SECOND ESSAY ON A GENERAL METHOD IN DYNAMICS

By

William Rowan Hamilton

(Philosophical Transactions of the Royal Society, part I for 1835, pp. 95–144.)

Edited by David R. Wilkins

2000

NOTE ON THE TEXT

This edition is based on the original publication in the *Philosophical Transactions of the Royal Society*, part I for 1835.

The following errors in the original published text have been corrected:

a term dt, missing from the integral (R^1.) has been inserted;

a factor t^2 has been inserted in the integrand in the second integral of equation (160.).

The *Second Essay on a General Method in Dynamics* has also been republished in *The Mathematical Papers of Sir William Rowan Hamilton, Volume II: Dynamics*, edited for the Royal Irish Academy by A. W. Conway and A. J. McConnell, and published by Cambridge University Press in 1940.

David R. Wilkins

Dublin, February 2000

Second Essay on a General Method in Dynamics. By William Rowan Hamilton*, Member of several Scientific Societies in Great Britain and in Foreign Countries, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland. Communicated by Captain* Beaufort*, R.N. F.R.S.*

Received October 29, 1834,—Read January 15, 1835.

[*Philosophical Transactions of the Royal Society*, part I for 1835, pp. 95–144.]

Introductory Remarks.

The former Essay* contained a general method for reducing all the most important problems of dynamics to the study of one characteristic function, one central or radical relation. It was remarked at the close of that Essay, that many eliminations required by this method in its first conception, might be avoided by a general transformation, introducing the time explicitly into a part S of the whole characteristic function V; and it is now proposed to fix the attention chiefly on this part S, and to call it the *Principal Function.* The properties of this part or function S, which were noticed briefly in the former Essay, are now more fully set forth; and especially its uses in questions of perturbation, in which it dispenses with many laborious and circuitous processes, and enables us to express accurately the disturbed configuration of a system by the rules of undisturbed motion, if only the initial components of velocities be changed in a suitable manner. Another manner of extending rigorously to disturbed motion the rules of undisturbed, by the gradual variation of elements, in number double the number of coordinates or other marks of position of the system, which was first invented by Lagrange, and was afterwards improved by Poisson, is considered in this Second Essay under a form perhaps a little more general; and the general method of calculation which has already been applied to other analogous questions in optics and in dynamics by the author of the present Essay, is now applied to the integration of the equations which determine these elements. This general method is founded chiefly on a combination of the principles of variations with those of partial differentials, and may furnish, when it shall be matured by the labours of other analysts, a separate branch of algebra, which may be called perhaps the *Calculus of Principal Functions*; because, in all the chief applications of algebra to physics, and in a very extensive class of purely mathematical questions, it reduces the determination of many mutually connected functions to the search and study of one principal or central relation. When applied to the integration of the equations of varying elements, it suggests, as is now shown, the consideration of a certain *Function of Elements*, which may be variously chosen, and may either be rigorously determined, or at least approached to, with an indefinite accuracy, by a corollary of the general method. And to illustrate all these new

* Philosophical Transactions for the year 1834, Second Part.

general processes, but especially those which are connected with problems of perturbation, they are applied in this Essay to a very simple example, suggested by the motions of projectiles, the parabolic path being treated as the undisturbed. As a more important example, the problem of determining the motions of a ternary or multiple system, with any laws of attraction or repulsion, and with one predominant mass, which was touched on in the former Essay, is here resumed in a new way, by forming and integrating the differential equations of a new set of varying elements, entirely distinct in theory (though little differing in practice) from the elements conceived by LAGRANGE, and having this advantage, that the differentials of all the new elements for *both* the disturbed and disturbing masses may be expressed by the coefficients of *one* disturbing function.

Transformations of the Differential Equations of Motion of an Attracting or Repelling System.

1. It is well known to mathematicians, that the differential equations of motion of any system of free points, attracting or repelling one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula:

$$\Sigma\,.m(x''\delta x + y''\delta y + z''\delta z) = \delta U : \tag{1.}$$

the sign of summation Σ extending to all the points of the system; m being, for any one such point, the constant called its mass, and $x\ y\ z$ being its rectangular coordinates; while $x''\ y''\ z''$ are the accelerations, or second differential coefficients taken with respect to the time, and δx, δy, δz are any arbitrary infinitesimal variations of those coordinates, and U is a certain *force-function*, introduced into dynamics by LAGRANGE, and involving the masses and mutual distances of the several points of the system. If the number of those points be n, the formula (1.) may be decomposed into $3n$ ordinary differential equations of the second order, between the coordinates and the time,

$$m_i x_i'' = \frac{\partial U}{\partial x_i}; \quad m_i y_i'' = \frac{\partial U}{\partial y_i}; \quad m_i z_i'' = \frac{\partial U}{\partial z_i} : \tag{2.}$$

and to integrate these differential equations of motion of an attracting or repelling system, or some transformations of these, is the chief and perhaps ultimately the only problem of mathematical dynamics.

2. To facilitate and generalize the solution of this problem, it is useful to express previously the $3n$ rectangular coordinates $x\ y\ z$ as functions of $3n$ other and more general marks of position $\eta_1\,\eta_2\,\ldots\,\eta_{3n}$; and then the differential equations of motion take this more general form, discovered by LAGRANGE,

$$\frac{d}{dt}\frac{\delta T}{\delta \eta_i'} - \frac{\delta T}{\delta \eta_i} = \frac{\delta U}{\delta \eta_i}, \tag{3.}$$

in which

$$T = \tfrac{1}{2}\,\Sigma\,.m(x'^2 + y'^2 + z'^2). \tag{4.}$$

For, from the equations (2.) or (1.),

$$\left.\begin{aligned}
\frac{\delta U}{\delta \eta_i} &= \Sigma\,.m\left(x''\frac{\delta x}{\delta \eta_i} + y''\frac{\delta y}{\delta \eta_i} + z''\frac{\delta z}{\delta \eta_i}\right)\\
&= \frac{d}{dt}\,\Sigma\,.m\left(x'\frac{\delta x}{\delta \eta_i} + y'\frac{\delta y}{\delta \eta_i} + z'\frac{\delta z}{\delta \eta_i}\right)\\
&\quad - \Sigma\,.m\left(x'\frac{d}{dt}\frac{\delta x}{\delta \eta_i} + y'\frac{d}{dt}\frac{\delta y}{\delta \eta_i} + z'\frac{d}{dt}\frac{\delta z}{\delta \eta_i}\right);
\end{aligned}\right\} \tag{5.}$$

in which

$$\left.\begin{aligned}
&\Sigma\,.m\left(x'\frac{\delta x}{\delta \eta_i} + y'\frac{\delta y}{\delta \eta_i} + z'\frac{\delta z}{\delta \eta_i}\right)\\
&= \Sigma\,.m\left(x'\frac{\delta x'}{\delta \eta_i'} + y'\frac{\delta y'}{\delta \eta_i'} + z'\frac{\delta z'}{\delta \eta_i'}\right) = \frac{\delta T}{\delta \eta_i'},
\end{aligned}\right\} \tag{6.}$$

and

$$\left.\begin{aligned}&\Sigma\,.m\left(x'\frac{d}{dt}\frac{\delta x}{\delta\eta_i}+y'\frac{d}{dt}\frac{\delta y}{\delta\eta_i}+z'\frac{d}{dt}\frac{\delta z}{\delta\eta_i}\right)\\ &=\Sigma\,.m\left(x'\frac{\delta x'}{\delta\eta_i}+y'\frac{\delta y'}{\delta\eta_i}+z'\frac{\delta z'}{\delta\eta_i}\right)=\frac{\delta T}{\delta\eta_i},\end{aligned}\right\}\tag{7.}$$

T being here considered as a function of the $6n$ quantities of the forms η' and η, obtained by introducing into its definition (4.), the values

$$x'=\eta'_1\frac{\delta x}{\delta\eta_1}+\eta'_2\frac{\delta x}{\delta\eta_2}+\cdots+\eta'_{3n}\frac{\delta x}{\delta\eta_{3n}},\ \text{\&c.}\tag{8.}$$

A different proof of this important transformation (3.) is given in the Mécanique Analytique.

3. The function T, being homogeneous of the second dimension with respect to the quantities η', must satisfy the condition

$$2T=\Sigma\,.\eta'\frac{\delta T}{\delta\eta'};\tag{9.}$$

and since the variation of the same function T may evidently be expressed as follows,

$$\delta T=\Sigma\left(\frac{\delta T}{\delta\eta'}\delta\eta'+\frac{\delta T}{\delta\eta}\delta\eta\right),\tag{10.}$$

we see that this variation may be expressed in this other way,

$$\delta T=\Sigma\left(\eta'\delta\frac{\delta T}{\delta\eta'}-\frac{\delta T}{\delta\eta}\delta\eta\right).\tag{11.}$$

If then we put, for abridgement,

$$\frac{\delta T}{\delta\eta'_1}=\varpi_1,\quad\ldots\quad\frac{\delta T}{\delta\eta'_{3n}}=\varpi_{3n},\tag{12.}$$

and consider T (as we may) as a function of the following form,

$$T=F(\varpi_1,\varpi_2,\ldots\varpi_{3n},\eta_1,\eta_2,\ldots\eta_{3n}),\tag{13.}$$

we see that

$$\frac{\delta F}{\delta\varpi_1}=\eta'_1,\quad\ldots\quad\frac{\delta F}{\delta\varpi_{3n}}=\eta'_{3n},\tag{14.}$$

and

$$\frac{\delta F}{\delta\eta_1}=-\frac{\delta T}{\delta\eta_1},\quad\ldots\quad\frac{\delta F}{\delta\eta_{3n}}=-\frac{\delta T}{\delta\eta_{3n}};\tag{15.}$$

and therefore that the general equation (3.) may receive this new transformation,

$$\frac{d\varpi_i}{dt} = \frac{\delta(U - F)}{\delta\eta_i}. \tag{16.}$$

If then we introduce, for abridgement, the following expression H,

$$H = F - U = F(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U(\eta_1, \eta_2, \dots \eta_{3n}), \tag{17.}$$

we are conducted to this new manner of presenting the differential equations of motion of a system of n points, attracting or repelling one another:

$$\left.\begin{array}{ll} \dfrac{d\eta_1}{dt} = \dfrac{\delta H}{\delta\varpi_1}; & \dfrac{d\varpi_1}{dt} = -\dfrac{\delta H}{\delta\eta_1}; \\ \dfrac{d\eta_2}{dt} = \dfrac{\delta H}{\delta\varpi_2}; & \dfrac{d\varpi_2}{dt} = -\dfrac{\delta H}{\delta\eta_2}; \\ \cdots\cdots\cdots & \\ \dfrac{d\eta_{3n}}{dt} = \dfrac{\delta H}{\delta\varpi_{3n}}; & \dfrac{d\varpi_{3n}}{dt} = -\dfrac{\delta H}{\delta\eta_{3n}}. \end{array}\right\} \tag{A.}$$

In this view, the problem of mathematical dynamics, for a system of n points, is to integrate a system (A.) of $6n$ ordinary differential equations of the first order, between the $6n$ variables η_i ϖ_i and the time t; and the solution of the problem must consist in assigning these $6n$ variables as functions of the time, and of their own initial values, which we may call e_i p_i. And all these $6n$ functions, or $6n$ relations to determine them, may be expressed, with perfect generality and rigour, by the method of the former Essay, or by the following simplified process.

Integration of the Equations of Motion, by means of one Principal Function.

4. If we take the variation of the definite integral

$$S = \int_0^t \left(\Sigma\,.\,\varpi \frac{\delta H}{\delta\varpi} - H \right) dt \tag{18.}$$

without varying t or dt, we find, by the Calculus of Variations,

$$\delta S = \int_0^t \delta S'\,.\,dt, \tag{19.}$$

in which

$$S' = \Sigma\,.\,\varpi \frac{\delta H}{\delta\varpi} - H, \tag{20.}$$

and therefore

$$\delta S' = \Sigma \left(\varpi\, \delta \frac{\delta H}{\delta\varpi} - \frac{\delta H}{\delta\eta} \delta\eta \right), \tag{21.}$$

that is, by the equations of motion (A.),

$$\delta S' = \Sigma\left(\varpi\,\delta\frac{d\eta}{dt} + \frac{d\varpi}{dt}\delta\eta\right) = \frac{d}{dt}\,\Sigma\,.\varpi\,\delta\eta; \tag{22.}$$

the variation of the integral S is therefore

$$\delta S = \Sigma(\varpi\,\delta\eta - p\,\delta e), \tag{23.}$$

(p and e being still initial values,) and it decomposes itself into the following $6n$ expressions, when S is considered as a function of the $6n$ quantities η_i e_i, (involving also the time,)

$$\left.\begin{array}{ll} \varpi_1 = \dfrac{\delta S}{\delta\eta_1}; & p_1 = -\dfrac{\delta S}{\delta e_1}; \\ \varpi_2 = \dfrac{\delta S}{\delta\eta_2}; & p_2 = -\dfrac{\delta S}{\delta e_2}; \\ \cdots\cdots\cdots & \\ \varpi_{3n} = \dfrac{\delta S}{\delta\eta_{3n}}; & p_{3n} = -\dfrac{\delta S}{\delta e_{3n}}; \end{array}\right\} \tag{B.}$$

which are evidently forms for the sought integrals of the $6n$ differential equations of motion (A.), containing only one unknown function S. The difficulty of mathematical dynamics is therefore reduced to the search and study of this one function S, which may for that reason be called the PRINCIPAL FUNCTION of motion of a system.

