligned}
\delta^2 V + \delta^2 V_{\prime} = {} & \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{\prime}}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)^2 + \left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{\prime}}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right)^2 \\
& + 2\left(\frac{\delta^2 V}{\delta x\,\delta y} + \frac{\delta^2 V_{\prime}}{\delta x\,\delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right);
\end{aligned} \tag{H^{17}}$$

the *condition of existence of a maximum or minimum, properly so called, of the function* $V + V_{\prime}$, is therefore,

$$Q > 0, \quad \text{if} \quad Q = \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{\prime}}{\delta x^2}\right)\left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{\prime}}{\delta y^2}\right) - \left(\frac{\delta^2 V}{\delta x\,\delta y} + \frac{\delta^2 V_{\prime}}{\delta x\,\delta y}\right)^2. \tag{I^{17}}$$

When we have on the contrary

$$Q < 0, \tag{K^{17}}$$

the variation of the second order $\delta^2 V + \delta^2 V_{\prime}$ admits of changing sign, in passing from one set of values of δx, δy, δz to another, that is, in passing from one near point B' to another; and since, to the accuracy of the second order,

$$V(A, B') + V(B', C) - V(A, C) = \tfrac{1}{2}(\delta^2 V + \delta^2 V_{\prime}), \tag{L^{17}}$$

we shall have the one or the other of the two following opposite inequalities

$$V(A, B') + V(B', C) > \ \text{or}\ < V(A, C), \tag{M^{17}}$$

according as the near point B' is in one or the other pair of opposite diedrate angles formed by *two separating planes* P' P'' determined by the following equation

$$\delta^2 V + \delta^2 V_{\prime} = 0, \tag{N^{17}}$$

which is, by (H^{17}), quadratic with respect to the ratio

$$\frac{\delta y - \dfrac{\beta}{\gamma}\,\delta z}{\delta x - \dfrac{\alpha}{\gamma}\,\delta z}.$$

These two separating planes P' P'' contain each the ray-line or element of the path (A, B, C) at B; and they divide the near points of less from those of greater action, or those of shorter from those of longer time, when the continuous integral $V + V' = V(A, C)$ is not greater than all, or less than all, the adjacent values of the sum $\Sigma \int v\,ds$. The directions of these planes depend on the positions of the points A, B, C; so that if we consider A and B as fixed, but suppose C to move along the prolongation (B, C) of the path (A, B), the separating planes P', P'', will in general revolve about the ray-line at B. The will even become imaginary, when by this motion of C the quantity Q becomes $>$ instead of < 0, so as to satisfy the condition of existence of a maximum or minimum of the function $V + V_{\prime}$; and in this transition from the real to the imaginary state the two separating planes P' P'' will close up into one real *transition-plane* P, determined by either of the two following equations,

$$\left.\begin{aligned} 0 &= \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{\prime}}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right) + \left(\frac{\delta^2 V}{\delta x\,\delta y} + \frac{\delta^2 V_{\prime}}{\delta x\,\delta y}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right), \\ 0 &= \left(\frac{\delta^2 V}{\delta x\,\delta y} + \frac{\delta^2 V_{\prime}}{\delta x\,\delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right) + \left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{\prime}}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right); \end{aligned}\right\} \tag{O^{17}}$$

while the corresponding position of the point C, which we may call by analogy a *transition-point*, will satisfy the condition

$$Q = 0, \quad \text{that is,} \quad \left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V_{\prime}}{\delta x^2}\right)\left(\frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V_{\prime}}{\delta y^2}\right) = \left(\frac{\delta^2 V}{\delta x\, \delta y} + \frac{\delta^2 V_{\prime}}{\delta x\, \delta y}\right)^2. \qquad (\mathrm{P}^{17})$$

We are now prepared to perceive a remarkable connexion between the transition-planes and transition-points to which we have been thus conducted by the consideration of the maxima and minima of the function $V + V_{\prime}$, and the condition of final and initial intersection of two near luminous paths. For these conditions of intersection may be obtained by supposing that not only the point B, having the co-ordinates $x\ y\ z$, is on a given path (A, C), so as to satisfy the equations (D^{17}), but that also an infinitely near point B', having for co-ordinates $x + \delta x$, $y + \delta y$, $z + \delta z$, is on another path of the same colour connecting the same extreme points A and C, so as to give the differential equations

$$\delta \frac{\delta V}{\delta x} = -\delta \frac{\delta V_{\prime}}{\delta x}, \quad \delta \frac{\delta V}{\delta y} = -\delta \frac{\delta V_{\prime}}{\delta y}, \quad \delta \frac{\delta V}{\delta z} = -\delta \frac{\delta V_{\prime}}{\delta z}: \qquad (\mathrm{Q}^{17})$$

and since these last equations may be reduced, by the relations (G^{17}), to the forms (O^{17}), we see that when the conditions of initial and final intersection of a given path (A, B, C) with a near path (A, B', C) are satisfied, and when we consider the initial point A as fixed, the near intermediate point B' must be in a transition-plane P of the form (O^{17}), and the final point of intersection C must be a transition-point of the form (P^{17}). Continuing therefore to regard the initial point A as the fixed origin of a system of luminous paths, polygon or curved, of any common colour, which undergo any number of refractions or reflexions, ordinary or extraordinary, and gradual or sudden, it is easy to see that we may consider these paths as touching a certain set of *caustic curves*, in the final state of the system, and therefore grouped into certain sets of consecutively intersecting paths, and as having for their loci certain corresponding sets of ray-surfaces, which may be called *caustic pencils*: and that *these caustic pencils are touched by the transition-planes* (O^{17}), while the *transition-points* (P^{17}) *are on the caustic curves*, and therefore on their loci the caustic surfaces. The transition-points are also evidently the points of consecutive intersection, or of vergency, of the luminous paths from A, in the final state of the system. And it is manifest, from the foregoing remarks, that these final points of intersection are also transition-points in the following other sense, that when the point C, in moving along the prolongation of the path (A, B), arrives at any one of these points of intersection, the condition of existence of maximum or minimum of the function $V + V_{\prime}$ begins or ceases to be satisfied.

The separating planes $P'\ P''$ have, when real, another remarkable property, that of containing the directions of mutual osculation, at the point B, of the two action-surfaces or waves determined by the equations

$$V = \text{const.}, \quad V_{\prime} = \text{const.}; \qquad (\mathrm{R}^{17})$$

for these equations may be put approximately under the following forms, (when we choose the point B for origin and the final direction of the path (A, B) for the positive semiaxis of

z, so as to have $\alpha = 0$, $\beta = 0$, $\gamma = 1$,)

$$\left.\begin{aligned} z &= px + qy + \tfrac{1}{2}rx^2 + sxy + \tfrac{1}{2}ty^2, \\ z_{\prime} &= p_{\prime}x + q_{\prime}y + \tfrac{1}{2}r_{\prime}x^2 + s_{\prime}xy + \tfrac{1}{2}t_{\prime}y^2, \end{aligned}\right\} \tag{S^{17}}$$

in which the coefficients have the following relations

$$\left.\begin{aligned} p_{\prime} &= p, \quad q_{\prime} = q, \\ r_{\prime} - r &= \frac{1}{v}\left(\frac{\delta^2 V_{\prime}}{\delta x^2} + \frac{\delta^2 V}{\delta x^2}\right), \\ s_{\prime} - s &= \frac{1}{v}\left(\frac{\delta^2 V_{\prime}}{\delta x\,\delta y} + \frac{\delta^2 V}{\delta x\,\delta y}\right), \\ t_{\prime} - t &= \frac{1}{v}\left(\frac{\delta^2 V_{\prime}}{\delta y^2} + \frac{\delta^2 V}{\delta y^2}\right), \end{aligned}\right\} \tag{T^{17}}$$

and therefore the planes

$$0 = (r_{\prime} - r)x^2 + 2(s_{\prime} - s)xy + (t_{\prime} - t)y^2, \tag{U^{17}}$$

which pass through the given ray-line at the point B, and contain the directions of osculation of the second order of the two touching surfaces (R^{17}) or (S^{17}), are the separating planes (N^{17}). We might also characterise these separating planes, or planes of osculation, as containing the directions of mutual intersection of the same two touching surfaces for which V and $V_{\prime}$ are constant; or as the planes in which the deflexures of these two surfaces are equal, the ray-line at B being made the axis of deflexion.

The comparison of the same two waves or action-surfaces (R^{17}) gives a new property of the planes and points of transition; for the equations which determine a plane and point of this kind may be put under the form

$$(r - r_{\prime})x + (s - s_{\prime})y = 0, \quad (s - s_{\prime})x + (t - t_{\prime})y = 0, \quad \text{or,} \quad \delta p_{\prime} = \delta p, \quad \delta q_{\prime} = \delta q : \tag{V^{17}}$$

they express, therefore, that when C is a transition-point, the two surfaces (R^{17}) touch one another not only at the point B, but in the whole extent of an infinitely small arc contained in the transition-plane.

The point C may be called the *focus* of the second wave or action-surface $V_{\prime}$, since all the corresponding paths of light (B', C) are supposed to meet in it; and in like manner the point A may be called the focus of the first surface V of the same kind, since all the paths (A, B') are supposed to diverge from A. The focus A and the point of osculation B remaining fixed, we may change the focus C, and thereby the directions of osculation; but there are, in general, certain *extreme or limiting positions for the osculating focus C, corresponding to extreme osculating waves or action-surfaces $V_{\prime}$*, and it is easy to show that *these extreme osculating foci coincide with the transition-points or points of vergency*: and that *the transition-planes or tangent-planes of the caustic pencils contain the directions of such extreme or limiting osculation.*

These theorems of intersection and osculation include several less general theorems of the same kind, assigned in former memoirs. It is easy also to see that they extend to the case when the order of the points A B C on a luminous path is different, so that B is not intermediate between A and C, and so that the paths (A, B) (A, B'), which go from A to the points B and B', coincide at those points with the paths (C, B) (C, B'), and not with the opposite paths (B, C) (B', C), that is, tend *from* the point C, not *to* it; observing only that we must then employ the *difference* instead of the *sum* of the two integrals $\int v\,ds$, or of the two functions V and $V_{\prime}$.

