A THEOREM CONCERNING POLYGONIC SYNGRAPHY

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By Sir William R. Hamilton.

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Professor Sir William Rowan Hamilton exhibited the following Theorem, to which he had been conducted by that theory of geometrical *syngraphy* of which he had lately submitted to the Academy a verbal and hitherto unreported sketch, and on which he hopes to return in a future communication.

Theorem. Let $A_1, A_2, \ldots A_n$ be any n points (in number odd or even) assumed at pleasure on the n successive sides of a closed polygon $BB_1B_2 \ldots B_{n-1}$ (plane or gauche), inscribed in any given surface of the second order. Take any three points, P, Q, R, on that surface, as initial points, and draw from each a system of n successive chords, passing in order through the n assumed points (A), and terminating in three other superficial and final points, P', Q', R'. Then there will be (in general) *another* inscribed and closed polygon, $CC_1C_2 \ldots C_{n-1}$, of which the n sides shall pass successively, in the same order, through the same n points (A); and of which the initial point C shall also be connected with the point B of the former polygon, by the relations

$$\frac{ael}{bc}\frac{\beta\gamma}{\alpha\epsilon\lambda} = \frac{a'e'l'}{b'c'}\frac{\beta'\gamma'}{\alpha'\epsilon'\lambda'}, \quad \frac{bfm}{ca}\frac{\gamma\alpha}{\beta\zeta\mu} = \frac{b'f'm'}{c'a'}\frac{\gamma'\alpha'}{\beta'\zeta'\mu'}, \quad \frac{cgn}{ab}\frac{\alpha\beta}{\gamma\eta\nu} = \frac{c'g'n'}{a'b'}\frac{\alpha'\beta'}{\gamma'\eta'\nu'};$$

where

$$\begin{aligned} a &= QR, & b &= RP, & c &= PQ, \\ e &= BP, & f &= BQ, & g &= BR, \\ l &= CP, & m &= CQ, & n &= CR, \\ a' &= Q'R', & b' &= R'P', & c' &= P'Q', \\ e' &= BP', & f' &= BQ', & g' &= BR', \\ l' &= CP', & m' &= CQ', & n' &= CR'; \end{aligned}$$

while $\alpha\,\beta\,\gamma\,\epsilon\,\zeta\,\eta\,\lambda\,\mu\,\nu$, and $\alpha'\,\beta'\,\gamma'\,\epsilon'\,\zeta'\,\eta'\,\lambda'\,\mu'\,\nu'$, denote the semidiameters of the surface, respectively parallel to the chords $a\,b\,c\,e\,f\,g\,l\,m\,n$, $a'\,b'\,c'\,e'\,f'\,g'\,l'\,m'\,n'$.

As a very particular *case* of this theorem, we may suppose that $PQ'RP'QR'$ is a plane hexagon in a conic, and BC its Pascal's line.