This function S was introduced in the first Essay under the form

$$S = \int_0^t (T + U)\,dt,$$

the symbols T and U having in this form their recent meanings; and it is worth observing, that when S is expressed by this definite integral, the conditions for its variation vanishing (if the final and initial coordinates and the time be given) are precisely the differential equations of motion (3.), under the forms assigned by LAGRANGE. The variation of this definite integral S has therefore the double property, of giving the differential equations of motion for any transformed coordinates when the extreme positions are regarded as fixed, and of giving the integrals of those differential equations when the extreme positions are treated as varying.

5. Although the function S seems to deserve the name here given it of *Principal Function*, as serving to express, in what appears the simplest way, the integrals of the equations of motion, and the differential equations themselves; yet the same analysis conducts to other functions, which also may be used to express the integrals of the same equations. Thus if we put

$$Q = \int_0^t \left(-\,\Sigma\,.\eta\frac{\delta H}{\delta\eta} + H\right) dt, \tag{24.}$$

and take the variation of this integral Q without varying t or dt, we find, by a similar process,

$$\delta Q = \Sigma(\eta\,\delta\varpi - e\,\delta p); \tag{25.}$$

so that if we consider Q as a function of the $6n$ quantities ϖ_i p_i and of the time, we shall have $6n$ expressions

$$\eta_i = +\frac{\delta Q}{\delta\varpi_i}, \quad e_i = -\frac{\delta Q}{\delta p_i}, \tag{26.}$$

which are other forms of the integrals of the equations of motion (A.), involving the function Q instead of S. We might also employ the integral

$$V = \int_0^t \Sigma\,.\varpi\frac{\delta H}{\delta\varpi}\,dt = \Sigma\int_e^\eta \varpi\,d\eta, \tag{27.}$$

which was called the *Characteristic Function* in the former Essay, and of which, when considered as a function of the $6n+1$ quantities η_i e_i H, the variation is

$$\delta V = \Sigma(\varpi\,\delta\eta - p\,\delta e) + t\,\delta H. \tag{28.}$$

And all these functions S, Q, V, are connected in such a way, that the forms and properties of any one may be deduced from those of any other.

Investigation of a Pair of Partial Differential Equations of the first Order, which the Principal Function must satisfy.

6. In forming the variation (23.), or the partial differential coefficients (B.) of the Principal Function S, the variation of the time was omitted; but it is easy to calculate the coefficient $\dfrac{\delta S}{\delta t}$ corresponding to this variation, since the evident equation

$$\frac{dS}{dt} = \frac{\delta S}{\delta t} + \Sigma\,\frac{\delta S}{\delta\eta}\frac{d\eta}{dt} \tag{29.}$$

gives, by (20.), and by (A.), (B.),

$$\frac{\delta S}{\delta t} = S' - \Sigma\,.\varpi\frac{\delta H}{\delta\varpi} = -H. \tag{30.}$$

It is evident also that this coefficient, or the quantity $-H$, is constant, so as not to alter during the motion of the system; because the differential equations of motion (A.) give

$$\frac{dH}{dt} = \Sigma\left(\frac{\delta H}{\delta\eta}\frac{d\eta}{dt} + \frac{\delta H}{\delta\varpi}\frac{d\varpi}{dt}\right) = 0. \tag{31.}$$

If, therefore, we attend to the equation (17.), and observe that the function F is necessarily rational and integer and homogeneous of the second dimension with respect to the quantities ϖ_i, we shall perceive that the principal function S must satisfy the two following equations between its partial differential coefficients of the first order, which offer the chief means of discovering its form:

$$\left.\begin{aligned} \frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \dots \frac{\delta S}{\delta \eta_{3n}}, \eta_1, \eta_2, \dots \eta_{3n}\right) &= U(\eta_1, \eta_2, \dots \eta_{3n}), \\ \frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \dots \frac{\delta S}{\delta e_{3n}}, e_1, e_2, \dots e_{3n}\right) &= U(e_1, e_2, \dots e_{3n}). \end{aligned}\right\} \tag{C.}$$

Reciprocally, if the form of S be known, the forms of these equations (C.) can be deduced from it, by elimination of the quantities e or η between the expressions of its partial differential coefficients; and thus we can return from the principal function S to the functions F and U, and consequently to the expression H, and the equations of motion (A.)

Analogous remarks apply to the functions Q and V, which must satisfy the partial differential equations,

$$\left.\begin{aligned} -\frac{\delta Q}{\delta t} + F\left(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \frac{\delta Q}{\delta \varpi_1}, \frac{\delta Q}{\delta \varpi_2}, \dots \frac{\delta Q}{\delta \varpi_{3n}}\right) &= U\left(\frac{\delta Q}{\delta \varpi_1}, \frac{\delta Q}{\delta \varpi_2}, \dots \frac{\delta Q}{\delta \varpi_{3n}}\right), \\ -\frac{\delta Q}{\delta t} + F\left(p_1, p_2, \dots, p_{3n}, -\frac{\delta Q}{\delta p_1}, -\frac{\delta Q}{\delta p_2}, \dots -\frac{\delta Q}{\delta p_{3n}}\right) &= U\left(-\frac{\delta Q}{\delta p_1}, -\frac{\delta Q}{\delta p_2}, \dots -\frac{\delta Q}{\delta p_{3n}}\right), \end{aligned}\right\} \tag{32.}$$

and

$$\left.\begin{aligned} F\left(\frac{\delta V}{\delta \eta_1}, \frac{\delta V}{\delta \eta_2}, \dots \frac{\delta V}{\delta \eta_{3n}}, \eta_1, \eta_2, \dots \eta_{3n}\right) &= H + U(\eta_1, \eta_2, \dots \eta_{3n}), \\ F\left(\frac{\delta V}{\delta e_1}, \frac{\delta V}{\delta e_2}, \dots \frac{\delta V}{\delta e_{3n}}, e_1, e_2, \dots e_{3n}\right) &= H + U(e_1, e_2, \dots e_{3n}). \end{aligned}\right\} \tag{33.}$$

General Method of improving an approximate Expression for the Principal Function in any Problem of Dynamics.

7. If we separate the principal function S into any two parts,

$$S_1 + S_2 = S, \tag{34.}$$

and substitute their sum for S in the first equation (C.), the function F, from its rational and integer and homogeneous form and dimension, may be expressed in this new way,

$$\left.\begin{aligned} F\left(\frac{\delta S}{\delta \eta_1}, \dots \frac{\delta S}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) &= F\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) \\ &\quad + F'\left(\frac{\delta S_1}{\delta \eta_1}\right)\frac{\delta S_2}{\delta \eta_1} + \dots + F'\left(\frac{\delta S_1}{\delta \eta_{3n}}\right)\frac{\delta S_2}{\delta \eta_{3n}} + F\left(\frac{\delta S_2}{\delta \eta_1}, \dots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) \\ &= F\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) - F\left(\frac{\delta S_2}{\delta \eta_1}, \dots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) \\ &\quad + F'\left(\frac{\delta S}{\delta \eta_1}\right)\frac{\delta S_2}{\delta \eta_1} + \dots + F'\left(\frac{\delta S}{\delta \eta_{3n}}\right)\frac{\delta S_2}{\delta \eta_{3n}}, \end{aligned}\right\} \tag{35.}$$

because

$$F'\left(\frac{\delta S_1}{\delta \eta_i}\right) = F'\left(\frac{\delta S}{\delta \eta_i}\right) - F'\left(\frac{\delta S_2}{\delta \eta_i}\right), \tag{36.}$$

and

$$\Sigma\,.F'\left(\frac{\delta S_2}{\delta \eta}\right)\frac{\delta S_2}{\delta \eta} = 2F\left(\frac{\delta S_2}{\delta \eta_1},\ldots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right); \tag{37.}$$

and since, by (A.) and (B.),

$$F'\left(\frac{\delta S}{\delta \eta_i}\right) = F'(\varpi_i) = \frac{\delta H}{\delta \varpi_i} = \frac{d\eta_i}{dt}, \tag{38.}$$

we easily transform the first equation (C.) to the following,

$$\left.\begin{aligned} \frac{dS_2}{dt} &= -\frac{\delta S_1}{\delta t} + U(\eta_1, \ldots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1},\ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) \\ &\quad + F\left(\frac{\delta S_2}{\delta \eta_1},\ldots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right), \end{aligned}\right\} \tag{D.}$$

which gives rigorously

$$\left.\begin{aligned} S_2 = \int_0^t &\left\{-\frac{\delta S_1}{\delta t} + U(\eta_1, \ldots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1},\ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right)\right\} dt \\ &+ \int_0^t F\left(\frac{\delta S_2}{\delta \eta_1},\ldots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) dt, \end{aligned}\right\} \tag{E.}$$

supposing only that the two parts S_1, S_2, like the whole principal function S, are chosen so as to vanish with the time.

This general and rigorous transformation offers a general method of improving an approximate expression for the principal function S, in any problem of dynamics. For if the part S_1 be such an approximate expression, then the remaining part S_2 will be small; and the homogeneous function F involving the squares and products of the coefficients of this small part, in the second definite integral (E.), will be in general also small, and of a higher order of smallness; we may therefore in general neglect this second definite integral, in passing to a second approximation, and may in general improve a first approximate expresssion S_1 by adding to it the following correction,

$$\Delta S_1 = \int_0^t \left\{-\frac{\delta S_1}{\delta t} + U(\eta_1, \ldots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1},\ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right)\right\} dt; \tag{F.}$$

in calculating which definite integral we may employ the following approximate forms for the integrals of the equations of motion,

$$p_1 = -\frac{\delta S_1}{\delta e_1}, \quad p_2 = -\frac{\delta S_1}{\delta e_2}, \quad \ldots \quad p_{3n} = -\frac{\delta S_1}{\delta e_{3n}}, \tag{39.}$$

expressing first, by these, the variables η_i as functions of the time and of the $6n$ constants $e_i\ p_i$, and then eliminating, after the integration, the $3n$ quantities p_i, by the same approximate forms. And when an improved expression, or second approximate value $S_1 + \Delta S_1$, for the principal function S, has been thus obtained, it may be substituted in like manner for the first approximate value S_1, so as to obtain a still closer approximation, and the process may be repeated indefinitely.

An analogous process applies to the indefinite improvement of a first approximate expression for the function Q or V.

Rigorous Theory of Perturbations, founded on the Properties of the Disturbing Part of the whole Principal Function.

8. If we separate the expression H (17.) into any two parts of the same kind,

$$H_1 + H_2 = H, \tag{40.}$$

in which

$$H_1 = F_1(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U_1(\eta_1, \eta_2, \dots \eta_{3n}), \tag{41.}$$

and

$$H_2 = F_2(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U_2(\eta_1, \eta_2, \dots \eta_{3n}), \tag{42.}$$

the functions $F_1\ F_2\ U_1\ U_2$ being such that

$$F_1 + F_2 = F, \quad U_1 + U_2 = U; \tag{43.}$$

the differential equations of motion (A.) will take this form,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i} + \frac{\delta H_2}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i} - \frac{\delta H_2}{\delta \eta_i}, \tag{G.}$$

and if the part H_2 and its coefficients be small, they will not differ much from these other differential equations,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i}; \tag{H.}$$

so that the rigorous integrals of the latter system will be approximate integrals of the former. Whenever then, by a proper choice of the predominant term H_1, a system of $6n$ equations such as (H.) has been formed and rigorously integrated, giving expressions for the $6n$ variables $\eta_i\ \varpi_i$ as functions of the time t, and of their own initial values $e_i\ p_i$, which may be thus denoted:

$$\eta_i = \phi_i(t, e_1, e_2, \dots e_{3n}, p_1, p_2, \dots p_{3n}), \tag{44.}$$

and

$$\varpi_i = \psi_i(t, e_1, e_2, \dots e_{3n}, p_1, p_2, \dots p_{3n}); \tag{45.}$$

the simpler motion thus defined by the rigorous integrals of (H.) may be called the *undisturbed motion* of the proposed system of n points, and the more complex motion expressed by the rigorous integrals of (G.) may be called by contrast the *disturbed motion* of that system; and to pass from one to the other, may be called a *Problem of Perturbation.*

9. To accomplish this passage, let us observe that the differential equations of undisturbed motion (H.), being of the same form as the original equations (A.), may have their integrals similarly expressed, that is, as follows:

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i}, \tag{I.}$$

S_1 being here the *principal function of undisturbed motion*, or the definite integral

$$S_1 = \int_0^t \left(\Sigma \,.\, \varpi \frac{\delta H_1}{\delta \varpi} - H_1 \right) dt, \tag{46.}$$

considered as a function of the time and of the quantities η_i e_i. In like manner if we represent by $S_1 + S_2$ the whole principal function of disturbed motion, the rigorous integrals of (G.) may be expressed by (B.), as follows:

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i} + \frac{\delta S_2}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i} - \frac{\delta S_2}{\delta e_i}. \tag{K.}$$