When the point C is on a given straight ray in a given uniform medium, we can easily prove, by the theory of the partial differential coefficients of the second order of the characteristic and related functions which was explained in former numbers, that the equation (P^{17}) becomes quadratic with respect to $z_{\prime}$ or $V_{\prime}$, and assigns, in general, two real or imaginary positions C_1, C_2, for the transition-point, or point of vergency; and that the equations (O^{17}) assign two corresponding real or imaginary transition-planes P_1 P_2, or tangent planes of caustic pencils. And when, besides, the points B, C, are both in one common or uniform medium, so that the paths (B, C) (B', C) are straight, then each of the caustic pencils, or ray-surfaces, composed of such straight paths consecutively intersecting each other and touching one caustic curve, becomes a *developable pencil*, and its tangent plane becomes a *plane of vergency*, of the kind considered in the sixteenth number. The relations also between the two planes of vergency in a final uniform medium, which were pointed out in the twenty-first number, may easily be deduced from the present more general view and from the recent theorems of osculation; for thus we are led to consider a series of waves or action-surfaces $V_{\prime}$, similar and similarly placed, and determined in shape but not in size or focus by the uniform medium, and then to seek the extreme or limiting surfaces of this set which osculate to the given surface V at the given point B; and since it can be shown that *in general among any series of surfaces, similar and similarly placed, but having arbitrary magnitudes, and osculating to a given surface at a given point, there are two extreme osculating surfaces*, real or imaginary, and that *the tangents which mark the two corresponding directions of calculation are conjugate tangents* (of the kind discovered by M. Dupin) *on each surface of the osculating series, and also on the given surface*, it follows as before that the conjugate planes of vergency in a final uniform medium are conjugate planes of deflexure of each medium-surface $V_{\prime}$, and also of the surface V determined by the whole combination. When the final medium is ordinary as well as uniform, then the osculating surfaces $V_{\prime}$ are spheres, and the directions of extreme osculation are the rectangular directions of the lines of curvature on the surface V, which is now perpendicular to the rays; in this case, therefore, and more generally when a given ray in a final uniform medium corresponds to an *umbilical point* or point of spheric curvature on the medium-surface $V_{\prime}$, the planes of vergency cut that surface, and the surface V to which it osculates, in two rectangular directions, because two conjugate tangents at an umbilical point are always perpendicular to each other: and, in like manner, the planes of vergency being conjugate planes of deflexure will (by the seventeenth number) be themselves rectangular, if the final ray whether ordinary or extraordinary be such that taking it for the axis of deflexion of the medium-surface $V_{\prime}$ the indicating cylinder of deflexion is circular.

The foregoing principles give us also the *law of osculation* of the variable medium-surface $V_{\prime}$ between its extreme positions, in a final uniform medium, namely, that *the distances of the variable osculating focus from the two points of vergency, are proportional to the squares of* *the sines of the inclinations of the variable plane of osculation to the two planes of vergency, multiplied respectively by certain constant factors.* A formula expressing this law was deduced in the First Supplement; but the constant and in general unequal factors, (in the formula ζ and 1,) for the squares of the sines of the inclinations, were inadvertently omitted in the enunciation. Our present methods would enable us to investigate without difficulty the law for the more complicated case, when the osculating focus C being still in a uniform medium, the point of osculation B is in another uniform medium, or even in an atmosphere ordinary or extraordinary.

We might extend the reasonings of the present number to the case of sudden reflexion or refraction, ordinary or extraordinary, and obtain analogous results, which would include, in like manner, the results of former memoirs. In this case we should find a certain analogous condition for the existence of a maximum or minimum of the function $\Sigma \int v\, ds$; and when this condition is not satisfied, we should have to consider *two pairs of separating planes*, which cross the tangent plane of the reflecting or refracting surface in one common pair of *separating lines*: the two pairs of planes passing together from the real to the imaginary state, and in this passage closing up into *two transition-planes, which touch the caustic pencil before and after the sudden reflexion or refraction*, and intersect in one common *transition-line, on the tangent plane of the reflector or refractor*, connected with a *transition-point upon the caustic curve of the pencil*, and with certain *extreme osculating waves or action-surfaces and focal reflectors or refractors*, of a kind easily discovered from the analogy of the foregoing results.

Formulæ for the Principal Foci and Principal Rays of a Straight or Curved System, Ordinary or Extraordinary. General method of investigating the Arrangement and Aberrations of the Rays, near a Principal Focus, or other point of vergency.

25. Among the various points of consecutive intersection of the rays of an optical system, there are in general certain eminent points of vergency, in which certain particular luminous paths are intersected each by all the infinitely near paths of the system. These eminent points and paths have been pointed out in my former memoirs, and have been called *principal foci*, and *principal rays*. They may be determined for straight final systems, by the characteristic function V, and by any three of the six following equations,

$$\left.\begin{aligned}
&\frac{\delta^2 V}{\delta x^2} + \frac{1}{R}\frac{\delta^2 v}{\delta \alpha^2} = 0, \quad && \frac{\delta^2 V}{\delta x\, \delta y} + \frac{1}{R}\frac{\delta^2 v}{\delta \alpha\, \delta \beta} = 0, \\
&\frac{\delta^2 V}{\delta y^2} + \frac{1}{R}\frac{\delta^2 v}{\delta \beta^2} = 0, \quad && \frac{\delta^2 V}{\delta y\, \delta z} + \frac{1}{R}\frac{\delta^2 v}{\delta \beta\, \delta \gamma} = 0, \\
&\frac{\delta^2 V}{\delta z^2} + \frac{1}{R}\frac{\delta^2 v}{\delta \gamma^2} = 0, \quad && \frac{\delta^2 V}{\delta z\, \delta x} + \frac{1}{R}\frac{\delta^2 v}{\delta \gamma\, \delta \alpha} = 0,
\end{aligned}\right\} \tag{W^{17}}$$

x, y, z being the co-ordinates of any point on a principal ray, and $x + \alpha R$, $y + \beta R$, $z + \gamma R$ being the co-ordinates of the principal focus; they may also be deduced from the auxiliary function W, when made homogeneous of the first dimension with respect to σ, τ, υ, by the equations

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta\sigma^2} + V\frac{\delta^2\Omega}{\delta\sigma^2} = 0, \quad \frac{\delta^2 W}{\delta\sigma\,\delta\tau} + V\frac{\delta^2\Omega}{\delta\sigma\,\delta\tau} = 0,\\
\frac{\delta^2 W}{\delta\tau^2} + V\frac{\delta^2\Omega}{\delta\tau^2} = 0, \quad \frac{\delta^2 W}{\delta\tau\,\delta\upsilon} + V\frac{\delta^2\Omega}{\delta\tau\,\delta\upsilon} = 0,\\
\frac{\delta^2 W}{\delta\upsilon^2} + V\frac{\delta^2\Omega}{\delta\upsilon^2} = 0, \quad \frac{\delta^2 W}{\delta\upsilon\,\delta\sigma} + V\frac{\delta^2\Omega}{\delta\upsilon\,\delta\sigma} = 0,
\end{aligned}\right\} \tag{X^{17}}$$

of which only three are distinct, and in which V corresponds to the focus: or from the function T, when expressed as depending on σ, τ, σ', τ', χ, by the following,

$$\begin{aligned}
&\left(\frac{\delta^2 S}{\delta\sigma'^2}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)^2 - 2\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'} + \frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\right)^2\right\}\\
&= \left(\frac{\delta^2 S}{\delta\sigma'\,\delta\tau'}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} - \frac{\delta^2 S}{\delta\sigma\,\delta\tau}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} + \frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\sigma'}\right)\right.\\
&\qquad\qquad\left. + \frac{\delta^2 S}{\delta\tau^2}\frac{\delta^2 S}{\delta\sigma\,\delta\sigma'}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\right\}\\
&= \left(\frac{\delta^2 S}{\delta\tau'^2}\right)^{-1}\left\{\frac{\delta^2 S}{\delta\sigma^2}\left(\frac{\delta^2 S}{\delta\tau\,\delta\tau'}\right)^2 - 2\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\frac{\delta^2 S}{\delta\tau\,\delta\tau'} + \frac{\delta^2 S}{\delta\tau^2}\left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau'}\right)^2\right\}\\
&= \frac{\delta^2 S}{\delta\sigma^2}\frac{\delta^2 S}{\delta\tau^2} - \left(\frac{\delta^2 S}{\delta\sigma\,\delta\tau}\right)^2,
\end{aligned} \tag{Y^{17}}$$

in which, as before, $S = T - z\upsilon + z'\upsilon'$.

