ON A THEOREM RESPECTING ELLIPSOIDS, OBTAINED BY THE METHOD OF QUATERNIONS

By

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By Sir William R. Hamilton.

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The following theorem respecting ellipsoids, obtained by the method of quaternions, was communicated by Sir William Rowan Hamilton, in a note to the Secretary of Council:

"On the mean axis of a given ellipsoid, as the major axis, describe an ellipsoid of revolution, of which the equatorial circle shall be touched by those tangents to the principal section of the given ellipsoid (in the plane of the focal hyperbola), which are parallel to the umbilicar diameters. In this equatorial circle, and in every smaller and parallel circle of the new ellipsoid thus constructed, conceive that indefinitely many quadrilaterals are inscribed, for each of which one pair of opposite sides shall be parallel to the given umbilicar diameters, while the other pair of opposite sides shall be parallel to the asymptotes of the focal hyperbola. Then *the intersection of the first pair of opposite sides of the inscribed quadrilateral will be a point on the surface of the given ellipsoid.*

"I may remark that the distance of either focus of the new ellipsoid from the common centre of the new and old ellipsoids, will be equal to the perpendicular let fall from either of the two points, which were called T and V in a recent note and diagram, on the umbilicar semidiameter AU, or on that semidiameter prolonged; while the distance of the umbilic U from the foot of either of these two perpendiculars, that is, the projection of either of the two equal tangents to the focal hyperbola, TU, UV, on the umbilicar semidiameter AU, or on that semidiameter prolonged, will be the minor semiaxis, or the radius of the equator, of the new ellipsoid (of revolution).

"This new ellipsoid *touches* the old one at the ends of the given mean axis; but it also *cuts* the same old or given ellipsoid, in a system of two ellipses, contained in planes perpendicular to the asymptotes of the focal hyperbola.

"If the semiaxes of the given ellipsoid be a, b, c, the common distance of the two *foci* of the new or derived ellipsoid (of revolution) from the common centre of the two ellipsoids, is expressed by the formula

$$e = \frac{\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}}{\sqrt{(a^2 - b^2 + c^2)}}. \tag{1}$$

"And I venture, although with diffidence, to propose the name of the TWO MEDIAL FOCI, for the two points thus determined on the mean axis $2b$ of the ellipsoid a, b, c. If their vectors be denoted by $\pm\epsilon$, the equation of that original ellipsoid may be thus written:

$$\mathrm{T}(\lambda_1 + \epsilon) + \mathrm{T}(\lambda_1 - \epsilon) = 2b; \tag{2}$$

or thus,

$$\mathrm{T}(\lambda_1 - \epsilon) = b + b^{-1}\,\mathrm{S}\,.\,\epsilon\lambda_1; \tag{3}$$

where

$$\lambda_1 = \frac{\rho\eta + \theta\rho}{\eta + \theta}; \quad \epsilon = \frac{2\mathrm{V}\,.\,\eta\theta}{\mathrm{T}(\eta + \theta)}; \tag{4}$$

η, θ, ρ, having the same signification as in notes recently read; while e may perhaps be called the MEDIAL ECCENTRICITY of the ellipsoid $a\,b\,c$.

"In a future communication I may be induced to return on the quaternion analysis employed, and to submit to the Academy some account of it."