Comparing the forms (44.) with the second set of equations (I.) for the integrals of undisturbed motion, we find that the following relations between the functions ϕ_i S_1 must be rigorously and *identically* true:

$$\eta_i = \phi_i \left(t, e_1, e_2, \ldots e_{3n}, -\frac{\delta S_1}{\delta e_1}, -\frac{\delta S_1}{\delta e_2}, \ldots -\frac{\delta S_1}{\delta e_{3n}} \right); \tag{47.}$$

and therefore, by (K.), that the integrals of disturbed motion may be put under the following forms,

$$\eta_i = \phi_i \left(t, e_1, e_2, \ldots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \ldots p_{3n} + \frac{\delta S_2}{\delta e_{3n}} \right). \tag{L.}$$

We may therefore calculate rigorously the disturbed variables η_i by the rules of undisturbed motion (44.), if without altering the time t, or the initial values e_i of those variables, which determine the initial configuration, we alter (in general) the initial velocities and directions, by adding to the elements p_i the following perturbational terms,

$$\Delta p_1 = \frac{\delta S_2}{\delta e_1}, \quad \Delta p_2 = \frac{\delta S_2}{\delta e_2}, \quad \ldots \quad \Delta p_{3n} = \frac{\delta S_2}{\delta e_{3n}} : \tag{M.}$$

a remarkable result, which includes the whole theory of perturbation. We might deduce from it the differential coefficients η_i', or the connected quantities ϖ_i, which determine the disturbed directions and velocities of motion at any time t; but a similar reasoning gives at once the general expression,

$$\varpi_i = \frac{\delta S_2}{\delta \eta_i} + \psi_i \left(t, e_1, e_2, \ldots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \ldots p_{3n} + \frac{\delta S_2}{\delta e_{3n}} \right), \tag{N.}$$

implying, that after altering the initial velocities and directions or the elements p_i as before, by the perturbational terms (M.), we may then employ the rules of undisturbed motion (45.) to calulate the velocities and directions at the time t, or the varying quantities ϖ_i, if we finally apply to these quantities thus calculated the following new corrections for perturbation:

$$\Delta\varpi_1 = \frac{\delta S_2}{\delta \eta_1}, \quad \Delta\varpi_2 = \frac{\delta S_2}{\delta \eta_2}, \quad \ldots \quad \Delta\varpi_{3n} = \frac{\delta S_2}{\delta \eta_{3n}}. \tag{O.}$$

Approximate expressions deduced from the foregoing rigorous Theory.

10. The foregoing theory gives indeed rigorous expressions for the perturbations, in passing from the simpler motion (H.) or (I.) to the more complex motion (G.) or (K.): but it may seem that these expressions are of little use, because they involve an unknown *disturbing function* S_2, (namely, the perturbational part of the whole principal function S,) and also unknown or disturbed coordinates or marks of position η_i. However, it was lately shown that whenever a first approximate form for the principal function S, such as here the principal function S_1 of undisturbed motion, has been found, the correction S_2 can in general be assigned, with an indefinitely increasing accuracy; and since the perturbations (M.) and (O.) involve the disturbed coordinates η_i only as they enter into the coefficients of this small disturbing function S_2, it is evidently permitted to substitute for these coordinates, at first, their undisturbed values, and then to correct the results by substituting more accurate expressions.

11. The function S_1 of undisturbed motion must satisfy rigorously two partial differential equations of the form (C.), namely,

$$\left.\begin{aligned} \frac{\delta S_1}{\delta t} + F_1\left(\frac{\delta S_1}{\delta \eta_1}, \ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) &= U_1(\eta_1, \ldots \eta_{3n}), \\ \frac{\delta S_1}{\delta t} + F_1\left(\frac{\delta S_1}{\delta e_1}, \ldots \frac{\delta S_1}{\delta e_{3n}}, e_1, \ldots e_{3n}\right) &= U_1(e_1, \ldots e_{3n}); \end{aligned}\right\} \tag{P.}$$

and therefore, by (D.), the disturbing function S_2 must satisfy rigorously the following other condition:

$$\left.\begin{aligned} \frac{dS_2}{dt} = U_2(\eta_1, \ldots \eta_{3n}) - F_2\left(\frac{\delta S_1}{\delta \eta_1}, \ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) \\ + F\left(\frac{\delta S_2}{\delta \eta_1}, \ldots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right), \end{aligned}\right\} \tag{Q.}$$

and may, on account of the homogeneity and dimension of F, be approximately expressed as follows:

$$S_2 = \int_0^t \left\{U_2(\eta_1, \ldots \eta_{3n}) - F_2\left(\frac{\delta S_1}{\delta \eta_1}, \ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right)\right\} dt, \tag{R.}$$

or thus, by (I.),

$$S_2 = \int_0^t \{U_2(\eta_1, \ldots \eta_{3n}) - F_2(\varpi_1, \ldots \varpi_{3n}, \eta_1, \ldots \eta_{3n})\} dt, \tag{S.}$$

that is, by (42.),

$$S_2 = -\int_0^t H_2 \, dt. \tag{T.}$$

In this expression, H_2 is given immediately as a function of the varying quantities $\eta_i \ \varpi_i$, but it may be considered in the same order of approximation as a known function of their initial values $e_i \ p_i$ and of the time t, obtained by substituting for $\eta_i \ \varpi_i$ their undisturbed values (44.), (45.) as functions of those quantities; its variation may therefore be expressed in either of the two following ways:

$$\delta H_2 = \Sigma \left(\frac{\delta H_2}{\delta \eta} \delta \eta + \frac{\delta H_2}{\delta \varpi} \delta \varpi \right), \tag{48.}$$

or

$$\delta H_2 = \Sigma \left(\frac{\delta H_2}{\delta e} \delta e + \frac{\delta H_2}{\delta p} \delta p \right) + \frac{\delta H_2}{\delta t} \delta t. \tag{49.}$$

Adopting the latter view, and effecting the integration (T.) with respect to the time, by treating the elements $e_i \ p_i$ as constant, we are afterwards to substitute for the quantities p_i their undisturbed expressions (39.) or (I.), and then we find for the variation of the disturbing function S_2 the expression

$$\delta S_2 = -H_2 \, \delta t + \Sigma \left(-\delta e \, . \int_0^t \frac{\delta H_2}{\delta e} dt + \delta \frac{\delta S_1}{\delta e} \, . \int_0^t \frac{\delta H_2}{\delta p} dt \right), \tag{50.}$$

which enables us to transform the perturbational terms (M.), (O.) into the following approximate forms:

$$\Delta p_i = -\int_0^t \frac{\delta H_2}{\delta e_i} dt + \Sigma \, . \frac{\delta^2 S_1}{\delta e \, \delta e_i} \int_0^t \frac{\delta H_2}{\delta p} dt, \tag{U.}$$

and

$$\Delta \varpi_i = \Sigma \, . \frac{\delta^2 S_1}{\delta e \, \delta \eta_i} \int_0^t \frac{\delta H_2}{\delta p} dt, \tag{V.}$$

containing only functions and quantities which may be regarded as given, by the theory of undisturbed motion.

12. In the same order of approximation, if the variation of the expression (44.) for an undisturbed coordinate η_i be thus denoted,

$$\delta \eta_i = \frac{\delta \eta_i}{\delta t} \delta t + \Sigma \left(\frac{\delta \eta_i}{\delta e} \delta e + \frac{\delta \eta_i}{\delta p} \delta p \right), \tag{51.}$$

the perturbation of that coordinate may be expressed as follows:

$$\Delta \eta_i = \Sigma \, . \frac{\delta \eta_i}{\delta p} \Delta p; \tag{W.}$$

that is, by (U.),

$$\left.\begin{aligned}\Delta\eta_i = &-\frac{\delta\eta_i}{\delta p_1}\int_0^t \frac{\delta H_2}{\delta e_1}dt - \frac{\delta\eta_i}{\delta p_2}\int_0^t \frac{\delta H_2}{\delta e_2}dt - \cdots - \frac{\delta\eta_i}{\delta p_{3n}}\int_0^t \frac{\delta H_2}{\delta e_{3n}}dt \\ &+ \left(\frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_1^2} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_1\,\delta e_2} + \cdots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_1\,\delta e_{3n}}\right)\int_0^t \frac{\delta H_2}{\delta p_1}dt \\ &+ \cdots\cdots \\ &+ \left(\frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_{3n}\,\delta e_1} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_{3n}\,\delta e_2} + \cdots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_{3n}^2}\right)\int_0^t \frac{\delta H_2}{\delta p_{3n}}dt.\end{aligned}\right\} \tag{52.}$$

Besides, the identical equation (47.) gives

$$\frac{\delta\eta_i}{\delta e_k} = \frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_k\,\delta e_1} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_k\,\delta e_2} + \cdots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_k\,\delta e_{3n}}; \tag{53.}$$

the expression (52.) may therefore be thus abridged,

$$\left.\begin{aligned}\Delta\eta_i = &-\frac{\delta\eta_i}{\delta p_1}\int_0^t \frac{\delta H_2}{\delta e_1}dt - \cdots - \frac{\delta\eta_i}{\delta p_{3n}}\int_0^t \frac{\delta H_2}{\delta e_{3n}}dt \\ &+ \frac{\delta\eta_i}{\delta e_1}\int_0^t \frac{\delta H_2}{\delta p_1}dt + \cdots + \frac{\delta\eta_i}{\delta e_{3n}}\int_0^t \frac{\delta H_2}{\delta p_{3n}}dt,\end{aligned}\right\} \tag{X.}$$

and shows that instead of the rigorous perturbational terms (M.) we may approximately employ the following,

$$\Delta p_i = -\int_0^t \frac{\delta H_2}{\delta e_i}dt, \tag{Y.}$$

in order to calculate the disturbed configuration at any time t by the rules of undisturbed motion, provided that besides thus altering the initial velocities and directions we alter also the initial configuration, by the formula

$$\Delta e_i = \int_0^t \frac{\delta H_2}{\delta p_i}dt. \tag{Z.}$$

It would not be difficult to calculate, in like manner, approximate expressions for the disturbed directions and velocities at any time t; but it is better to resume, in another way, the rigorous problem of perturbation.

Other Rigorous Theory of Perturbation, founded on the properties of the disturbing part of the constant of living force, and giving formulæ for the Variation of Elements more analogous to those already known.

13. Suppose that the theory of undisturbed motion has given the $6n$ constants $e_i\ p_i$ or any combinations of these, $\kappa_1, \kappa_2, \ldots, \kappa_{6n}$, as functions of the $6n$ variables $\eta_i\ \varpi_i$, and of the time t, which may be thus denoted:

$$\kappa_i = \chi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \tag{54.}$$

and which give reciprocally expressions for the variables $\eta_i\ \varpi_i$ in terms of these elements and of the time, analogous to (44.) and (45.), and capable of being denoted similarly,

$$\eta_i = \phi_i(t, \kappa_1, \kappa_2, \ldots \kappa_{6n}), \quad \varpi_i = \psi_i(t, \kappa_1, \kappa_2, \ldots \kappa_{6n}); \tag{55.}$$

then, the total differential coefficient of every such *element* or function κ_i, taken with respect to the time, (both as it enters explicitly and implicitly into the expressions (54.),) must vanish in the undisturbed motion; so that, by the differential equations of such motion (H.), the following general relation must be rigorously and *identically* true:

$$0 = \frac{\delta\kappa_i}{\delta t} + \Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta H_1}{\delta\varpi} - \frac{\delta\kappa_i}{\delta\varpi}\frac{\delta H_1}{\delta\eta}\right). \tag{56.}$$

In passing to disturbed motion, if we retain the equation (54.) as a *definition* of the quantity κ_i, that quantity will no longer be constant, but it will continue to satisfy the inverse relations (55.), and may be called, by analogy, a *varying element* of the motion; and its total differential coefficient, taken with respect to the time, may, by the identical equation (56.), and by the differential equations of disturbed motion (G.), be rigorously expressed as follows:

$$\frac{d\kappa_i}{dt} = \Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta H_2}{\delta\varpi} - \frac{\delta\kappa_i}{\delta\varpi}\frac{\delta H_2}{\delta\eta}\right). \tag{A^1.}$$

14. This result (A^1.) contains the whole theory of the gradual variation of the elements of disturbed motion of a system; but it may receive an advantageous transformation, by the substitution of the expressions (55.) for the variables $\eta_i\ \varpi_i$ as functions of the time and of the elements; since it will thus conduct to a system of $6n$ rigorous and ordinary differential equations of the first order between those varying elements and the time. Expressing, therefore, the quantity H_2 as a function of these latter variables, its variation δH_2 takes this new form,

$$\delta H_2 = \Sigma\,.\frac{\delta H_2}{\delta\kappa}\delta\kappa + \frac{\delta H_2}{\delta t}\delta t, \tag{57.}$$

and gives, by comparison with the form (48.), and by (54.),

$$\frac{\delta H_2}{\delta\eta_r} = \Sigma\,.\frac{\delta H_2}{\delta\kappa}\frac{\delta\kappa}{\delta\eta_r}; \quad \frac{\delta H_2}{\delta\varpi_r} = \Sigma\,.\frac{\delta H_2}{\delta\kappa}\frac{\delta\kappa}{\delta\varpi_r}; \tag{58.}$$

and thus the general equation (A^1.) is transformed to the following,

$$\frac{d\kappa_i}{dt} = a_{i,1}\frac{\delta H_2}{\delta\kappa_1} + a_{i,2}\frac{\delta H_2}{\delta\kappa_2} + \cdots + a_{i,6n}\frac{\delta H_2}{\delta\kappa_{6n}}, \tag{B^1.}$$

in which

$$a_{i,s} = \Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta\kappa_s}{\delta\varpi} - \frac{\delta\kappa_i}{\delta\varpi}\frac{\delta\kappa_s}{\delta\eta}\right): \tag{C^1.}$$

so that it only remains to eliminate the variables $\eta\ \varpi$ from the expressions of these latter coefficients. Now it is remarkable that this elimination removes the symbol t also, and leaves the coefficients $a_{i,s}$ expressed as functions of the elements κ alone, not explicitly involving the time. This general theorem of dynamics, which is, perhaps, a little more extensive than the analogous results discovered by LAGRANGE and by POISSON, since it does not limit the disturbing terms in the differential equations of motion to depend on the configuration only, may be investigated in the following way.