When the final medium is variable, we may employ the following equations,

$$\left.\begin{aligned}
&\frac{\dfrac{\delta^2 W}{\delta\sigma^2}}{\left(\dfrac{\delta\Omega}{\delta\sigma}\right)^2} = \frac{\dfrac{\delta^2 W}{\delta\tau^2}}{\left(\dfrac{\delta\Omega}{\delta\tau}\right)^2} = \frac{\dfrac{\delta^2 W}{\delta\upsilon^2}}{\left(\dfrac{\delta\Omega}{\delta\upsilon}\right)^2} = \frac{\dfrac{\delta^2 W}{\delta\sigma\,\delta\tau}}{\dfrac{\delta\Omega}{\delta\sigma}\dfrac{\delta\Omega}{\delta\tau}} = \frac{\dfrac{\delta^2 W}{\delta\tau\,\delta\upsilon}}{\dfrac{\delta\Omega}{\delta\tau}\dfrac{\delta\Omega}{\delta\upsilon}} = \frac{\dfrac{\delta^2 W}{\delta\upsilon\,\delta\sigma}}{\dfrac{\delta\Omega}{\delta\upsilon}\dfrac{\delta\Omega}{\delta\sigma}}\\
&\qquad = -\left(\frac{\delta\Omega}{\delta\sigma}\frac{\delta\Omega}{\delta x} + \frac{\delta\Omega}{\delta\tau}\frac{\delta\Omega}{\delta y} + \frac{\delta\Omega}{\delta\upsilon}\frac{\delta\Omega}{\delta z}\right)^{-1},\\
\text{or,}\quad &\frac{1}{\alpha^2}\frac{\delta^2 W}{\delta\sigma^2} = \frac{1}{\beta^2}\frac{\delta^2 W}{\delta\tau^2} = \frac{1}{\gamma^2}\frac{\delta^2 W}{\delta\upsilon^2} = \frac{1}{\alpha\beta}\frac{\delta^2 W}{\delta\sigma\,\delta\tau} = \frac{1}{\beta\gamma}\frac{\delta^2 W}{\delta\tau\,\delta\upsilon} = \frac{1}{\gamma\alpha}\frac{\delta^2 W}{\delta\upsilon\,\delta\sigma}\\
&\qquad = \left(\alpha\frac{\delta\upsilon}{\delta x} + \beta\frac{\delta\upsilon}{\delta y} + \gamma\frac{\delta\upsilon}{\delta z}\right)^{-1},
\end{aligned}\right\} \tag{Z^{17}}$$

of which only three are distinct, but which are sufficient to determine the *principal foci and principal rays of a curved system, ordinary or extraordinary*, by the auxiliary function W, considered as depending on σ, τ, υ, x', y', z', χ, in confirmity to the new view of that function, proposed in the present Supplement. The new function T might also be employed for the same purpose, but with somewhat less facility.

It was remarked, in a former number, that at a point of vergency the general linear expressions for the relations of near rays fail; but the more complex expressions by which these linear forms must be replaced at a principal focus or other point of vergency, and generally when it is proposed to determine the aberrational corrections of the first approximate or limiting relations, can always be obtained without difficulty by developing to the required order of accuracy the general and rigorous equations which we have given for a luminous path. An example of such deduction will occur, when we come to consider the theory of *instruments of revolution*, which on account of its extent and importance must be reserved for a future occasion.

Combination of the foregoing View of Optics with the Undulatory Theory of Light. The quantities σ, τ, υ, *or* $\frac{\delta V}{\delta x}$, $\frac{\delta V}{\delta y}$, $\frac{\delta V}{\delta z}$, *that is, the Partial Differential Coefficients of the First Order of the Characteristic Function* V, *taken with respect to the Final Co-ordinates, are, in the Undulatory Theory of Light, the Components of Normal Slowness of Propagation of a Wave. The Fundamental Formula* (A) *may easily be explained and proved by the principles of the same theory.*

26. It remains, for the execution of the design announced at the beginning of this Supplement, to illustrate the mathematical view of optics proposed in this and in former memoirs, by connecting it more closely with the undulatory theory of light.

For this purpose we shall begin by examining the undulatory meanings of the symbols σ, τ, υ, of which, in the present Supplement, we have made so frequent a use, and which we have defined by the equations (E),

$$\sigma = \frac{\delta V}{\delta x}, \quad \tau = \frac{\delta V}{\delta y}, \quad \upsilon = \frac{\delta V}{\delta z},$$

V being the undulatory time of propagation of light of some given colour, from some origin x', y', z', to a point x, y, z, through any combination of media. It is evident that these quantities σ, τ, υ are proportional to the direction-cosines of the normal to the wave for which the time V is constant, and which has for its differential equation

$$0 = \delta V = \sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z; \tag{A^{18}}$$

and if, as in the second number, we denote $(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$ by ω, these direction-cosines themselves will be $\sigma\omega$, $\tau\omega$, $\upsilon\omega$; and ω will be the *normal velocity*, because the infinitesimal time δV, during which the wave propagates itself in the direction of its own normal through the infinitesimal line δl, from the point x, y, z, to the point $x + \sigma\omega \,.\, \delta l$, $y + \tau\omega \,.\, \delta l$, $z + \upsilon\omega \,.\, \delta l$, is

$$\delta V = \sigma \,.\, \sigma\omega \,.\, \delta l + \tau \,.\, \tau\omega \,.\, \delta l + \upsilon \,.\, \upsilon\omega \,.\, \delta l = \frac{1}{\omega}\,\delta l: \tag{B^{18}}$$

we may therefore call the quantities σ, τ, υ, *the components of normal slowness*, because they are equal to the reciprocal of the normal velocity, that is, to *the normal slowness, multiplied respectively by the direction-cosines of the normal*, that is, by the cosines of the angles which it makes with the rectangular axes of co-ordinates.

Such then may be said to be the optical meaning of our quantities σ, τ, υ, in the theory of the propagation of light by waves. And we might easily deduce from this meaning, and from the first principles of the undulatory theory, the general expression (A) for the variation of the characteristic function V, which has been proposed in the present and former memoirs, as fundamental in mathematical optics. For it is an immediate consequence of the dynamical ideas of the undulatory theory of light, that for a plane wave of a given direction and colour, in a given uniform medium, the normal velocity of propagation is determined, or at least restricted to a finite variety of values; so that this normal velocity may be considered as a function of its cosines of direction, involving also the colour, and depending for its form on the nature of the uniform medium, and on the positions of the axes of co-ordinates, to which the angles of direction are referred: and if the medium be variable instead of uniform, and the wave curved instead of plane, we must suppose that the normal velocity ω is still a function of its direction-cosines $\sigma(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\tau(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\upsilon(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, and of the colour χ, involving also, in this more general case, the co-ordinates x, y, z. In this manner we are conducted, by the principles of the undulatory theory, to a relation between σ, τ, υ, x, y, z, χ, of the kind already often employed in the present Supplement, namely,

$$0 = \Omega = (\sigma^2+\tau^2+\upsilon^2)^{\frac{1}{2}}\omega - 1, \tag{M}$$

$\Omega+1$ being a homogeneous function of σ, τ, υ, of the first dimension, which satisfies therefore the condition

$$\sigma\frac{\delta\Omega}{\delta\sigma} + \tau\frac{\delta\Omega}{\delta\tau} + \upsilon\frac{\delta\Omega}{\delta\upsilon} = \Omega + 1,$$

and which involves also in general the co-ordinates x, y, z, and the colour χ, and depends for its form on the optical properties of the medium in which the point $x\ y\ z$ is placed. *To connect now*, for any given point and colour, the *velocity and direction of the ray with the direction of the normal of the wave*, we may suppose, at first, that the medium is uniform, and that the wave is plane. The two positions of this plane wave, at the time V, and at the time $V+\Delta V$, may be denoted by the equations

$$\left.\begin{array}{ll} \text{First} & \sigma x + \tau y + \upsilon z = V + W, \\ \text{Second} & \sigma\Delta x + \tau\Delta y + \upsilon\Delta z = \Delta V, \end{array}\right\} \tag{C^{18}}$$

in which σ, τ, υ, W, are constants; and by the principles of the same undulatory theory, if the point $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, on the second plane wave, corresponding to the time $V+\Delta V$, be upon the ray that passes through the point $x\ y\ z$ of the first plane wave, it will be also on all the other infinitely near plane waves which correspond to the same time $V+\Delta V$, these other waves having passed through the point $x\ y\ z$ at the time V, and having made infinitely small angles with the first plane wave; we are therefore to find the co-ordinates $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, of the second point upon the ray, by seeking the intersection of the second wave (C^{18}) with all those other waves which are obtained from it by assigning to σ, τ, υ, any infinitely small variations consistent with the relation

$$0 = \delta\Omega = \frac{\delta\Omega}{\delta\sigma}\delta\sigma + \frac{\delta\Omega}{\delta\tau}\delta\tau + \frac{\delta\Omega}{\delta\upsilon}\delta\upsilon;$$

and thus we find

$$\frac{\alpha}{v} = \frac{\Delta x}{\Delta V} = \frac{\delta\Omega}{\delta\sigma}, \quad \frac{\beta}{v} = \frac{\Delta y}{\Delta V} = \frac{\delta\Omega}{\delta\tau}, \quad \frac{\gamma}{v} = \frac{\Delta z}{\Delta V} = \frac{\delta\Omega}{\delta\upsilon}, \tag{D^{18}}$$

as in the second number of this Supplement, and therefore

$$\begin{aligned} v &= \alpha\sigma + \beta\tau + \gamma\upsilon, \\ 0 &= \alpha\,\delta\sigma + \beta\,\delta\tau + \gamma\,\delta\upsilon, \\ \delta v &= \sigma\,\delta\alpha + \tau\,\delta\beta + \upsilon\,\delta\gamma, \end{aligned}$$

and finally

$$\frac{\delta v}{\delta\alpha} = \sigma, \quad \frac{\delta v}{\delta\beta} = \tau, \quad \frac{\delta v}{\delta\gamma} = \upsilon, \tag{E^{18}}$$

if we denote by v the reciprocal of the undulatory velocity with which the light is propagated along the ray, and by α, β, γ, the cosines of the angles which the ray makes with the axes of co-ordinates. We see, therefore, by the foregoing reasoning, which it is easy to extend to the case of curved waves and of variable media, that *the components σ, τ, υ, of normal slowness of a wave, or the partial differential coefficients of the first order of the time-function V, are equal to the partial differential coefficients of the first order,* $\frac{\delta v}{\delta\alpha}$, $\frac{\delta v}{\delta\beta}$, $\frac{\delta v}{\delta\gamma}$, *of the undulatory slowness V of propagation along the ray, when this latter slowness is expressed as a homogeneous function of the first dimension of the direction-cosines α β γ of the ray*: which is the general theorem of mathematical optics, expressed by our fundamental formula (A).

That general theorem does not appear to have been perceived by other writers; nor do they seem to have distinctly thought of the components of normal slowness, nor of the function of which these components are partial differential coefficients, that is, the time V of propagation of light from one variable point to another, through any combination of uniform or variable media, considered as depending on the final and initial co-ordinates and on the colour: much less do those who have hitherto written upon light, appear to have thought of *the time-function V as a* CHARACTERISTIC FUNCTION, *to the study of which may be reduced all the problems of mathematical optics*. But the problem of connecting by general equations the direction and velocity of a ray with the direction and with the law of normal velocity of a wave, has been elegantly resolved by M. CAUCHY, in the 50th *Livraison* of the *Exercices de Mathématiques*: and the formulæ which have been there deduced by considering the normal velocity as a homogeneous function of the first dimension of its three cosines of direction, may easily be shown to agree with the equations (D^{18}).

Theory of FRESNEL. *New Formulæ, founded on that theory, for the Velocities and Polarisations of a Plane Wave or Wave-Element. New method of deducing the Equation of* FRESNEL'S *Curved Wave, propagated from a Point in a Uniform Medium with Three Unequal Elasticities. Lines of Single Ray-Velocity, and of Single Normal-Velocity, discovered by* FRESNEL.

27. Let us now consider more particularly the undulatory theory of FRESNEL.

In that theory, the small displacements of the vibrating etherial points are confined to the surface of the wave, the ether being supposed to be sensibly incompressible, and so to resist and prevent any sensible normal vibration: and the tangential forces, which regulate the tangential or transversal vibrations, result in general from the elasticity of the ether, combined with this normal resistance. It is also supposed that the etherial medium has in general three principal unequal elasticities, corresponding to displacements in the directions of three rectangular *axes of elasticity*; in such a manner that if we take these for the axes of co-ordinates, any small component displacements δx, δy, δz parallel to these three axes will produce elastic forces $-a^2\,\delta x$, $-b^2\delta y$, $-c^2\,\delta z$ parallel to the same axes, and equal to the displacements taken with contrary signs and multiplied by certain constant positive factors a^2, b^2, c^2; and any small resultant displacement, δl, in any other direction, having δx, δy, δz for its components or projections, will produce a corresponding elastic force $-E\,\delta l$, of which the components are $-a^2\,\delta x$, $-b^2\,\delta y$, $-c^2\,\delta z$, and which has not in general the same direction as the displacement δl, nor a direction exactly opposite to that. Light, polarised in any plane, P, is supposed to correspond to vibrations perpendicular to that plane, and propagated without change of direction; and in order that a vibration should thus preserve its direction unchanged, while the plane wave or wave-element to which it belongs is propagated through the uniform medium with a normal velocity ω, it is necessary and sufficient that the elastic force $-E\,\delta l$, when combined with a normal resistance arising from the incompressibility of the ether, should produce a tangential force $-\omega^2\,\delta l$, in the direction opposite to the displacement δl, and equal to this displacement taken with a contrary sign, and multiplied by the square of the normal velocity of propagation, so that its components are $-\omega^2\,\delta x$, $-\omega^2\,\delta y$, $-\omega^2\,\delta z$: that is, we must have the equations

$$\frac{1}{\sigma}(\omega^2 - a^2)\delta x = \frac{1}{\tau}(\omega^2 - b^2)\delta y = \frac{1}{\upsilon}(\omega^2 - c^2)\delta z, \tag{F^{18}}$$

in which σ, τ, υ, are, as before, the components of normal slowness, so that the equation of the wave-element containing the transversal vibration is

$$\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z = 0. \tag{A^{18}}$$

These equations (A^{18}) (F^{18}) *suffice in general to determine, on* Fresnel's *principles, the velocities of propagation and the planes of polarisation for any given wave-element in any known crystallised medium.*

Thus, eliminating the components of displacement δx, δy, δz, between the equations (A^{18}) (F^{18}), we find the following *law of the normal velocity* ω, considered as depending on the normal direction, that is, on the ratios of σ, τ, υ,

$$\frac{\sigma^2}{\omega^2 - a^2} + \frac{\tau^2}{\omega^2 - b^2} + \frac{\upsilon^2}{\omega^2 - c^2} = 0. \tag{G^{18}}$$

To deduce hence the *direction and velocity of a ray, for any given normal direction and normal velocity*, compatible with the foregoing law, that is, for any given values of the components of normal slowness σ, τ, υ, compatible with the relation (G^{18}), we are to make, by (M),

$$\omega^2 = \frac{(\Omega + 1)^2}{\sigma^2 + \tau^2 + \upsilon^2}, \tag{H^{18}}$$

and we then find, by (I), or by (D^{18}), the following expressions for the components of the velocity of the ray,

$$\left.\begin{aligned}
\frac{\alpha}{v} &= \frac{\delta\Omega}{\delta\sigma} = \frac{\sigma\omega^2}{\Omega+1}\,\frac{\lambda^2-a^2}{\omega^2-a^2},\\
\frac{\beta}{v} &= \frac{\delta\Omega}{\delta\tau} = \frac{\tau\omega^2}{\Omega+1}\,\frac{\lambda^2-b^2}{\omega^2-b^2},\\
\frac{\gamma}{v} &= \frac{\delta\Omega}{\delta\upsilon} = \frac{\upsilon\omega^2}{\Omega+1}\,\frac{\lambda^2-c^2}{\omega^2-c^2},
\end{aligned}\right\} \tag{I^{18}}$$

if we put for abridgment

$$\lambda^2 = \frac{\left(\dfrac{a^2\sigma}{\omega^2-a^2}\right)^2 + \left(\dfrac{b^2\tau}{\omega^2-b^2}\right)^2 + \left(\dfrac{c^2\upsilon}{\omega^2-c^2}\right)^2}{\left(\dfrac{a\sigma}{\omega^2-a^2}\right)^2 + \left(\dfrac{b\tau}{\omega^2-b^2}\right)^2 + \left(\dfrac{c\upsilon}{\omega^2-c^2}\right)^2}. \tag{K^{18}}$$

And to deduce the *law of the velocity* $\frac{1}{v}$ *of the ray, considered as depending on its own direction*, that is, on the cosines $\alpha\ \beta\ \gamma$ of its inclinations to the semiaxes $a\ b\ c$ of elasticity, we are to eliminate (according to the general method of the second number) the ratios of $\sigma\ \tau\ \upsilon$ between the three expressions (I^{18}), and so deduce the relations between the three components of velocity $\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}$; now the equations ($I^{18}$) given evidently, by ($K^{18}$),

$$\frac{a^2\alpha^2}{\lambda^2-a^2} + \frac{b^2\beta^2}{\lambda^2-b^2} + \frac{c^2\gamma^2}{\lambda^2-c^2} = 0; \tag{L^{18}}$$

they give also, when we attend to (G^{18}),

$$\left(\frac{\alpha}{v}\right)^2 + \left(\frac{\beta}{v}\right)^2 + \left(\frac{\gamma}{v}\right)^2 = \lambda^2 : \tag{M^{18}}$$

λ therefore is the velocity of the ray, or the radius vector of the curved *unit-wave*, propagated in all directions from the origin of co-ordinates during the unit of time; and the *equation of the wave* in rectangular co-ordinates $x\ y\ z$, parallel to the axes of elasticity, is

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2} + \frac{b^2y^2}{x^2+y^2+z^2-b^2} + \frac{b^2z^2}{x^2+y^2+z^2-c^2} = 0, \tag{N^{18}}$$

or, when freed from fractions,

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)+a^2b^2c^2 = a^2(b^2+c^2)x^2+b^2(c^2+a^2)y^2+c^2(a^2+b^2)z^2. \tag{O^{18}}$$

This method of determining the equation of Fresnel's *Wave*, will perhaps be thought simpler than that which was employed by the illustrious discover, and than others which have since been proposed.

Reciprocally to determine by our general methods the normal direction and velocity, or the components of normal slowness σ, τ, υ, for any proposed direction and velocity of a ray compatible with this form of the wave, that is, for any values of $\alpha\ \beta\ \gamma\ \lambda$ compatible with the relation (L^{18}), we are to substitute for the ray-velocity λ in that relation its value (M^{18}), and we find, by (E^{18}),

$$\left.\begin{aligned} \sigma &= \frac{\delta v}{\delta\alpha} = \frac{\alpha}{v} \cdot \frac{1 - a^2\nu^2}{\lambda^2 - a^2}, \\ \tau &= \frac{\delta v}{\delta\beta} = \frac{\beta}{v} \cdot \frac{1 - b^2\nu^2}{\lambda^2 - b^2}, \\ \upsilon &= \frac{\delta v}{\delta\gamma} = \frac{\gamma}{v} \cdot \frac{1 - c^2\nu^2}{\lambda^2 - c^2}, \end{aligned}\right\} \tag{P^{18}}$$

if we put for abridgment

$$\nu^2 = \frac{\left(\dfrac{\alpha}{\lambda^2 - a^2}\right)^2 + \left(\dfrac{\beta}{\lambda^2 - b^2}\right)^2 + \left(\dfrac{\gamma}{\lambda^2 - c^2}\right)^2}{\left(\dfrac{a\alpha}{\lambda^2 - a^2}\right)^2 + \left(\dfrac{b\beta}{\lambda^2 - b^2}\right)^2 + \left(\dfrac{c\gamma}{\lambda^2 - c^2}\right)^2}. \tag{Q^{18}}$$

It is easy to see that the value of ν thus determined is the normal slowness, or reciprocal of ω, because the expressions (P^{18}) give, by (L^{18}),

$$\sigma^2 + \tau^2 + \upsilon^2 = \nu^2; \tag{R^{18}}$$

and since the same expressions given also evidently, by (Q^{18}),

$$\frac{\sigma^2}{1 - a^2\nu^2} + \frac{\tau^2}{1 - b^2\nu^2} + \frac{\upsilon^2}{1 - c^2\nu^2} = 0, \tag{S^{18}}$$

we easily deduce the law (G^{18}) of dependence of the normal velocity on the normal direction, from the form of FRESNEL'S wave, as we had deduced the latter from the former.