15. The sign of summation Σ in $(\mathrm{C}^1.)$, like the same sign in those other analogous equations in which it has already occurred without an index in this Essay, refers not to the expressed indices, such as here i, s, in the quantity to be summed, but to an index which is not expressed and which may be here called r; so that if we introduce for greater clearness this variable index and its limits, the expression $(\mathrm{C}^1.)$ becomes

$$a_{i,s} = \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta\kappa_i}{\delta\eta_r} \frac{\delta\kappa_s}{\delta\varpi_r} - \frac{\delta\kappa_i}{\delta\varpi_r} \frac{\delta\kappa_s}{\delta\eta_r} \right) : \tag{59.}$$

and its total differential coefficient, taken with respect to time, may be separated into the two following parts,

$$\left.\begin{aligned} \frac{d}{dt} a_{i,s} &= \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta\kappa_i}{\delta\eta_r} \frac{d}{dt} \frac{\delta\kappa_s}{\delta\varpi_r} - \frac{\delta\kappa_s}{\delta\eta_r} \frac{d}{dt} \frac{\delta\kappa_i}{\delta\varpi_r} \right) \\ &+ \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta\kappa_s}{\delta\varpi_r} \frac{d}{dt} \frac{\delta\kappa_i}{\delta\eta_r} - \frac{\delta\kappa_i}{\delta\varpi_r} \frac{d}{dt} \frac{\delta\kappa_s}{\delta\eta_r} \right), \end{aligned}\right\} \tag{60.}$$

which we shall proceed to calculate separately, and then to add them together. By the definition (54.), and the differential equations of disturbed motion (G.),

$$\frac{d}{dt} \frac{\delta\kappa_i}{\delta\varpi_r} = \frac{\delta^2\kappa_i}{\delta t\, \delta\varpi_r} + \Sigma_{(u)\,1}^{\;3n} \left\{ \frac{\delta^2\kappa_i}{\delta\eta_u\, \delta\varpi_r} \left(\frac{\delta H_1}{\delta\varpi_u} + \frac{\delta H_2}{\delta\varpi_u} \right) - \frac{\delta^2\kappa_i}{\delta\varpi_u\, \delta\varpi_r} \left(\frac{\delta H_1}{\delta\eta_u} + \frac{\delta H_2}{\delta\eta_u} \right) \right\}, \tag{61.}$$

in which, by the identical equation (56.),

$$\frac{\delta^2\kappa_i}{\delta t\, \delta\varpi_r} = -\frac{\delta}{\delta\varpi_r} \Sigma_{(u)\,1}^{\;3n} \left(\frac{\delta\kappa_i}{\delta\eta_u} \frac{\delta H_1}{\delta\varpi_u} - \frac{\delta\kappa_i}{\delta\varpi_u} \frac{\delta H_1}{\delta\eta_u} \right); \tag{62.}$$

we have therefore

$$\frac{d}{dt} \frac{\delta\kappa_i}{\delta\varpi_r} = \Sigma_{(u)\,1}^{\;3n} \left(\frac{\delta^2\kappa_i}{\delta\eta_u\, \delta\varpi_r} \frac{\delta H_2}{\delta\varpi_u} - \frac{\delta^2\kappa_i}{\delta\varpi_u\, \delta\varpi_r} \frac{\delta H_2}{\delta\eta_u} + \frac{\delta\kappa_i}{\delta\varpi_u} \frac{\delta^2 H_1}{\delta\eta_u \delta\varpi_r} - \frac{\delta\kappa_i}{\delta\eta_u} \frac{\delta^2 H_1}{\delta\varpi_u \delta\varpi_r} \right), \tag{63.}$$

and $\dfrac{d}{dt} \dfrac{\delta\kappa_s}{\delta\varpi_r}$ may be found from this, by merely changing i to s: so that

$$\left.\begin{aligned} &\Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta\kappa_i}{\delta\eta_r} \frac{d}{dt} \frac{\delta\kappa_s}{\delta\varpi_r} - \frac{\delta\kappa_s}{\delta\eta_r} \frac{d}{dt} \frac{\delta\kappa_i}{\delta\varpi_r} \right) \\ &\quad = \Sigma_{(r,u)\,1,1}^{\;3n,3n} \left\{ \left(\frac{\delta\kappa_s}{\delta\eta_r} \frac{\delta^2\kappa_i}{\delta\varpi_u\, \delta\varpi_r} - \frac{\delta\kappa_i}{\delta\eta_r} \frac{\delta^2\kappa_s}{\delta\varpi_u\, \delta\varpi_r} \right) \frac{\delta H_2}{\delta\eta_u} \right. \\ &\qquad + \left(\frac{\delta\kappa_i}{\delta\eta_r} \frac{\delta^2\kappa_s}{\delta\eta_u\, \delta\varpi_r} - \frac{\delta\kappa_s}{\delta\eta_r} \frac{\delta^2\kappa_i}{\delta\eta_u\, \delta\varpi_r} \right) \frac{\delta H_2}{\delta\varpi_u} \\ &\qquad \left. + \left(\frac{\delta\kappa_i}{\delta\eta_r} \frac{\delta\kappa_s}{\delta\varpi_u} - \frac{\delta\kappa_s}{\delta\eta_r} \frac{\delta\kappa_i}{\delta\varpi_u} \right) \frac{\delta^2 H_1}{\delta\eta_u\, \delta\varpi_r} + \left(\frac{\delta\kappa_s}{\delta\eta_r} \frac{\delta\kappa_i}{\delta\eta_u} - \frac{\delta\kappa_i}{\delta\eta_r} \frac{\delta\kappa_s}{\delta\eta_u} \right) \frac{\delta^2 H_1}{\delta\varpi_u\, \delta\varpi_r} \right\}, \end{aligned}\right\} \tag{64.}$$

and similarly,

$$\left.\begin{aligned}
&\Sigma_{(r)\,1}^{\;3n}\left(\frac{\delta\kappa_s}{\delta\varpi_r}\frac{d}{dt}\frac{\delta\kappa_i}{\delta\eta_r}-\frac{\delta\kappa_i}{\delta\varpi_r}\frac{d}{dt}\frac{\delta\kappa_s}{\delta\eta_r}\right)\\
&\quad=\Sigma_{(r,u)\,1,1}^{\;3n,3n}\left\{\left(\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\eta_u\,\delta\eta_r}-\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\eta_u\,\delta\eta_r}\right)\frac{\delta H_2}{\delta\varpi_u}\right.\\
&\qquad+\left(\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\varpi_u\,\delta\eta_r}-\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\varpi_u\,\delta\eta_r}\right)\frac{\delta H_2}{\delta\eta_u}\\
&\qquad\left.+\left(\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta\kappa_s}{\delta\eta_u}-\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta\kappa_i}{\delta\eta_u}\right)\frac{\delta^2 H_1}{\delta\varpi_u\,\delta\eta_r}+\left(\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta\kappa_i}{\delta\varpi_u}-\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta\kappa_s}{\delta\varpi_u}\right)\frac{\delta^2 H_1}{\delta\eta_u\,\delta\eta_r}\right\}.
\end{aligned}\right\}\tag{65.}$$

Adding, therefore, the two last expressions, and making the reductions which present themselves, we find, by (60.),

$$\frac{d}{dt}a_{i,s}=\Sigma_{(u)\,1}^{\;3n}\left(A_{i,s}^{(u)}\frac{\delta H_2}{\delta\eta_u}+B_{i,s}^{(u)}\frac{\delta H_2}{\delta\varpi_u}\right),\tag{D^1.}$$

in which

$$\left.\begin{aligned}
A_{i,s}^{(u)}&=\Sigma_{(r)\,1}^{\;3n}\left(\frac{\delta\kappa_s}{\delta\eta_r}\frac{\delta^2\kappa_i}{\delta\varpi_u\,\delta\varpi_r}-\frac{\delta\kappa_i}{\delta\eta_r}\frac{\delta^2\kappa_s}{\delta\varpi_u\,\delta\varpi_r}+\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\varpi_u\,\delta\eta_r}-\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\varpi_u\,\delta\eta_r}\right),\\
B_{i,s}^{(u)}&=\Sigma_{(r)\,1}^{\;3n}\left(\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\eta_u\,\delta\eta_r}-\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\eta_u\,\delta\eta_r}+\frac{\delta\kappa_i}{\delta\eta_r}\frac{\delta^2\kappa_s}{\delta\eta_u\,\delta\varpi_r}-\frac{\delta\kappa_s}{\delta\eta_r}\frac{\delta^2\kappa_i}{\delta\eta_u\,\delta\varpi_r}\right);
\end{aligned}\right\}\tag{66.}$$

and since this general form (D^1.) for $\dfrac{d}{dt}a_{i,s}$ contains no term independent of the disturbing quantities $\dfrac{\delta H_2}{\delta\eta}$, $\dfrac{\delta H_2}{\delta\varpi}$, it is easy to infer from it the important consequence already mentioned, namely, that the coefficients $a_{i,s}$, in the differentials (B^1.) of the elements, may be expressed as functions of those elements alone, not explicitly involving the time.

It is evident also, that these coefficients $a_{i,s}$ have the property

$$a_{s,i}=-a_{i,s}\tag{67.}$$

and

$$a_{i,i}=0;\tag{68.}$$

the term proportional to $\dfrac{\delta H_2}{\delta\kappa_i}$ disappears therefore from the expression (B^1.) for $\dfrac{d\kappa_i}{dt}$; and the term

$$\frac{\delta H_2}{\delta\kappa_i}\,.\,a_{i,s}\,.\,\frac{\delta H_2}{\delta\kappa_s}\quad\text{in}\quad\frac{\delta H_2}{\delta\kappa_i}\frac{d\kappa_i}{dt},$$

destroys the term

$$\frac{\delta H_2}{\delta\kappa_s}\,.\,a_{s,i}\,.\,\frac{\delta H_2}{\delta\kappa_i}\quad\text{in}\quad\frac{\delta H_2}{\delta\kappa_s}\frac{d\kappa_s}{dt},$$

when these terms are added together; we have, therefore,

$$\Sigma . \frac{\delta H_2}{\delta \kappa} \frac{d\kappa}{dt} = 0, \tag{E^1.}$$

or

$$\frac{dH_2}{dt} = \frac{\delta H_2}{\delta t}; \tag{F^1.}$$

that is, in taking the first total differential coefficient of the disturbing expression H_2 with respect to the time, the elements may be treated as constant.

Simplification of the differential equations which determine these gradually varying elements, in any problem of Perturbation; and Integration of the simplified equations by means of certain Functions of Elements.

16. The most natural choice of these elements is that which makes them correspond, in undisturbed motion, to the initial quantities e_i p_i. These quantities, by the differential equations (H.), may be expressed in undisturbed motion as follows,

$$e_i = \eta_i - \int_0^t \frac{\delta H_1}{\delta \varpi_i} dt, \quad p_i = \varpi_i + \int_0^t \frac{\delta H_1}{\delta \eta_i} dt; \tag{69.}$$

and if we suppose them found, by elimination, under the forms

$$\left. \begin{aligned} e_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ p_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \end{aligned} \right\} \tag{70.}$$

it is easy to see that the following equations must be rigorously and identically true, for all values of η_i ϖ_i,

$$\left. \begin{aligned} 0 &= \Phi_i(0, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ 0 &= \Psi_i(0, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}). \end{aligned} \right\} \tag{71.}$$

When, therefore, in passing to disturbed motion, we establish the equations of definition,

$$\left. \begin{aligned} \kappa_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ \lambda_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \end{aligned} \right\} \tag{72.}$$

introducing $6n$ varying elements κ_i λ_i, of which the set λ_i would have been represented in our recent notation as follows:

$$\lambda_i = \kappa_{3n+i}; \tag{73.}$$

we see that all the partial differential coefficients of the forms $\frac{\delta \kappa_i}{\delta \eta_r}$, $\frac{\delta \kappa_i}{\delta \varpi_r}$, $\frac{\delta \lambda_i}{\delta \eta_r}$, $\frac{\delta \lambda_i}{\delta \varpi_r}$, vanish when $t = 0$, except the following:

$$\frac{\delta \kappa_i}{\delta \eta_i} = 1, \quad \frac{\delta \lambda_i}{\delta \varpi_i} = 1; \tag{74.}$$

and, therefore, that when t is made $= 0$, in the coefficients $a_{i,s}$, (59.), all those coefficients vanish, except the following:

$$a_{r,3n+r} = 1, \quad a_{3n+r,r} = -1. \tag{75.}$$

But it has been proved that these coefficients $a_{i,s}$, when expressed as functions of the elements, do not contain the time explicitly; and the supposition $t = 0$ introduces no relation between those $6n$ elements κ_i λ_i, which still remain independent: the coefficients $a_{i,s}$, therefore, could not acquire the values 1, 0, -1, by the supposition $t = 0$, unless they had those values constantly, and independently of that supposition. The differential equations of the forms (B¹.) may therefore be expressed, for the present system of varying elements, in the following simpler way:

$$\frac{d\kappa_i}{dt} = \frac{\delta H_2}{\delta \lambda_i}; \quad \frac{d\lambda_i}{dt} = -\frac{\delta H_2}{\delta \kappa_i}; \tag{G^1.}$$

and an easy verification of these expressions is offered by the formula (E¹.), which takes now this form,

$$\Sigma \left(\frac{\delta H_2}{\delta \kappa} \frac{d\kappa}{dt} + \frac{\delta H_2}{\delta \lambda} \frac{d\lambda}{dt} \right) = 0. \tag{H^1.}$$