The equations (L^{18}) (M^{18}) which gave us the equation of the wave in rectangular co-ordinates, give also the following polar equation for the reciprocal of its radius-vector, that is, for the slowness v of the ray,

$$\begin{aligned} 0 = v^4 - v^2\{\alpha^2(b^{-2} + c^{-2}) + \beta^2(c^{-2} + a^{-2}) + \gamma^2(a^{-2} + b^{-2})\} \\ + (\alpha^2 + \beta^2 + \gamma^2)(\alpha^2 b^{-2} c^{-2} + \beta^2 c^{-2} a^{-2} + \gamma^2 a^{-2} b^{-2}), \end{aligned} \tag{T^{18}}$$

and therefore the following double expression for the square of this slowness,

$$\begin{aligned} v^2 &= \tfrac{1}{2}(c^{-2} + a^{-2})(\alpha^2 + \beta^2 + \gamma^2) \\ &\quad + \tfrac{1}{2}(c^{-2} - a^{-2})\{A'A'' \pm \sqrt{\alpha^2 + \beta^2 + \gamma^2 - A'^2}\sqrt{\alpha^2 + \beta^2 + \gamma^2 - A''^2}\}, \end{aligned} \tag{U^{18}}$$

if we put for abridgment

$$\left.\begin{aligned} A' &= \alpha\sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}} + \gamma\sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}}, \\ A'' &= \alpha\sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}} - \gamma\sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}} : \end{aligned}\right\} \tag{V^{18}}$$

supposing therefore $a^2 > b^2 > c^2$, the *polar equation of the wave* may be put under the form

$$\rho^{-2} = \tfrac{1}{2}(c^{-2} + a^{-2}) + \tfrac{1}{2}(c^{-2} - a^{-2})\cos.((\rho\rho') \pm (\rho\rho'')), \tag{W^{18}}$$

ρ being the radius-vector or velocity and $(\rho\rho')$ $(\rho\rho'')$ being the angles which this radius ρ makes with two constant radii ρ', ρ'', determined by the following cosines of their inclinations to the semiaxes of $x\ y\ z$, or of $a\ b\ c$,

$$\rho'_a = \rho''_a = \sqrt{\frac{b^{-2}-a^{-2}}{c^{-2}-a^{-2}}}, \quad \rho'_b = \rho''_b = 0, \quad \rho'_c = -\rho''_c = \sqrt{\frac{c^{-2}-b^{-2}}{c^{-2}-a^{-2}}}. \tag{X^{18}}$$

The expression (W^{18}), for the reciprocal of the square of the velocity of a ray, has been assigned by FRESNEL, who has also remarked that it gives always two unequal velocities unless the direction ρ of the ray coincide with some one of the four directions $\pm\rho'$, $\pm\rho''$, which are opposite two by two, and situated in the plane ac of the extreme axes of elasticity. FRESNEL has shown in like manner that any given normal direction corresponds to two unequal normal velocities, except four particular directions, which we may call $\pm\omega'$, $\pm\omega''$, and which are determined by the following cosines of direction,

$$\omega'_a = -\omega''_a = \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \omega'_b = \omega''_b = 0, \quad \omega'_c = \omega''_c = \sqrt{\frac{b^2-c^2}{a^2-c^2}} : \tag{Y^{18}}$$

and in fact it is easy to establish the following expression for the double value of the square of the normal velocity, analogous to the expression (W^{18}),

$$\omega^2 = \tfrac{1}{2}(a^2 + c^2) + \tfrac{1}{2}(a^2 - c^2)\cos.((\omega\omega') \pm (\omega\omega'')), \tag{Z^{18}}$$

which cannot reduce itself to a single value, unless the sine of $(\omega\omega')$ or of $(\omega\omega'')$ vanishes. FRESNEL has given the name of *optic axes* sometimes to the one and sometimes to the other of the two sets of directions (X^{18}) (Y^{18}); but to prevent the confusion which might arise from this double use of a term, we shall, for the present, call the set $\pm\rho'$, $\pm\rho''$, by the longer but more expressive name of the directions or *lines of single ray-velocity*: and similarly we shall call the set $\pm\omega'$, $\pm\omega''$, the directions or *lines of single normal velocity.*

New Properties of FRESNEL'S *Wave. This Wave has Four Conoidal Cusps, at the Ends of the Lines of Single Ray-Velocity; it has also Four Circles of Contact, of which each is contained on a Touching Plane of Single Normal-Velocity. The Lines of Single Ray-Velocity may therefore be called Cusp-Rays; and the Lines of Single Normal-Velocity may be called Normals of Circular Contact.*

28. The reasonings of the foregoing number suppose that the axes of co-ordinates coincide with the axes of elasticity; but it is easy to extend the result thus obtained, to any other axes of co-ordinates, by the formulæ of transformation which were given in the thirteenth number. We shall content ourselves at present with considering two remarkable transformations of this kind, suggested by the two foregoing sets of lines of single velocity, which conduct to some new properties of FRESNEL'S wave, and to some new consequences of his theory.

The polar equation (W^{18}) of the wave may be put under the form

$$1 = \tfrac{1}{2}(c^{-2} + a^{-2})\rho^2 + \tfrac{1}{2}(c^{-2} - a^{-2})\{r'r'' \pm \sqrt{\rho^2 - r'^2}\sqrt{\rho^2 - r''^2}\}, \tag{A^{19}}$$

if we put for abridgment

$$r' = A'\rho = x\rho'_a + z\rho'_c, \quad r'' = A''\rho = x\rho''_a + z\rho''_c, \tag{B^{19}}$$

so that r', r'', are the projections of the radius-vector ρ on the directions ρ', ρ'', of single ray-velocity; and if we take new rectangular co-ordinates, $x_\prime$ $y_\prime$ $z_\prime$, such that the plane of $x_\prime z_\prime$ is still the plane ac of the extreme axes of elasticity, but that the positive semiaxis of $z_\prime$ coincides with the line ρ', we may employ the following formulæ of transformation

$$x = x_\prime\rho'_c + z_\prime\rho'_a, \quad y = y_\prime, \quad z = -x_\prime\rho'_a + z_\prime\rho'_c, \tag{C^{19}}$$

which give

$$\rho^2 = x_\prime^2 + y_\prime^2 + z_\prime^2, \quad r' = z_\prime, \quad r'' = x_\prime \sin.(\rho'\rho'') + z_\prime \cos.(\rho'\rho''), \tag{D^{19}}$$

and change the equation (A^{19}) of the wave to the form

$$\begin{aligned} 1 = b^{-2}z_\prime^2 + \tfrac{1}{2}z_\prime x_\prime(c^{-2} - a^{-2})\sin.(\rho'\rho'') + \tfrac{1}{2}(c^{-2} + a^{-2})(x_\prime^2 + y_\prime^2) \\ \pm \tfrac{1}{2}(c^{-2} - a^{-2})\sqrt{x_\prime^2 + y_\prime^2}\sqrt{(z_\prime \sin.(\rho'\rho'') - x_\prime \cos.(\rho'\rho''))^2 + y_\prime^2}. \end{aligned} \tag{E^{19}}$$

This equation enables us easily to examine the shape of the wave near the end of the radius ρ', that is, near the point having for its new co-ordinates

$$x_\prime = 0, \quad y_\prime = 0, \quad z_\prime = b; \tag{F^{19}}$$

for it takes, near that point, the following approximate form,

$$z_\prime = b - \tfrac{1}{2}b^2\sqrt{c^{-2} - b^{-2}}\,\sqrt{b^{-2} - a^{-2}}\,(x_\prime \pm \sqrt{x_\prime^2 + y_\prime^2}), \tag{G^{19}}$$

which shows that at the point (F^{19}) *the wave has a conoidal cusp*, and is touched not by one determined tangent plane but by *a tangent cone of the second degree*, represented rigorously by the equation (G^{19}). FRESNEL does not appear to have been aware of the existence of this tangent cone to his wave; he seems to have thought that at the end of a radius ρ' of single ray-velocity the wave was touched only by two right lines, contained in the plane of ac, namely, by the tangents to a certain circle and ellipse, the intersections of the wave with that plane: but it is evident from the foregoing transformation that every other section of the wave, made by a plane containing the radius-vector ρ', is touched, at the end of that radius, by two tangent lines, contained on the cone (G^{19}). It is evident also that there are *four such conoidal cusps*, at the ends of the four lines of single ray-velocity, $\pm\rho'$, $\pm\rho''$. They are determined by the following co-ordinates, when referred to the axes of elasticity,

$$x = \pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad y = 0, \quad z = \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}; \tag{H^{19}}$$

and they are the four intersections of FRESNEL'S circle and ellipse, in the plane of ac, which have for their equations in that plane

$$x^2 + z^2 = b^2, \quad a^2x^2 + c^2z^2 = a^2c^2. \tag{I^{19}}$$

Again, if we employ the following new formulæ of transformation,

$$x = x_{\prime\prime}\omega'_c + z_{\prime\prime}\omega'_a, \quad y = y_{\prime\prime}, \quad z = -x_{\prime\prime}\omega'_a + z_{\prime\prime}\omega'_c, \tag{K^{19}}$$

so as to pass to a new system of rectangular co-ordinates such that the plane of $x_{\prime\prime}z_{\prime\prime}$ coincides with the plane of ac, and the positive semiaxis of $z_{\prime\prime}$ with the line ω' of single normal velocity, we find a new transformed equation of the wave, which may be thus written,

$$(x_{\prime\prime}^2 + y_{\prime\prime}^2 + x_{\prime\prime}z_{\prime\prime}b^{-2}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2})^2 = Q(1 - z_{\prime\prime}^2 b^{-2}), \tag{L^{19}}$$

if we put for abridgment

$$Q = (a^2 + c^2)\rho^2 + (a^2 - c^2)r'r'' - a^2c^2(1 + z_{\prime\prime}^2 b^{-2}); \tag{M^{19}}$$

and hence it is easy to prove that *the plane*

$$z_{\prime\prime} = b, \tag{N^{19}}$$

which is perpendicular to the line ω' at its extremity, touches the wave in the whole extent of a circle; the equation of this circle of contact being, in its own plane,

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + x_{\prime\prime}b^{-1}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2} = 0. \tag{O^{19}}$$

It is evident that there are *four such circles of plane contact at the ends of the four lines $\pm\omega'$, $\pm\omega''$, of single normal-velocity.* They are all equal to each other, and the common magnitude of their diameters is $b^{-1}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}$. The same conclusions may be drawn from FRESNEL'S equation of the wave in co-ordinates $x\ y\ z$, referred to the axes of elasticity: the equations of the *four planes of circular contact* being, in these co-ordinates,

$$z\sqrt{b^2 - c^2} \pm x\sqrt{a^2 - b^2} = \pm b\sqrt{a^2 - c^2}. \tag{P^{19}}$$

FRESNEL however does not himself appear to have suspected the existence of these circles of contact, nor do they seem to have been since perceived by any other person. We shall find that the circles and cusps, pointed out in the present number, conduct to some remarkable theoretical conclusions respecting the laws of refraction in biaxal crystals.

New Consequences of FRESNEL'S *Principles. It follows from those Principles, that Crystals of sufficient Biaxal Energy ought to exhibit two kinds of Conical Refraction, an External and an Internal: a Cusp-Ray giving an External Cone of Rays, and a Normal of Circular Contact being connected with an Internal Cone.*

29. The general formulæ for reflexion or refraction, ordinary or extraordinary, which we have deduced from the nature of the characteristic function V, become simply

$$\Delta\sigma = 0, \quad \Delta\tau = 0, \tag{Q^{19}}$$

when we take for the plane of xy the tangent plane to the reflecting or refracting surface; they show therefore that *the components of normal slowness parallel to this tangent plane are not changed*, which is a new and general form for the laws of reflexion and refraction. It is easy to combine this general theorem with FRESNEL'S law of velocity, and so to deduce new consequences from that law with respect to biaxal crystals.

For this deduction, our theorem may be expressed as follows,

$$0 = \Delta\left(a_t \frac{\delta v}{\delta\alpha} + b_t \frac{\delta v}{\delta\beta} + c_t \frac{\delta v}{\delta\gamma}\right), \tag{R^{19}}$$

in which v is the undulatory slowness of a ray considered as a homogeneous function of the first dimension of the cosines $\alpha\ \beta\ \gamma$ of its inclinations to any three rectangular semiaxes $a\ b\ c$, while Δ refers to the changes produced by reflexion or refraction, the unaltered trinomial to which it is prefixed being the component of normal slowness in the direction of any line t on the tangent plane of the reflecting or refracting surface, and $a_t\ b_t\ c_t$ being the cosines of the inclinations of this line to the semiaxes $a\ b\ c$: and in order to combine this theorem with the principles of FRESNEL, we have only to suppose that the rectangular semiaxes $a\ b\ c$ in each medium are the semiaxes of elasticity of that medium, and that the form of the function v is determined as in the twenty-seventh number.

Thus, to calculate the refraction of light on entering from a vacuum into a biaxal crystal $a\ b\ c$ bounded by a plane face F, we may denote by $\alpha_0\ \beta_0\ \gamma_0$ the cosines of the inclinations of the internal or incident ray to two rectangular lines s, t upon the face F, and to the inward normal, and we shall have the two equations following,

$$\left.\begin{aligned} \alpha_0 &= a_s \frac{\delta v}{\delta\alpha} + b_s \frac{\delta v}{\delta\beta} + c_s \frac{\delta v}{\delta\gamma} \ \ (= \sigma a_s + \tau b_s + \upsilon c_s), \\ \beta_0 &= a_t \frac{\delta v}{\delta\alpha} + b_t \frac{\delta v}{\delta\beta} + c_t \frac{\delta v}{\delta\gamma} \ \ (= \sigma a_t + \tau b_t + \upsilon c_t), \end{aligned}\right\} \tag{S^{19}}$$

which contain the required connexions between $\alpha_0\ \beta_0\ \gamma_0$ and $\alpha\ \beta\ \gamma$, that is, between the external and internal directions. In this manner we find in general two incident rays for one refracted, and two refracted for one incident; because a given system of values of $\alpha\ \beta\ \gamma$, that is, a given direction of the internal ray, corresponds in general to two systems of values of the internal components of normal slowness $\sigma\ \tau\ \upsilon$, and therefore to two systems of values $\alpha_0\ \beta_0\ \gamma_0$, that is, to two external directions; while, reciprocally, a given system of two linear relations between σ, τ, υ, deduced by (S^{19}) from a given external direction, corresponds in general to two directions of the internal ray. But there are two remarkable exceptions, connected with the two sets of lines of single velocity, and with the conoidal cusps and circles of contact on FRESNEL'S wave.

For we have seen that at a conoidal cusp the tangent plane to the wave is indeterminate; it is evident therefore that a *cusp-ray* must correspond to an infinite variety of systems of components of normal slowness σ, τ, υ, within the biaxal crystal, and therefore also to an infinite variety of systems of direction-cosines α_0 β_0 γ_0 of the external ray; so that *this one internal cusp-ray must correspond to an external cone of rays, according to a new theoretical law of light* which may be called EXTERNAL CONICAL REFRACTION.

And again, at a circle of contact, the wave has one common tangent plane for all the points of that circle, and therefore the infinite variety of internal rays which correspond to these different points have all one common wave-normal, which may be called a *normal of circular contact*, and all these internal rays have one common system of components of normal slowness σ τ υ within the crystal, and consequently correspond to one common external ray: so that *this one external ray is connected with an internal cone of rays, according to another new theoretical law of light*, which may be called INTERNAL CONICAL REFRACTION.

To develope, somewhat more fully, these two new consequences from FRESNEL'S principles, let us begin by considering *external conical refraction*: and let us seek the equation of the external cone of rays, corresponding to the internal cusp-ray ρ'. The approximate equation (G^{19}) of the wave, near the end of this cusp-ray, in the transformed co-ordinates $x_{\prime}$ $y_{\prime}$ $z_{\prime}$, gives the following approximate expression for the undulatory slowness v of a near ray, considered as a homogeneous function of the first dimension of the cosines $\alpha_{\prime}$ $\beta_{\prime}$ $\gamma_{\prime}$ of its inclinations to the positive semiaxes of these co-ordinates $x_{\prime}$ $y_{\prime}$ $z_{\prime}$,

$$v = b^{-1}\gamma_{\prime} + r_{\prime}(\alpha_{\prime} \pm \sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}), \tag{T^{19}}$$

in which

$$r_{\prime} = \tfrac{1}{2}b\sqrt{c^{-2} - b^{-2}}\sqrt{b^{-2} - a^{-2}}; \tag{U^{19}}$$

it gives therefore by our general method, the following components of normal slowness parallel to the same semiaxes of $x_{\prime}$ $y_{\prime}$ $z_{\prime}$,

$$\left.\begin{aligned} \sigma\rho'_c - \upsilon\rho'_a = \sigma_{\prime} = \frac{\delta v}{\delta\alpha_{\prime}} &= r_{\prime} \pm \frac{r_{\prime}\alpha_{\prime}}{\sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}}, \\ \tau = \tau_{\prime} = \frac{\delta v}{\delta\beta_{\prime}} &= \quad \pm \frac{r_{\prime}\beta_{\prime}}{\sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}}, \\ \sigma\rho'_a + \upsilon\rho'_c = \upsilon_{\prime} = \frac{\delta v}{\delta\gamma_{\prime}} &= b^{-1}, \end{aligned}\right\} \tag{V^{19}}$$

the expressions for $\sigma_{\prime}$ $\tau_{\prime}$ becoming indefinitely more accurate as $\alpha_{\prime}$ $\beta_{\prime}$ diminish, that is, as the near internal ray approaches to the cusp-ray ρ', and the expression for $\upsilon_{\prime}$ being rigorous: *the relations between the components of normal slowness σ τ υ of the cusp-ray ρ'* are therefore

$$(\sigma\rho'_c - \upsilon\rho'_a)^2 + \tau^2 = 2r_{\prime}(\sigma\rho'_c - \upsilon\rho'_a), \quad \sigma\rho'_a + \upsilon\rho'_c = b^{-1}, \tag{W^{19}}$$

and *the equation (in $\alpha_0\ \beta_0$) of the external cone of rays corresponding to the one internal cusp-ray ρ' is to be found by eliminating these three internal components $\sigma\ \tau\ \upsilon$ between the two relations* (W^{19}) *and the two equations of refraction* (S^{19}).

For example, if the internal cusp-ray ρ' coincide with the inward normal to the refracting face F of the crystal, we may take, for the semiaxes s, t upon that face, the projection of a, and the semiaxis b of elasticity; and then the equations of refraction (S^{19}) becoming

$$\alpha_0 = \sigma\rho'_c - \upsilon\rho'_a, \quad \beta_0 = \tau, \tag{X^{19}}$$

we have, by (W^{19}), the following polar equation of the external cone of rays,

$$\alpha_0^2 + \beta_0^2 = 2r_{\prime}\alpha_0; \tag{Y^{19}}$$

or, in rectangular co-ordinates, an equation of the fourth degree,

$$(x_0^2 + y_0^2)^2 = 4r_{\prime}^2 x_0^2(x_0^2 + y_0^2 + z_0^2). \tag{Z^{19}}$$

This cone is nearly circular in all the known biaxal crystals, because the coefficient $r_{\prime}$ is small, by (U^{19}), when the biaxal energy is weak, that is, when the semiaxes of elasticity $a\ b\ c$ are nearly equal to each other: and rigorously the external cone (Z^{19}) meets the concentric sphere of radius unity in a curve contained on a circular cylinder of radius $= r_{\prime}$, one side of this cylinder coinciding with a ray of the cone.