17. The initial values of the varying elements κ_i λ_i are evidently e_i p_i, by the definitions (72.), and by the identical equations (71.); the problem of integrating rigorously the equations of disturbed motion (G.), between the variables η_i ϖ_i and the time, or of determining these variables as functions of the time and of their own initial values e_i p_i, is therefore rigorously transformed into the problem of integrating the equations (G¹.), or of determining the $6n$ elements κ_i λ_i as functions of the time and of the same initial values. The chief advantage of this transformation is, that if the perturbations be small, the new variables (namely the elements,) alter but little: and that, since the new differential equations are of the same form as the old, they may be integrated by a similar method. Considering, therefore, the definite integral

$$E = \int_0^t \left(\Sigma \,.\lambda \frac{\delta H_2}{\delta \lambda} - H_2 \right) dt, \tag{76.}$$

as a function of the time and of the $6n$ quantities $\kappa_1, \kappa_2, \ldots \kappa_{3n}, e_1, e_2, \ldots e_n$, and observing that its variation, taken with respect to the latter quantities, may be shown by a process similar to that of the fourth number of this Essay to be

$$\delta E = \Sigma(\lambda \,\delta\kappa - p\,\delta e), \tag{I^1.}$$

we find that the rigorous integrals of the differential equations (G¹.) may be expressed in the following manner:

$$\lambda_i = \frac{\delta E}{\delta \kappa_i}, \quad p_i = -\frac{\delta E}{\delta e_i}, \tag{K^1.}$$

in which there enters only one unknown *function of elements E*, to the search and study of which single function the problem of perturbation is reduced by this new method.

We might also have put

$$C = \int_0^t \left(- \Sigma \,.\kappa \frac{\delta H_2}{\delta \kappa} + H_2 \right) dt, \tag{77.}$$

and considered this definite integral C as a function of the time and of the $6n$ quantities $\lambda_i \; p_i$; and then we should have found the following other forms for the integrals of the differential equations of varying elements,

$$\kappa_i = + \frac{\delta C}{\delta \lambda_i}, \quad e_i = - \frac{\delta C}{\delta p_i}, \tag{L^1.}$$

And each of these *functions of elements*, C and E, must satisfy a certain partial differential equation, analogous to the first equation of each pair mentioned in the sixth number of this Essay, and deduced on similar principles.

18. Thus, it is evident, by the form of the function E, and by the equations (K^1.), (G^1.), and (76.), that the partial differential coefficient of this function, taken with respect to the time, is

$$\frac{\delta E}{\delta t} = \frac{dE}{dt} - \Sigma \,. \frac{\delta E}{\delta \kappa} \frac{d\kappa}{dt} = - H_2; \tag{M^1.}$$

and therefore that if we separate this function E into any two parts

$$E_1 + E_2 = E, \tag{N^1.}$$

and if, for greater clearness, we put the expression H_2 under the form

$$H_2 = H_2(t, \kappa_1, \kappa_2, \ldots \kappa_{3n}, \lambda_1, \lambda_2, \ldots \lambda_{3n}), \tag{O^1.}$$

we shall have rigorously the partial differential equation

$$0 = \frac{\delta E_1}{\delta t} + \frac{\delta E_2}{\delta t} + H_2 \left(t, \kappa_1, \ldots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1} + \frac{\delta E_2}{\delta \kappa_1}, \ldots \frac{\delta E_1}{\delta \kappa_{3n}} + \frac{\delta E_2}{\delta \kappa_{3n}} \right) : \tag{P^1.}$$

which gives, approximately, by (G^1.) and (K^1), when the part E_2 is small, and when we neglect the squares and products of its partial differential coefficients,

$$0 = \frac{dE_2}{dt} + \frac{\delta E_1}{\delta t} + H_2 \left(t, \kappa_1, \ldots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1}, \ldots \frac{\delta E_1}{\delta \kappa_{3n}} \right). \tag{Q^1.}$$

Hence, in the same order of approximation, if the part E_1, like the whole function E, be chosen so as to vanish with the time, we shall have

$$E_2 = - \int_0^t \left\{ \frac{\delta E_1}{\delta t} + H_2 \left(t, \kappa_1, \ldots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1}, \ldots \frac{\delta E_1}{\delta \kappa_{3n}} \right) \right\} dt : \tag{R^1.}$$

and thus a first approximate expression E_1 can be successively and indefinitely corrected.

Again, by (L¹.) and (G¹.), and by the definition (77.),

$$\frac{\delta C}{\delta t} = \frac{dC}{dt} - \Sigma \, . \frac{\delta C}{\delta \lambda} \frac{d\lambda}{dt} = H_2; \tag{S^1.}$$

the function C must therefore satisfy rigorously the partial differential equation,

$$\frac{\delta C}{\delta t} = H_2 \left(t, \frac{\delta C}{\delta \lambda_1}, \ldots \frac{\delta C}{\delta \lambda_{3n}}, \lambda_1, \ldots \lambda_{3n} \right) : \tag{T^1.}$$

and if we put

$$C = C_1 + C_2, \tag{U^1.}$$

and suppose that the part C_2 is small, then the rigorous equation

$$\frac{\delta C_1}{\delta t} + \frac{\delta C_2}{\delta t} = H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1} + \frac{\delta C_2}{\delta \lambda_1}, \ldots \frac{\delta C_1}{\delta \lambda_{3n}} + \frac{\delta C_2}{\delta \lambda_{3n}}, \lambda_1, \ldots \lambda_{3n} \right) \tag{V^1.}$$

becomes approximately, by (G¹.) and (L¹.),

$$\frac{dC_2}{dt} = -\frac{\delta C_1}{\delta t} + H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1}, \ldots \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \ldots \lambda_{3n} \right), \tag{W^1.}$$

and gives by integration,

$$C_2 = \int_0^t \left\{ -\frac{\delta C_1}{\delta t} + H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1}, \ldots \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \ldots \lambda_{3n} \right) \right\} dt, \tag{X^1.}$$

the parts C_1 and C_2 being supposed to vanish separately when $t = 0$, like the whole function of elements C.

And to obtain such a first approximation, E_1 or C_1, to either of these two functions of elements E, C, we may change, in the definitions (76.), (77.), the varying elements κ λ, to their initial values e p, and then eliminate one set of these initial values by the corresponding set of the following approximate equations, deduced from the formulæ (G¹.):

$$\kappa_i = e_i + \int_0^t \frac{\delta H_2}{\delta p_i} dt; \tag{Y^1.}$$

and

$$\lambda_i = p_i - \int_0^t \frac{\delta H_2}{\delta e_i} dt. \tag{Z^1.}$$

It is easy also to see that these two functions of elements C and E are connected with each other, and with the disturbing function S_2, so that the form of any one may be deduced from that of any other, when the function S_1 of undisturbed motion is known.

Analogous formulæ for the motion of a Single Point.

19. Our general method in dynamics, though intended chiefly for the study of attracting and repelling systems, is not confined to such, but may be used in all questions to which the law of living forces applies. And all the analysis of this Essay, but especially the theory of perturbations, may usefully be illustrated by the following analogous reasonings and results respecting the motion of a single point.

Imagine then such a point, having for its three rectangular coordinates $x\ y\ z$, and moving in an orbit determined by three ordinary differential equations of the second order of forms analogous to the equations (2.), namely,

$$x'' = \frac{\delta U}{\delta x}; \quad y'' = \frac{\delta U}{\delta y}; \quad z'' = \frac{\delta U}{\delta z}; \tag{78.}$$

U being any given function of the coordinates not expressly involving the time: and let us establish the following definition, analogous to (4.),

$$T = \tfrac{1}{2}(x'^2 + y'^2 + z'^2), \tag{79.}$$

$x'\ y'\ z'$ being the first, and $x''\ y''\ z''$ being the second differential coefficients of the coordinates, considered as functions of the time t. If we express, for greater generality or facility, the rectangular coordinates $x\ y\ z$ as functions of three other marks of position $\eta_1\ \eta_2\ \eta_3$, T will become a homogeneous function of the second dimension of their first differential coefficients $\eta_1'\ \eta_2'\ \eta_3'$ taken with respect to time; and if we put, for abridgement,

$$\varpi_1 = \frac{\delta T}{\delta \eta_1'}, \quad \varpi_2 = \frac{\delta T}{\delta \eta_2'}, \quad \varpi_3 = \frac{\delta T}{\delta \eta_3'}, \tag{80.}$$

T may considered also as a function of the form

$$T = F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3), \tag{81.}$$

which will be homogeneous of the second dimension with respect to $\varpi_1\ \varpi_2\ \varpi_3$. We may also put, for abridgement,

$$T = F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3) - U(\eta_1, \eta_2, \eta_3) = H; \tag{82.}$$

and then, instead of the three differential equations of the second order (78.), we may employ the six following of the first order, analogous to the equations (A.), and obtained by a similar reasoning,

$$\left.\begin{aligned} \frac{d\eta_1}{dt} &= +\frac{\delta H}{\delta \varpi_1}, & \frac{d\eta_2}{dt} &= +\frac{\delta H}{\delta \varpi_2}, & \frac{d\eta_3}{dt} &= +\frac{\delta H}{\delta \varpi_3}, \\ \frac{d\varpi_1}{dt} &= -\frac{\delta H}{\delta \eta_1}, & \frac{d\varpi_2}{dt} &= -\frac{\delta H}{\delta \eta_2}, & \frac{d\varpi_3}{dt} &= -\frac{\delta H}{\delta \eta_3}, \end{aligned}\right\} \tag{83.}$$

20. The rigorous integrals of these six differential equations may be expressed under the following forms, analogous to (B.),

$$\left.\begin{aligned}\varpi_1 &= \frac{\delta S}{\delta \eta_1}, & \varpi_2 &= \frac{\delta S}{\delta \eta_2}, & \varpi_3 &= \frac{\delta S}{\delta \eta_3},\\ p_1 &= -\frac{\delta S}{\delta e_1}, & p_2 &= -\frac{\delta S}{\delta e_2}, & p_3 &= -\frac{\delta S}{\delta e_3},\end{aligned}\right\} \tag{84.}$$

in which $e_1\ e_2\ e_3\ p_1\ p_2\ p_3$ are the initial values, or values at the time 0, of $\eta_1\ \eta_2\ \eta_3\ \varpi_1\ \varpi_2\ \varpi_3$; and S is the definite integral

$$S = \int_0^t \left(\varpi_1 \frac{\delta H}{\delta \varpi_1} + \varpi_2 \frac{\delta H}{\delta \varpi_2} + \varpi_3 \frac{\delta H}{\delta \varpi_3} - H\right) dt, \tag{85.}$$

considered as a function of $\eta_1\ \eta_2\ \eta_3\ e_1\ e_2\ e_3$ and t. The quantity H does not change in the course of the motion, and the function S must satisfy the following pair of partial differential equations of the first order, analogous to the pair (C.),

$$\left.\begin{aligned}\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \frac{\delta S}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right) &= U(\eta_1, \eta_2, \eta_3);\\ \frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \frac{\delta S}{\delta e_3}, e_1, e_2, e_3\right) &= U(e_1, e_2, e_3).\end{aligned}\right\} \tag{86.}$$

This important function S, which may be called the *principal function* of the motion, may hence be rigorously expressed under the following form, obtained by reasonings analogous to those of the seventh number of this Essay:

$$\left.\begin{aligned}S = S_1 &+ \int_0^t \left\{-\frac{\delta S_1}{\delta t} + U(\eta_1, \eta_2, \eta_3) - F\left(\frac{\delta S_1}{\delta \eta_1}, \frac{\delta S_1}{\delta \eta_2}, \frac{\delta S_1}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right)\right\} dt\\ &+ \int_0^t F\left(\frac{\delta S}{\delta \eta_1} - \frac{\delta S_1}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2} - \frac{\delta S_1}{\delta \eta_2}, \frac{\delta S}{\delta \eta_3} - \frac{\delta S_1}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right) dt;\end{aligned}\right\} \tag{87.}$$

S_1 being any arbitrary function of the same quantities $\eta_1\ \eta_2\ \eta_3\ e_1\ e_2\ e_3\ t$, so chosen as to vanish with the time. And if this arbitrary function S_1 be chosen so as to be a first approximate value of the principal function S, we may neglect, in a second approximation, the second definite integral in (87.).