With respect to the *internal conical refraction*, the *equation of the internal cone of rays corresponding to the internal wave-normal* ω', or *normal of circular contact*, is always, by (N^{19}) (O^{19}),

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + 2r_{\prime\prime}x_{\prime\prime}z_{\prime\prime} = 0, \text{ if } r_{\prime\prime} = \tfrac{1}{2}b^{-2}\sqrt{a^2-b^2}\sqrt{b^2-c^2}, \tag{A^{20}}$$

when referred to the rectangular co-ordinates $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}$ by the transformation (K^{19}); and in the simpler rectangular co-ordinates $x\ y\ z$ which are parallel to the axes of elasticity the equation of this cone is

$$(x\omega'_c - z\omega'_a)^2 + y^2 + 2r_{\prime\prime}(x\omega'_c - z\omega'_a)(x\omega'_a + z\omega'_c) = 0, \tag{B^{20}}$$

in which we may change the co-ordinates $x\ y\ z$ to the direction-cosines $\alpha\ \beta\ \gamma$ of an internal ray of the cone: while the one external ray corresponding is determined by the following direction-cosines

$$\alpha_0 = b^{-1}\omega'_s, \quad \beta_0 = b^{-1}\omega'_t; \tag{C^{20}}$$

or by the ordinary law of proportional sines, since the internal wave-normal of circular contact ω', which is one ray of the internal cone, is connected with the external ray by this ordinary law, if we take as the refracting index of the crystal the reciprocal b^{-1} of the mean semiaxis of elasticity. It is evident hence that if the internal cone emerge at a new plane face, it will *emerge a cylinder*, whether the two faces be parallel or inclined, that is, whether the crystal be a plate or a prism.

Theory of Conical Polarisation. Lines of Vibration. These Lines, on FRESNEL'S *Wave, are the Intersections of Two Series of Concentric and Co-axal Ellipsoids.*

30. A given direction of a wave-normal in a biaxal crystal corresponds in general to two directions of vibration, and therefore to two planes of polarisation, determined by the equations (F^{18}), namely one for each of the two values ω_1^2, ω_2^2 of the square of the normal velocity deduced by (G^{18}) from the given system of ratios of σ, τ, υ; and these two directions of vibration, or the two planes of polarisation, that is, the two normal planes of the wave perpendicular to these vibrations, are perpendicular to each other, since we can easily deduce from (G^{18}) the following relation between ω_1^2, ω_2^2,

$$\frac{\sigma^2}{(\omega_1^2 - a^2)(\omega_2^2 - a^2)} + \frac{\tau^2}{(\omega_1^2 - b^2)(\omega_2^2 - b^2)} + \frac{\upsilon^2}{(\omega_1^2 - c^2)(\omega_2^2 - c^2)} = 0 : \qquad (D^{20})$$

which general rectangularity of the two vibrations on any one plane wave has been otherwise established by FRESNEL, and is an important result of his theory. But besides this general *double polarisation* connected with the general *double refraction* in biaxal crystals, we may consider two other kinds which may be called *conical polarisation*, connected with the two kinds of *conical refraction*, which were pointed out in the foregoing number.

To examine the law of the conical polarisation connected with the internal conical refraction, and therefore with the planes of circular contact, we may employ the co-ordinates $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}$ defined by (K^{19}), and thus transform the general equations of polarisation (A^{18}) (F^{18}) into the following equally general,

$$\left.\begin{array}{c} \dfrac{\omega_c'\,\delta x_{\prime\prime} + \omega_a'\,\delta z_{\prime\prime}}{\omega_c'\sigma_{\prime\prime} + \omega_a'\upsilon_{\prime\prime}}(\omega^2 - a^2) = \dfrac{\delta y_{\prime\prime}}{\tau_{\prime\prime}}(\omega^2 - b^2) = \dfrac{-\omega_a'\,\delta x_{\prime\prime} + \omega_c'\,\delta z_{\prime\prime}}{-\omega_a'\sigma_{\prime\prime} + \omega_c'\upsilon_{\prime\prime}}(\omega^2 - c^2), \\ \sigma_{\prime\prime}\,\delta x_{\prime\prime} + \tau_{\prime\prime}\,\delta y_{\prime\prime} + \upsilon_{\prime\prime}\,\delta z_{\prime\prime} = 0; \end{array}\right\} \qquad (E^{20})$$

which give, for the projection of a vibration on the plane $x_{\prime\prime}y_{\prime\prime}$ of single normal velocity, the rigorous formula

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}} = \frac{(\omega^2 - a^2)(\omega^2 - c^2)}{\omega^2 - b^2}\,\frac{\tau_{\prime\prime}}{\upsilon_{\prime\prime}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2} + \sigma_{\prime\prime}(\omega^2 + b^2 - a^2 - c^2)}, \qquad (F^{20})$$

and for any plane wave slightly inclined to this plane of $x_{\prime\prime}y_{\prime\prime}$ the following approximate relation between the components of normal slowness,

$$\upsilon_{\prime\prime} = b^{-1} + r_{\prime\prime}(\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}), \qquad (G^{20})$$

retaining the meaning (A^{20}) of $r_{\prime\prime}$; and if we attend to the general connexions, established in this Supplement, between the direction-cosines of a ray and the components of normal slowness of a wave, we easily deduce from (G^{20}), by differentiation, the following other relations,

$$\frac{\alpha_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta\upsilon_{\prime\prime}}{\delta\sigma_{\prime\prime}} = -r_{\prime\prime}\left(1 \pm \frac{\sigma_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}}\right), \quad \frac{\beta_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta\upsilon_{\prime\prime}}{\delta\tau_{\prime\prime}} = \frac{\mp r_{\prime\prime}\tau_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}}; \qquad (H^{20})$$

and finally for the vibrations of a near wave

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}} = \frac{\tau_{\prime\prime}}{\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}} = \frac{\beta_{\prime\prime}}{\alpha_{\prime\prime}}. \tag{I^{20}}$$

This formula contains the theory of the conical polarisation connected with internal conical refraction. It shows that *the vibrations at the circle of contact on* FRESNEL'S *wave are in the chords of that circle drawn from the extremity of the normal* ω' *of single velocity*; and therefore that *the corresponding planes of polarisation all pass through another parallel normal at the opposite point of the circle.* The plane of polarisation, therefore, in passing from one position to another, *revolves only half as rapidly* as the revolving radius, so that the angle between any two planes of polarisation is only *half* the angle between the two corresponding radii of this circle on FRESNEL'S wave. And if we suppose that the direction of the external incident ray coincides with the wave-normal ω', and therefore also with the normal to the refracting face of the crystal, then the small internal components of normal slowness $\sigma_{\prime\prime}$ $\tau_{\prime\prime}$, parallel to this refracting face, are equal (by our general theorem of refraction) to the small external direction-cosines α_0 β_0 of the inclinations of a near incident ray to the semiaxes of $x_{\prime\prime}$ and $y_{\prime\prime}$; from which it follows, by (I^{20}) that *the plane of external incidence containing this near incident ray revolves twice as rapidly as the corresponding plane of refraction.*

For the other kind of conical polarisation, connected with the external conical refraction, and therefore with the conoidal cusps on FRESNEL'S wave, we find by a similar process,

$$\frac{\delta y_{\prime}}{\delta x_{\prime}} = \frac{\tau_{\prime}}{\sigma_{\prime}} = \frac{\beta_{\prime}}{\alpha_{\prime} \pm \sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}}, \tag{K^{20}}$$

and

$$\delta z_{\prime} = -2br_{\prime}\,\delta x_{\prime}, \tag{L^{20}}$$

$r_{\prime}$ having the meaning (U^{19}). The formula (K^{20}) shows that the normal plane to the wave, containing any vibration near the cusp, contains either the cusp-ray itself, or a line parallel to this ray; so that the direction of any near vibration coincides with or is parallel to the projection of the cusp-ray on the corresponding tangent plane of the wave, or of the cone which touches it at the cusp: and the formula (L^{20}) shows that all these near vibrations are parallel to one common plane, which is easily seen to be perpendicular to the plane of ac, and to contain the tangent at the cusp to the elliptic section (I^{19}) of the wave, made by this latter plane; so that *all the planes of polarisation near the cusp, contain, or are parallel to, the normal of this elliptic section.* And the direction of any near vibration on the wave, or on its tangent cone, may be obtained by cutting the corresponding tangent plane of this wave or cone by a plane perpendicular to this elliptic normal.

If the cusp-ray be incident perpendicularly on a refracting face of the crystal, then the internal components $\sigma_{\prime}$ $\tau_{\prime}$ are equal to the direction-cosines α_0 β_0 of the corresponding ray of the emerging external cone; and therefore, by (K^{20}), the plane of refraction of this external ray contains the internal vibration, and therefore also, by FRESNEL'S principles, the external vibration corresponding: so that, *in the external conical polarisation, produced by the perpendicular internal incidence of a cusp-ray, the plane of polarisation of an external ray is perpendicular to its plane of refraction; and therefore revolves about half as rapidly as* *the plane containing this emergent ray and passing through the approximate axis of the nearly circular emergent cone, when the biaxal energy is small.* We see also, by (K^{20}), that the plane containing the cusp-ray and containing or parallel to a near internal ray, revolves with double the rapidity of the plane containing the cusp-ray and parallel to the near wave-normal; and therefore, in the case of perpendicular incidence of the cusp-ray, the plane of incidence of a near internal ray revolves with double the rapidity of the plane of external refraction, which, as we have seen, contains here the external vibrations.

In general, the equations of polarisation (F^{18}), which we have deduced from Fresnel's principles, conduct, by (I^{18}) (L^{18}), to the following simple formula

$$a^2\alpha\,\delta x + b^2\beta\,\delta y + c^2\gamma\,\delta z = 0, \tag{M^{20}}$$

δx, δy, δz being still the components of displacement parallel to the semiaxes a, b, c, and α, β, γ being still the cosines of the inclinations of the ray to the same semiaxes of elasticity: and this formula (M^{20}), when combined with the equation of transversal vibrations,

$$\delta V = 0, \quad \text{or,} \quad \sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z = 0, \tag{A^{18}}$$

determines easily the direction of vibration for any given direction and velocity of a ray, that is, for any point of Fresnel's curved wave propagated from a luminous origin within a biaxal crystal. And we easily see that *on any wave in a biaxal crystal*, whether propagated from within or from without, the differential equation (M^{20}) determines *a series of lines of vibration*, having the property that at any point of such a line the vibration is in the direction of the line itself. To find these lines on Fresnel's wave (O^{18}), we may change $\alpha\ \beta\ \gamma$ to $x\ y\ z$ in the differential equation (M^{20}), and we then find, by integration,

$$a^2x^2 + b^2y^2 + c^2z^2 = \varepsilon^4, \tag{N^{20}}$$

ε being an arbitrary constant; and since this integral, when combined with the equation (O^{18}) of the wave itself, gives

$$(a^4+\varepsilon^4)x^2 + (b^4+\varepsilon^4)y^2 + (c^4+\varepsilon^4)z^2 = (a^2+b^2+c^2)\varepsilon^4 - a^2b^2c^2, \tag{O^{20}}$$

we see that *the lines of vibration on* Fresnel's *wave, propagated from a point in a biaxal crystal, are the intersections of two series* (N^{20}) (O^{20}) *of concentric and co-axal ellipsoids.*

By this general integration, extending to the whole wave, or by integrating the approximate equations for vibrations near the conoidal cusps and circles of contact, obtained from (K^{20}) (I^{20}) by changing the direction-cosines of a ray to the proportional co-ordinates of the wave, we find that near a cusp the lines of vibration coincide nearly with small parabolic arcs on the tangent cone of the wave, in planes perpendicular to the elliptic normal already mentioned; and that in crossing a circle of contact the course of each line of vibration is directed towards that point of the circle which is the end of the corresponding wave-normal of single velocity, that is, towards the foot of the perpendicular let fall from the centre of the wave on the plane of circular contact.

In any Uniform Medium, the Curved Wave propagated from a point is connected with a certain other surface, which may be called the surface of components, by relations discovered by M. CAUCHY, *and by some new relations connected with a General Theorem of Reciprocity. This new Theorem of Reciprocity gives a new construction for the Wave, in any Undulatory Theory of Light: and it connects the Cusps and Circles of Contact on* FRESNEL'S *Wave, with Circles and Cusps of the same kind on the Surface of Components.*

31. The theory of the wave propagated from a point in any uniform medium may be much illustrated by comparing this wave with a certain other surface which appears to have been first discovered by M. CAUCHY, who has pointed out some of its properties in the *Livraison* already referred to. In that *Livraison*, M. CAUCHY has treated of the propagation of plane waves in a system of mutually attracting or repelling particles; and has been conducted to a relation between the normal velocity of propagation, which he calls s, and the cosines of its inclinations to the positive semiaxes of x, y, z, which cosines he denotes by a, b, c. The relation thus found being expressed by equating to zero a certain homogeneous function (of the sixth dimension) of s, a, b, c, it has suggested to M. CAUCHY the consideration of s as a homogeneous function of the first dimension of the cosines a, b, c, whereas we have preferred to treat the normal velocity (denoted in this Supplement by ω) as a homogeneous function of its cosines of direction of the dimension zero; a difference in method which makes no real difference in the results, because the relation existing between the cosines (namely, that the sum of their squares is unity,) permits us to transform in an infinite variety of ways any equation into which they enter. M. CAUCHY deduces from his view of the relation between the normal velocity and cosines of normal direction, the following equations between the time t and the co-ordinates x y z of a ray from the origin of co-ordinates,

$$\frac{x}{t} = \frac{ds}{da}, \quad \frac{y}{t} = \frac{ds}{db}, \quad \frac{z}{t} = \frac{ds}{dc},$$

which were alluded to in the twenty-sixth number of the present Supplement, as substantially equivalent to our equations (D^{18}). He deduces also an equation of the form

$$F\left(\frac{a}{s}, \frac{b}{s}, \frac{c}{s}\right) = 0,$$

which he constructs by a surface having $\frac{a}{s}$, $\frac{b}{s}$, $\frac{c}{s}$, for its co-ordinates. Our methods suggest immediately the same surface, as the construction of the same equation under the form

$$\Omega(\sigma, \tau, \upsilon) = 0,$$

which has been so frequently employed in this Supplement; and from the optical meanings that we have pointed out for the co-ordinates σ, τ, υ, of this surface $\Omega = 0$, we shall call it the surface of components of normal slowness, or simply *the surface of components*. M. CAUCHY shows that this surface is connected with the curved wave propagated from the origin of co-ordinates in the unit of time, (which we have called the *unit-wave* and may denote by the equation

$$V = 1,)$$

by two remarkable relations, which can easily be deduced from our formulæ, and may be thus enunciated: first, *the sum of the products of their corresponding co-ordinates*, or, in other words, *the product of any two corresponding radii multiplied by the cosine of the included angle, is unity*; and secondly, *the wave is the enveloppe of the planes which cut perpendicularly the radii of the surface of components at distances from the centre equal to the reciprocals of those radii.*

To these two relations, discovered by M. Cauchy, we may add a third, not less remarkable, which he does not seem to have perceived: namely, that *the surface of components is the enveloppe of the planes which cut perpendicularly the radii of the wave at distances from its centre equal to the reciprocals of those radii*, that is, equal to the slowness of the rays. For it is a general theorem of reciprocity between surfaces, which can easily be deduced from the evident coexistence of the three equations

$$\left.\begin{aligned} xx' + yy' + zz' &= 1,\\ x\,\delta x' + y\,\delta y' + z\,\delta z' &= 0,\\ x'\,\delta x + y'\,\delta y + z'\,\delta z &= 0, \end{aligned}\right\} \qquad (\mathrm{P}^{20})$$

that *if one surface B be deduced from another A by drawing radii vectores to the latter from an arbitrary origin O, and altering the lengths of these radii to their reciprocals without changing their directions, and seeking the enveloppe B of the planes perpendicular at the extremities to these altered radii of A, then reciprocally, the surface A may be deduced from B by a repetition of the same construction*, employing the same origin O, and the same arbitrary unit of length. For example, if the surface A be formed by the revolution of an ellipse about its major axis, and if we place the arbitrary origin O at one focus of this ellipsoid A, and take the arbitrary unit equal to the semiaxis minor, the enveloped surface B will be a sphere, having its diameter equal to the axis major of the ellipsoid, and its centre on that axis major, the interval between the centres of the two surfaces being bisected by the origin O; and if from this excentric origin we draw radii to the sphere B, and change these unequal radii to their reciprocals, and draw perpendicular planes at the extremities of these new radii, the enveloppe of the planes so drawn will be the ellipsoid A. Another particular case of this general theory of *reciprocal surfaces*, namely the case of two concentric and co-axal ellipsoids, referred to their centre as origin, and having the semiaxes of one equal to the reciprocals of those of the other, has been perceived by Mr. Mac Cullagh, and elegantly proved by him, in the Second Part of the Sixteenth Volume of the *Transactions of the Royal Irish Academy.*

This general theorem of reciprocity, when applied to the unit-wave and surface of components, gives *a new construction for the unit-wave in any uniform medium, and for any law of velocity*: namely, that *the wave is the locus of the points obtained by letting fall perpendiculars from the centre on the tangent planes of the surface of components, and then altering the lengths of these perpendiculars to their reciprocals, without altering their directions.*

It follows also from this general theory of *reciprocal surfaces*, that *a conoidal cusp on any surface A corresponds in general to a curve of plane contact on the reciprocal surface B*, and reciprocally; and, accordingly the cusps and circles on Fresnel's wave are connected with circles and cusps on the corresponding surface of components, which latter surface is indeed deducible from the former by merely changing the semiaxes of elasticity $a\ b\ c$ to their reciprocals. And it was in fact by this general theorem that I was led to discover the four circles of contact on FRESNEL'S wave, by concluding that this wave must touch four planes in curves instead of points of contact, as soon as I had perceived the existence of four conoidal cusps on the surface of components, by obtaining (in some investigations respecting the aberrations of biaxal lenses) the formula (G^{20}), which is the approximate equation of such a cusp. I easily found also that there were *only four* such cusps on each of the two reciprocal surfaces, and therefore concluded that there were *only four* curves of plane contact on each. I may mention that though I have taken care to attribute to M. CAUCHY the discovery of the surface of components, yet before I met the *Exercices de Mathématiques*, I was familiar, in my own investigations, with the existence and with the foregoing properties of this surface: it is indeed immediately suggested by the first principles of my view of optics, since it constructs the fundamental partial differential equation

$$\Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}\right) = 0$$

which my characteristic function V must satisfy in a final uniform medium.

The surface of components possesses many other interesting properties, for example the following, that in a final uniform medium any two conjugate planes of vergency (E^{15}) are perpendicular to two conjugate tangents on it: which is analogous to the less simple relations considered in the twenty-first number. But the length to which this Supplement has extended, confines me here to remarking, that the general equations of reflexion or refraction,

$$\Delta\sigma = 0, \quad \Delta\tau = 0, \qquad (\mathrm{Q}^{19})$$

may be thus enunciated; *the corresponding points* (σ, τ, υ, and $\sigma + \Delta\sigma$, $\tau + \Delta\tau$, $\upsilon + \Delta\upsilon$) *upon the surface or surfaces of components* ($0 = \Omega$, $0 = \Omega + \Delta\Omega$,) *before and after any reflexion or refraction, ordinary or extraordinary, are situated on one common perpendicular to the plane which touches the reflecting or refracting surface at the point of reflexion or refraction*; a new geometrical relation, which gives a new and general construction to determine a reflected or refracted ray, simpler in many cases than the construction proposed by HUYGHENS.