21. A first approximation of this kind can be obtained, whenever, by separating the expression H, (82.) into a predominant and a smaller part, H_1 and H_2, and by neglecting the part H_2, we have changed the differential equations (83.) to others, namely,

$$\left.\begin{aligned}\frac{d\eta_1}{dt} &= \frac{\delta H_1}{\delta \varpi_1}, & \frac{d\eta_2}{dt} &= \frac{\delta H_1}{\delta \varpi_2}, & \frac{d\eta_3}{dt} &= \frac{\delta H_1}{\delta \varpi_3},\\ \frac{d\varpi_1}{dt} &= -\frac{\delta H_1}{\delta \eta_1}, & \frac{d\varpi_2}{dt} &= -\frac{\delta H_1}{\delta \eta_2}, & \frac{d\varpi_3}{dt} &= -\frac{\delta H_1}{\delta \eta_3},\end{aligned}\right\} \tag{88.}$$

and have succeeded in integrating rigorously these simplified equations, belonging to a simpler motion, which may be called the *undisturbed motion* of the point. For the principal function of such undisturbed motion, namely, the definite integral

$$S_1 = \int_0^t \left(\varpi_1 \frac{\delta H_1}{\delta \varpi_1} + \varpi_2 \frac{\delta H_1}{\delta \varpi_2} + \varpi_3 \frac{\delta H_1}{\delta \varpi_3} - H_1 \right) dt, \tag{89.}$$

considered as a function of η_1 η_2 η_3 e_1 e_2 e_3 t, will then be an approximate value for the original function of disturbed motion S, which original function corresponds to the more complex differential equations,

$$\left.\begin{aligned} \frac{d\eta_1}{dt} &= \frac{\delta H_1}{\delta \varpi_1} + \frac{\delta H_2}{\delta \varpi_1}, & \frac{d\eta_2}{dt} &= \frac{\delta H_1}{\delta \varpi_2} + \frac{\delta H_2}{\delta \varpi_2}, & \frac{d\eta_3}{dt} &= \frac{\delta H_1}{\delta \varpi_3} + \frac{\delta H_2}{\delta \varpi_3}, \\ \frac{d\varpi_1}{dt} &= -\frac{\delta H_1}{\delta \eta_1} - \frac{\delta H_2}{\delta \eta_1}, & \frac{d\varpi_2}{dt} &= -\frac{\delta H_1}{\delta \eta_2} - \frac{\delta H_2}{\delta \eta_2}, & \frac{d\varpi_3}{dt} &= -\frac{\delta H_1}{\delta \eta_3} - \frac{\delta H_2}{\delta \eta_3}. \end{aligned}\right\} \tag{90.}$$

The function S_1 of undisturbed motion must satisfy a pair of partial differential equations of the first order, analogous to the pair (86.); and the integrals of undisturbed motion may be represented thus,

$$\left.\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1}, & \varpi_2 &= \frac{\delta S_1}{\delta \eta_2}, & \varpi_3 &= \frac{\delta S_1}{\delta \eta_3}, \\ p_1 &= -\frac{\delta S_1}{\delta e_1}, & p_2 &= -\frac{\delta S_1}{\delta e_2}, & p_3 &= -\frac{\delta S_1}{\delta e_3} : \end{aligned}\right\} \tag{91.}$$

while the integrals of disturbed motion may be expressed with equal rigour under the following analogous forms,

$$\left.\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1}, & \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2}, & \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3}, \\ p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1}, & p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2}, & p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3}, \end{aligned}\right\} \tag{92.}$$

if S_1 denote the rigorous correction of S_1, or the disturbing part of the whole principal function S. And by the foregoing general theory of approximation, this disturbing part or function S_2 may be approximately represented by the definite integral (T.),

$$S_2 = -\int_0^t H_2 \, dt; \tag{93.}$$

in calculating which definite integral the equations (91.) may be employed.

22. If the integrals of undisturbed motion (91.) have given

$$\left.\begin{aligned} \eta_1 &= \phi_1(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_2 &= \phi_2(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_3 &= \phi_3(t, e_1, e_2, e_3, p_1, p_2, p_3), \end{aligned}\right\} \tag{94.}$$

and

$$\left.\begin{aligned}\varpi_1 &= \psi_1(t, e_1, e_2, e_3, p_1, p_2, p_3),\\ \varpi_2 &= \psi_2(t, e_1, e_2, e_3, p_1, p_2, p_3),\\ \varpi_3 &= \psi_3(t, e_1, e_2, e_3, p_1, p_2, p_3),\end{aligned}\right\} \tag{95.}$$

then the integrals of disturbed motion (92.) may be rigorously transformed as follows,

$$\left.\begin{aligned}\eta_1 &= \phi_1\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\\ \eta_2 &= \phi_2\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\\ \eta_3 &= \phi_3\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\end{aligned}\right\} \tag{96.}$$

and

$$\left.\begin{aligned}\varpi_1 &= \frac{\delta S_2}{\delta \eta_1} + \psi_1\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\\ \varpi_2 &= \frac{\delta S_2}{\delta \eta_2} + \psi_2\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\\ \varpi_3 &= \frac{\delta S_2}{\delta \eta_3} + \psi_3\left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3}\right),\end{aligned}\right\} \tag{97.}$$

S_2 being here the rigorous disturbing function. And the perturbations of position, at any time t, may be approximately expressed by the following formula,

$$\left.\begin{aligned}\Delta\eta_1 = {}& \frac{\delta \eta_1}{\delta e_1}\int_0^t \frac{\delta H_2}{\delta p_1}dt + \frac{\delta \eta_1}{\delta e_2}\int_0^t \frac{\delta H_2}{\delta p_2}dt + \frac{\delta \eta_1}{\delta e_3}\int_0^t \frac{\delta H_2}{\delta p_3}dt\\ & - \frac{\delta \eta_1}{\delta p_1}\int_0^t \frac{\delta H_2}{\delta e_1}dt - \frac{\delta \eta_1}{\delta p_2}\int_0^t \frac{\delta H_2}{\delta e_2}dt - \frac{\delta \eta_1}{\delta p_3}\int_0^t \frac{\delta H_2}{\delta e_3}dt,\end{aligned}\right\} \tag{98.}$$

together with two similar formulæ for the perturbations of the two other coordinates, or marks of position η_2, η_3. In these formulæ, the coordinates and H_2 are supposed to be expressed, by the theory of undisturbed motion, as functions of the time t, and of the constants e_1 e_2 e_3 p_1 p_2 p_3.

23. Again, if the integrals of undisturbed motion have given, by elimination, expressions for these constants, of the forms

$$\left.\begin{aligned}e_1 &= \eta_1 + \Phi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\ e_2 &= \eta_2 + \Phi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\ e_3 &= \eta_3 + \Phi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\end{aligned}\right\} \tag{99.}$$

and

$$\left.\begin{aligned}p_1 &= \varpi_1 + \Psi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\ p_2 &= \varpi_2 + \Psi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\ p_3 &= \varpi_3 + \Psi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3);\end{aligned}\right\} \tag{100.}$$

and if, for disturbed motion, we establish the definitions

$$\left.\begin{aligned}
\kappa_1 &= \eta_1 + \Phi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\
\kappa_2 &= \eta_2 + \Phi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\
\kappa_3 &= \eta_3 + \Phi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),
\end{aligned}\right\} \tag{101.}$$

and

$$\left.\begin{aligned}
\lambda_1 &= \varpi_1 + \Psi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\
\lambda_2 &= \varpi_2 + \Psi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3),\\
\lambda_3 &= \varpi_3 + \Psi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3);
\end{aligned}\right\} \tag{102.}$$

we shall have, for such disturbed motion, the following rigorous equations, of the forms (94.) and (95.),

$$\left.\begin{aligned}
\eta_1 &= \phi_1(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3),\\
\eta_2 &= \phi_2(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3),\\
\eta_3 &= \phi_3(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3),
\end{aligned}\right\} \tag{103.}$$

and

$$\left.\begin{aligned}
\varpi_1 &= \psi_1(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3),\\
\varpi_2 &= \psi_2(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3),\\
\varpi_3 &= \psi_3(t, \kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3);
\end{aligned}\right\} \tag{104.}$$

and may call the quantities κ_1 κ_2 κ_3 λ_1 λ_2 λ_3 the 6 *varying elements* of the motion. To determine these six varying elements, we may employ the six following rigorous equations in ordinary differentials of the first order, in which H_2 is supposed to have been expressed by (103.) and (104.) as a function of the elements and of the time:

$$\left.\begin{aligned}
\frac{d\kappa_1}{dt} &= \frac{\delta H_2}{\delta \lambda_1}, & \frac{d\kappa_2}{dt} &= \frac{\delta H_2}{\delta \lambda_2}, & \frac{d\kappa_3}{dt} &= \frac{\delta H_2}{\delta \lambda_3},\\
\frac{d\lambda_1}{dt} &= -\frac{\delta H_2}{\delta \kappa_1}, & \frac{d\lambda_2}{dt} &= -\frac{\delta H_2}{\delta \kappa_2}, & \frac{d\lambda_3}{dt} &= -\frac{\delta H_2}{\delta \kappa_3};
\end{aligned}\right\} \tag{105.}$$

and the rigorous integrals of these 6 equations may be expressed in the following manner,

$$\left.\begin{aligned}
\lambda_1 &= \frac{\delta E}{\delta \kappa_1}, & \lambda_2 &= \frac{\delta E}{\delta \kappa_2}, & \lambda_3 &= \frac{\delta E}{\delta \kappa_3},\\
p_1 &= -\frac{\delta E}{\delta e_1}, & p_2 &= -\frac{\delta E}{\delta e_2}, & p_3 &= -\frac{\delta E}{\delta e_3},
\end{aligned}\right\} \tag{106.}$$

the constants e_1 e_2 e_3 p_1 p_2 p_3 retaining their recent meanings, and being therefore the initial values of the elements κ_1 κ_2 κ_3 λ_1 λ_2 λ_3; while the function E, which may be called the *function of elements*, because its form determines the laws of their variations, is the definite integral

$$E = \int_0^t \left(\lambda_1 \frac{\delta H_2}{\delta \lambda_1} + \lambda_2 \frac{\delta H_2}{\delta \lambda_2} + \lambda_3 \frac{\delta H_2}{\delta \lambda_3} - H_2 \right) dt, \tag{107.}$$

considered as depending on κ_1 κ_2 κ_3 e_1 e_2 e_3 and t. The integrals of the equations (105.) may also be expressed in this other way,

$$\left.\begin{aligned} \kappa_1 &= +\frac{\delta C}{\delta \lambda_1}, & \kappa_2 &= +\frac{\delta C}{\delta \lambda_2}, & \kappa_3 &= +\frac{\delta C}{\delta \lambda_3}, \\ e_1 &= -\frac{\delta C}{\delta p_1}, & e_2 &= -\frac{\delta C}{\delta p_2}, & e_3 &= -\frac{\delta C}{\delta p_3}, \end{aligned}\right\} \tag{108.}$$

C being the definite integral

$$C = -\int_0^t \left(\kappa_1 \frac{\delta H_2}{\delta \kappa_1} + \kappa_2 \frac{\delta H_2}{\delta \kappa_2} + \kappa_3 \frac{\delta H_2}{\delta \kappa_3} - H_2 \right) dt, \tag{109.}$$

regarded as a function of λ_1 λ_2 λ_3 p_1 p_2 p_3 and t: and it is easy to prove that each of these two *functions of elements*, C and E, must satisfy a partial differential equation of the first order, which can be previously assigned, and which may assist in discovering the forms of these two functions, and especially in improving an approximate expression for either. All these results for the motion of a single point, are analogous to the results already deduced in this Essay, for an attracting or repelling system.

Mathematical Example, suggested by the motion of Projectiles.

24. If the three marks of position η_1 η_2 η_3 of the moving point are the rectangular coordinates themselves, and if the function U has the form

$$U = -g\eta_3 - \tfrac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \tag{110.}$$

g, μ, ν being constants; then the expression

$$H = \tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g\eta_3 + \tfrac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\} \tag{111.}$$

is that which must be substituted in the general forms (83.), in order to form the 6 differential equations of motion of the first order, namely,

$$\left.\begin{aligned} \frac{d\eta_1}{dt} &= \varpi_1, & \frac{d\eta_2}{dt} &= \varpi_2, & \frac{d\eta_3}{dt} &= \varpi_3, \\ \frac{d\varpi_1}{dt} &= -\mu^2\eta_1, & \frac{d\varpi_2}{dt} &= -\mu^2\eta_2, & \frac{d\varpi_3}{dt} &= -g - \nu^2\eta_3. \end{aligned}\right\} \tag{112.}$$

These differential equations have for their rigorous integrals the six following,

$$\left.\begin{aligned} \eta_1 &= e_1 \cos \mu t + \frac{p_1}{\mu} \sin \mu t, \\ \eta_2 &= e_2 \cos \mu t + \frac{p_2}{\mu} \sin \mu t, \\ \eta_3 &= e_3 \cos \nu t + \frac{p_3}{\nu} \sin \nu t - \frac{g}{\nu^2} \operatorname{vers} \nu t, \end{aligned}\right\} \tag{113.}$$

and

$$\left.\begin{aligned}\varpi_1 &= p_1 \cos\mu t - \mu e_1 \sin\mu t,\\ \varpi_2 &= p_2 \cos\mu t - \mu e_2 \sin\mu t,\\ \varpi_3 &= p_3 \cos\nu t - \left(\nu e_3 + \frac{g}{\nu}\right)\sin\nu t;\end{aligned}\right\} \tag{114.}$$

$e_1\ e_2\ e_3\ p_1\ p_2\ p_3$ being still the initial values of $\eta_1\ \eta_2\ \eta_3\ \varpi_1\ \varpi_2\ \varpi_3$.

Employing these rigorous integral equations to calculate the function S, that is, by (85.) and (110.) (111.), the definite integral

$$S = \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} + U\right) dt, \tag{115.}$$

we find

$$\left.\begin{aligned}\tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) = \tfrac{1}{4}&\left\{p_1^2 + p_2^2 + p_3^2 + \mu^2(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\\ &+ \tfrac{1}{4}\{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\}\cos 2\mu t - \tfrac{1}{2}\mu(e_1 p_1 + e_2 p_2)\sin 2\mu t\\ &+ \tfrac{1}{4}\left\{p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\cos 2\nu t - \tfrac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right)p_3 \sin 2\nu t,\end{aligned}\right\} \tag{116.}$$

and

$$\left.\begin{aligned}U = \frac{g^2}{2\nu^2} - \tfrac{1}{4}&\left\{p_1^2 + p_2^2 + p_3^2 + \mu^2(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\\ &+ \tfrac{1}{4}\{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\}\cos 2\mu t - \tfrac{1}{2}\mu(e_1 p_1 + e_2 p_2)\sin 2\mu t\\ &+ \tfrac{1}{4}\left\{p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\cos 2\nu t - \tfrac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right)p_3 \sin 2\nu t,\end{aligned}\right\} \tag{117.}$$

and therefore,

$$\left.\begin{aligned}S = \frac{g^2 t}{2\nu^2} &+ \{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\}\frac{\sin 2\mu t}{4\mu} - \tfrac{1}{2}(e_1 p_1 + e_2 p_2)\operatorname{vers} 2\mu t\\ &+ \left\{p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\frac{\sin 2\nu t}{4\nu} - \tfrac{1}{2}p_3\left(e_3 + \frac{g}{\nu^2}\right)\operatorname{vers} 2\nu t.\end{aligned}\right\} \tag{118.}$$

In order, however, to express this function S, as supposed by our general method, in terms of the final and initial coordinates and of the time, we must employ the analogous expressions for the constants $p_1\ p_2\ p_3$, deduced from the integrals (113.), namely, the following:

$$\left.\begin{aligned}p_1 &= \frac{\mu\eta_1 - \mu e_1 \cos\mu t}{\sin\mu t},\\ p_2 &= \frac{\mu\eta_2 - \mu e_2 \cos\mu t}{\sin\mu t},\\ p_3 &= \frac{\nu\eta_3 + \frac{g}{\nu} - \left(\nu e_3 + \frac{g}{\nu}\right)\cos\nu t}{\sin\nu t};\end{aligned}\right\} \tag{119.}$$

and then we find

$$\left.\begin{aligned} S = \frac{g^2 t}{2\nu^2} + \frac{\mu}{2} \cdot \frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2}{\tan \mu t} + \frac{\nu}{2} \cdot \frac{(\eta_3 - e_3)^2}{\tan \nu t} \\ - \mu(\eta_1 e_1 + \eta_2 e_2) \tan \frac{\mu t}{2} - \nu \left(\eta_3 + \frac{g}{\nu^2}\right)\left(e_3 + \frac{g}{\nu^2}\right) \tan \frac{\nu t}{2}. \end{aligned}\right\} \qquad (120.)$$

This *principal function* S satisfies the following pair of partial differential equations of the first order, of the kind (86.),

$$\left.\begin{aligned} \frac{\delta S}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S}{\delta \eta_1}\right)^2 + \left(\frac{\delta S}{\delta \eta_2}\right)^2 + \left(\frac{\delta S}{\delta \eta_3}\right)^2\right\} = -g\eta_3 - \frac{\mu^2}{2}(\eta_1^2 + \eta_2^2) - \frac{\nu^2}{2}\eta_3^2, \\ \frac{\delta S}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S}{\delta e_1}\right)^2 + \left(\frac{\delta S}{\delta e_2}\right)^2 + \left(\frac{\delta S}{\delta e_3}\right)^2\right\} = -g e_3 - \frac{\mu^2}{2}(e_1^2 + e_2^2) - \frac{\nu^2}{2}e_3^2, \end{aligned}\right\} \qquad (121.)$$

and *if its form had been previously found*, by the help of this pair, or in any other way, *the integrals of the equations of motion might (by our general method) have been deduced from it*, under the forms,

$$\left.\begin{aligned} \varpi_1 &= \frac{\delta S}{\delta \eta_1} = \mu(\eta_1 - e_1) \operatorname{cotan} \mu t - \mu e_1 \tan \frac{\mu t}{2}, \\ \varpi_2 &= \frac{\delta S}{\delta \eta_2} = \mu(\eta_2 - e_2) \operatorname{cotan} \mu t - \mu e_2 \tan \frac{\mu t}{2}, \\ \varpi_3 &= \frac{\delta S}{\delta \eta_3} = \nu(\eta_3 - e_3) \operatorname{cotan} \nu t - \left(\nu e_3 + \frac{g}{\nu}\right) \tan \frac{\nu t}{2}, \end{aligned}\right\} \qquad (122.)$$

and

$$\left.\begin{aligned} p_1 &= -\frac{\delta S}{\delta e_1} = \mu(\eta_1 - e_1) \operatorname{cotan} \mu t + \mu \eta_1 \tan \frac{\mu t}{2}, \\ p_2 &= -\frac{\delta S}{\delta e_2} = \mu(\eta_2 - e_2) \operatorname{cotan} \mu t + \mu \eta_2 \tan \frac{\mu t}{2}, \\ p_3 &= -\frac{\delta S}{\delta e_3} = \nu(\eta_3 - e_3) \operatorname{cotan} \nu t + \left(\nu \eta_3 + \frac{g}{\nu}\right) \tan \frac{\nu t}{2} : \end{aligned}\right\} \qquad (123.)$$

the last of these two sets of equations coinciding with the set (119.), or (113.), and conducting, when combined with the first set, (122.), to the other former set of integrals, (114.).

25. Suppose now, to illustrate the theory of perturbation, that the constants μ, ν are small, and that, after separating the expression (111.) for H into two parts,

$$H_1 = \tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g\eta_3, \qquad (124.)$$

and

$$H_2 = \tfrac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2 \eta_3^2\}, \qquad (125.)$$

we suppress at first the small part H_2, and so form, by (88.), these other and simpler differential equations of a motion which we shall call *undisturbed*:

$$\left.\begin{array}{lll} \dfrac{d\eta_1}{dt} = \varpi_1, & \dfrac{d\eta_2}{dt} = \varpi_2, & \dfrac{d\eta_3}{dt} = \varpi_3, \\ \dfrac{d\varpi_1}{dt} = 0, & \dfrac{d\varpi_2}{dt} = 0, & \dfrac{d\varpi_3}{dt} = -g. \end{array}\right\} \tag{126.}$$

These new equations have for their rigorous integrals, of the forms (94.) and (95.),

$$\eta_1 = e_1 + p_1 t, \quad \eta_2 = e_2 + p_2 t, \quad \eta_3 = e_3 + p_3 t - \tfrac{1}{2} g t^2, \tag{127.}$$

and

$$\varpi_1 = p_1, \quad \varpi_2 = p_2, \quad \varpi_3 = p_3 - gt; \tag{128.}$$

and the *principal function* S_1 of the same undisturbed motion is, by (89.),

$$\left.\begin{aligned} S_1 &= \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} - g\eta_3 \right) dt \\ &= \int_0^t \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - ge_3 - 2gp_3 t + g^2 t^2 \right) dt \\ &= \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - ge_3 \right) t - gp_3 t^2 + \tfrac{1}{3} g^2 t^3, \end{aligned}\right\} \tag{129.}$$

or finally, by (127.),

$$S_1 = \frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2 + (\eta_3 - e_3)^2}{2t} - \tfrac{1}{2} gt(\eta_3 + e_3) - \tfrac{1}{24} g^2 t^3. \tag{130.}$$

This function satisfies, as it ought, the following pair of partial differential equations,

$$\left.\begin{aligned} \frac{\delta S_1}{\delta t} + \frac{1}{2}\left\{ \left(\frac{\delta S_1}{\delta \eta_1}\right)^2 + \left(\frac{\delta S_1}{\delta \eta_2}\right)^2 + \left(\frac{\delta S_1}{\delta \eta_3}\right)^2 \right\} &= -g\eta_3, \\ \frac{\delta S_1}{\delta t} + \frac{1}{2}\left\{ \left(\frac{\delta S_1}{\delta e_1}\right)^2 + \left(\frac{\delta S_1}{\delta e_2}\right)^2 + \left(\frac{\delta S_1}{\delta e_3}\right)^2 \right\} &= -ge_3. \end{aligned}\right\} \tag{131.}$$

And if, by the help of this pair, or in any other way, the form (130.) of this *principal function* S_1 had been found, the integral equations (127.) and (128.) might have been deduced from it, by our general method, as follows:

$$\left.\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} = \frac{\eta_1 - e_1}{t}, \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} = \frac{\eta_2 - e_2}{t}, \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} = \frac{\eta_3 - e_3}{t} - \tfrac{1}{2} gt, \end{aligned}\right\} \tag{132.}$$

and

$$\left.\begin{aligned}
p_1 &= -\frac{\delta S_1}{\delta e_1} = \frac{\eta_1 - e_1}{t},\\
p_2 &= -\frac{\delta S_1}{\delta e_2} = \frac{\eta_2 - e_2}{t},\\
p_3 &= -\frac{\delta S_1}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \tfrac{1}{2}gt,
\end{aligned}\right\} \tag{133.}$$

the latter of these two sets coinciding with (127.), and the former set conducting to (128.).

26. Returning now from this simpler motion to the more complex motion first mentioned, and denoting by S_2 the *disturbing part* or function which must be added to S_1 in order to make up the whole principal function S of that more complex motion; we have, by applying our general method, the following rigorous expression for this disturbing function,

$$S_2 = -\int_0^t H_2\,dt + \int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_2}{\delta\eta_1}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_2}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_3}\right)^2\right\} dt, \tag{134.}$$

in which we may, approximately, neglect the second definite integral, and calculate the first by the help of the equations of undisturbed motion. In this manner we find, approximately, by (125.), (127.),

$$-H_2 = -\frac{\mu^2}{2}\{(e_1 + p_1 t)^2 + (e_2 + p_2 t)^2\} - \frac{\nu^2}{2}(e_3 + p_3 t - \tfrac{1}{2}gt^2)^2, \tag{135.}$$

and therefore, by integration,

$$\left.\begin{aligned}
S_2 = {}& -\tfrac{1}{2}\{\mu^2(e_1^2 + e_2^2) + \nu^2 e_3^2\}t - \tfrac{1}{2}\{\mu^2(e_1 p_1 + e_2 p_2) + \nu^2 e_3 p_3\}t^2\\
& - \tfrac{1}{6}\{\mu^2(p_1^2 + p_2^2) + \nu^2(p_3^2 - g e_3)\}t^3 + \tfrac{1}{8}\nu^2 g p_3 t^4 - \tfrac{1}{40}\nu^2 g^2 t^5,
\end{aligned}\right\} \tag{136.}$$

or, by (133.),

$$\left.\begin{aligned}
S_2 = {}& -\frac{\mu^2 t}{6}(\eta_1^2 + e_1\eta_1 + e_1^2 + \eta_2^2 + e_2\eta_2 + e_2^2)\\
& - \frac{\nu^2 t}{6}\{\eta_3^2 + e_3\eta_3 + e_3^2 + \tfrac{1}{4}g(\eta_3 + e_3)t^2 + \tfrac{1}{40}g^2 t^4\} :
\end{aligned}\right\} \tag{137.}$$

the error being of the fourth order, with respect to the small quantities μ, ν. And neglecting this small error, we can deduce, by our general method, approximate forms for the integrals of the equations of disturbed motion, from the corrected function $S_1 + S_2$, as follows:

$$\left.\begin{aligned}
\varpi_1 &= \frac{\delta S_1}{\delta\eta_1} + \frac{\delta S_2}{\delta\eta_1} = \frac{\eta_1 - e_1}{t} - \frac{\mu^2 t}{3}(\eta_1 + \tfrac{1}{2}e_1),\\
\varpi_2 &= \frac{\delta S_1}{\delta\eta_2} + \frac{\delta S_2}{\delta\eta_2} = \frac{\eta_2 - e_2}{t} - \frac{\mu^2 t}{3}(\eta_2 + \tfrac{1}{2}e_2),\\
\varpi_3 &= \frac{\delta S_1}{\delta\eta_3} + \frac{\delta S_2}{\delta\eta_3} = \frac{\eta_3 - e_3}{t} - \tfrac{1}{2}gt - \frac{\nu^2 t}{3}(\eta_3 + \tfrac{1}{2}e_3 + \tfrac{1}{8}gt^2);
\end{aligned}\right\} \tag{138.}$$

and

$$\left.\begin{aligned}
p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1} = \frac{\eta_1 - e_1}{t} + \frac{\mu^2 t}{3}(e_1 + \tfrac{1}{2}\eta_1),\\
p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2} = \frac{\eta_2 - e_2}{t} + \frac{\mu^2 t}{3}(e_2 + \tfrac{1}{2}\eta_2),\\
p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \tfrac{1}{2}gt + \frac{\nu^2 t}{3}(e_3 + \tfrac{1}{2}\eta_3 + \tfrac{1}{8}gt^2);
\end{aligned}\right\} \tag{139.}$$

or, in the same order of approximation,

$$\left.\begin{aligned}
\eta_1 &= e_1 + p_1 t - \tfrac{1}{2}\mu^2 t^2 (e_1 + \tfrac{1}{3}p_1 t),\\
\eta_2 &= e_2 + p_2 t - \tfrac{1}{2}\mu^2 t^2 (e_2 + \tfrac{1}{3}p_2 t),\\
\eta_3 &= e_3 + p_3 t - \tfrac{1}{2}gt^2 - \tfrac{1}{2}\nu^2 t^2 (e_3 + \tfrac{1}{3}p_3 t - \tfrac{1}{12}gt^2),
\end{aligned}\right\} \tag{140.}$$

and

$$\left.\begin{aligned}
\varpi_1 &= p_1 - \mu^2 t(e_1 + \tfrac{1}{2}p_1 t),\\
\varpi_2 &= p_2 - \mu^2 t(e_2 + \tfrac{1}{2}p_2 t),\\
\varpi_3 &= p_3 - gt - \nu^2 t(e_3 + \tfrac{1}{2}p_3 t - \tfrac{1}{6}gt^2).
\end{aligned}\right\} \tag{141.}$$

Accordingly, if we develope the rigorous integrals of disturbed motion, (113.) and (114.), as far as the squares (inclusive) of the small quantities μ and ν, we are conducted to these approximate integrals; and if we develope the rigorous expression (120.) for the principal function of such motion, to the same degree of accuracy, we obtain the sum of the two expressions (130.) and (137.).

27. To illustrate still further, in the present example, our general method of successive approximation, let S_3 denote the small unknown correction of the approximate expression (137.), so that we shall now have, rigorously, for the present disturbed motion,

$$S = S_1 + S_2 + S_3, \tag{142.}$$

S_1 and S_2 being here determined rigorously by (130.) and (137). Then, substituting $S_1 + S_2$ for S_1 in the general transformation (87.), we find, rigorously, in the present question,

$$\left.\begin{aligned}
S_3 = &-\int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_2}{\delta\eta_1}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_2}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_3}\right)^2\right\} dt\\
&+\int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_3}{\delta\eta_1}\right)^2 + \left(\frac{\delta S_3}{\delta\eta_2}\right)^2 + \left(\frac{\delta S_3}{\delta\eta_3}\right)^2\right\} dt :
\end{aligned}\right\} \tag{143.}$$

and if we neglect only terms of the eighth and higher dimensions with respect to the small quantities μ, ν, we may confine ourselves to the first of these two definite integrals, and may employ, in calculating it, the approximate expressions (140.) for the coordinates of disturbed motion. In this manner we obtain the very approximate expression,

$$\left.\begin{aligned} S_3 = &-\frac{\mu^4}{18}\int_0^t t^2\{(\eta_1 + \tfrac{1}{2}e_1)^2 + (\eta_2 + \tfrac{1}{2}e_2)^2\}dt \\ &-\frac{\nu^4}{18}\int_0^t t^2(\eta_3 + \tfrac{1}{2}e_3 + \tfrac{1}{8}gt^2)^2 dt \\ = &-\frac{\mu^4 t^3}{360}(4\eta_1^2 + 7\eta_1 e_1 + 4e_1^2 + 4\eta_2^2 + 7\eta_2 e_2 + 4e_2^2) \\ &-\frac{\nu^4 t^3}{360}(4\eta_3^2 + 7\eta_3 e_3 + 4e_3^2) - \frac{\nu^4 g t^5}{240}(\eta_3 + e_3) - \frac{17\nu^4 g^2 t^7}{40320} \\ &-\frac{\mu^6 t^5}{945}(\eta_1^2 + \tfrac{31}{16}\eta_1 e_1 + e_1^2 + \eta_2^2 + \tfrac{31}{16}\eta_2 e_2 + e_2^2) \\ &-\frac{\nu^6 t^5}{945}(\eta_3^2 + \tfrac{31}{16}\eta_3 e_3 + e_3^2) - \frac{17\nu^6 g t^7}{40320}(\eta_3 + e_3) - \frac{31\nu^6 g^2 t^9}{725760}; \end{aligned}\right\} \tag{144.}$$

which is accordingly the sum of the tems of the fourth and sixth dimensions in the development of the rigorous expression (120.), and gives, by our general method, correspondingly approximate expressions for the integrals of disturbed motion, under the forms

$$\left.\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1} + \frac{\delta S_3}{\delta \eta_1}, \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2} + \frac{\delta S_3}{\delta \eta_2}, \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3} + \frac{\delta S_3}{\delta \eta_3}, \end{aligned}\right\} \tag{145.}$$

and

$$\left.\begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1} - \frac{\delta S_3}{\delta e_1}, \\ p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2} - \frac{\delta S_3}{\delta e_2}, \\ p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3} - \frac{\delta S_3}{\delta e_3}. \end{aligned}\right\} \tag{146.}$$

28. To illustrate by the same example the theory of gradually varying elements, let us establish the following definitions, for the present disturbed motion,

$$\left.\begin{aligned} &\kappa_1 = \eta_1 - \varpi_1 t, \quad &&\kappa_2 = \eta_2 - \varpi_2 t, \quad &&\kappa_3 = \eta_3 - \varpi_3 t - \tfrac{1}{2}gt^2, \\ &\lambda_1 = \varpi_1, &&\lambda_2 = \varpi_2, &&\lambda_3 = \varpi_3 + gt, \end{aligned}\right\} \tag{147.}$$

and let us call these six quantities κ_1 κ_2 κ_3 λ_1 λ_2 λ_3 the *varying elements* of that motion, by analogy to the six constant quantities e_1 e_2 e_3 p_1 p_2 p_3, which may, for the undisturbed motion, be represented in a similar way, namely, by (127.) and (128.),

$$\left.\begin{aligned} &e_1 = \eta_1 - \varpi_1 t, \quad &&e_2 = \eta_2 - \varpi_2 t, \quad &&e_3 = \eta_3 - \varpi_3 t - \tfrac{1}{2}gt^2, \\ &p_1 = \varpi_1, &&p_2 = \varpi_2, &&p_3 = \varpi_3 + gt. \end{aligned}\right\} \tag{148.}$$

We shall then have rigorously, for the six disturbed variables $\eta_1\ \eta_2\ \eta_3\ \varpi_1\ \varpi_2\ \varpi_3$, expressions of the same forms as in the integrals (127.) and (128.) of undisturbed motion, but with variable instead of constant elements, namely, the following:

$$\left.\begin{array}{lll} \eta_1 = \kappa_1 + \lambda_1 t, & \eta_2 = \kappa_2 + \lambda_2 t, & \eta_3 = \kappa_3 + \lambda_3 t - \frac{1}{2}gt^2, \\ \varpi_1 = \lambda_1, & \varpi_2 = \lambda_2, & \varpi_3 = \lambda_3 - gt; \end{array}\right\} \qquad (149.)$$

and the rigorous determination of the six varying elements $\kappa_1\ \kappa_2\ \kappa_3\ \lambda_1\ \lambda_2\ \lambda_3$, as functions of the time and of their own initial values $e_1\ e_2\ e_3\ p_1\ p_2\ p_3$, depends on the integration of the 6 following equations, in ordinary differentials of the first order, of the forms (105.):

$$\left.\begin{array}{l} \dfrac{d\kappa_1}{dt} = \dfrac{\delta H_2}{\delta\lambda_1} = \mu^2 t(\kappa_1 + \lambda_1 t), \\ \dfrac{d\kappa_2}{dt} = \dfrac{\delta H_2}{\delta\lambda_2} = \mu^2 t(\kappa_2 + \lambda_2 t), \\ \dfrac{d\kappa_3}{dt} = \dfrac{\delta H_2}{\delta\lambda_3} = \nu^2 t(\kappa_3 + \lambda_3 t - \frac{1}{2}gt^2), \end{array}\right\} \qquad (150.)$$

and

$$\left.\begin{array}{l} \dfrac{d\lambda_1}{dt} = -\dfrac{\delta H_2}{\delta\kappa_1} = -\mu^2(\kappa_1 + \lambda_1 t), \\ \dfrac{d\lambda_2}{dt} = -\dfrac{\delta H_2}{\delta\kappa_2} = -\mu^2(\kappa_2 + \lambda_2 t), \\ \dfrac{d\lambda_3}{dt} = -\dfrac{\delta H_2}{\delta\kappa_3} = -\nu^2(\kappa_3 + \lambda_3 t - \frac{1}{2}gt^2), \end{array}\right\} \qquad (151.)$$

H_2 being here the expression

$$H_2 = \frac{\mu^2}{2}\{(\kappa_1 + \lambda_1 t)^2 + (\kappa_2 + \lambda_2 t)^2\} + \frac{\nu^2}{2}(\kappa_3 + \lambda_3 t - \tfrac{1}{2}gt^2)^2, \qquad (152.)$$

which is obtained from (125.) by substituting for the disturbed coordinates $\eta_1\ \eta_2\ \eta_3$ their values (149.), as functions of the varying elements and of the time. It is not difficult to integrate rigorously this system of equations (150.) and (151.); and we shall soon have occasion to state their complete and accurate integrals: but we shall continue for a while to treat these rigorous integrals as unknown, that we may take this oppportunity to exemplify our general method of indefinite approximation, for all such dynamical systems, founded on the properties of the *functions of elements* C and E. Of these two functions either may be employed, and we shall use here the function C.

29. This function, by (109.) and (152.), may rigorously be expressed as follows:

$$C = \frac{\mu^2}{2}\int_0^t (\lambda_1^2 t^2 - \kappa_1^2 + \lambda_2^2 t^2 - \kappa_2^2)\,dt + \frac{\nu^2}{2}\int_0^t \{(\lambda_3 t - \tfrac{1}{2}gt^2)^2 - \kappa_3^2\}\,dt; \qquad (153.)$$

and has therefore the following for a first approximate value, obtained by treating the elements κ_1 κ_2 κ_3 λ_1 λ_2 λ_3 as constant and equal to their initial values e_1 e_2 e_3 p_1 p_2 p_3,

$$\left.\begin{aligned} C = -\frac{t}{2}\{\mu^2(e_1^2+e_2^2)+\nu^2 e_3^2\} + \frac{t^3}{6}\{\mu^2(p_1^2+p_2^2)+\nu^2 p_3^2\} \\ -\frac{t^4}{8}\nu^2 g p_3 + \frac{t^5}{40}\nu^2 g^2. \end{aligned}\right\} \tag{154.}$$

In like manner we have, as first approximations, of the kind expressed by the general formula (Z^1.), the following results deduced from the equations (151.),

$$\left.\begin{aligned} \lambda_1 &= p_1 - \mu^2(e_1 t + \tfrac{1}{2}p_1 t^2), \\ \lambda_2 &= p_2 - \mu^2(e_2 t + \tfrac{1}{2}p_2 t^2), \\ \lambda_3 &= p_3 - \nu^2(e_3 t + \tfrac{1}{2}p_3 t^2 - \tfrac{1}{6}g t^3), \end{aligned}\right\} \tag{155.}$$

and therefore, as approximations of the same kind,

$$\left.\begin{aligned} e_1 &= -\tfrac{1}{2}p_1 t - \frac{\lambda_1 - p_1}{\mu^2 t}, \\ e_2 &= -\tfrac{1}{2}p_2 t - \frac{\lambda_2 - p_2}{\mu^2 t}, \\ e_3 &= -\tfrac{1}{2}p_3 t + \tfrac{1}{6}g t^2 - \frac{\lambda_3 - p_3}{\nu^2 t}. \end{aligned}\right\} \tag{156.}$$

Substituting these values for the initial constants e_1 e_2 e_3 in the approximate value (154.) for the function of elements C, we obtain the following approximate expression C_1 for that function, of the form supposed by our theory:

$$\left.\begin{aligned} C_1 = -\frac{1}{2t}\left\{\frac{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2}{\mu^2} + \frac{(\lambda_3-p_3)^2}{\nu^2}\right\} \\ -\frac{t}{2}\{(\lambda_1-p_1)p_1 + (\lambda_2-p_2)p_2 + (\lambda_3-p_3)(p_3 - \tfrac{1}{3}g t)\} \\ +\frac{t^3}{24}\{\mu^2(p_1^2+p_2^2)+\nu^2 p_3^2\} - \frac{t^4}{24}\nu^2 g p_3 + \frac{t^5}{90}\nu^2 g^2. \end{aligned}\right\} \tag{157.}$$

The rigorous function C must satisfy, in the present question, by the principles of the eighteenth number, the partial differential equation,

$$\frac{\delta C}{\delta t} = \frac{\mu^2}{2}\left\{\left(\frac{\delta C}{\delta \lambda_1} + \lambda_1 t\right)^2 + \left(\frac{\delta C}{\delta \lambda_2} + \lambda_2 t\right)^2\right\} + \frac{\nu^2}{2}\left(\frac{\delta C}{\delta \lambda_3} + \lambda_3 t - \tfrac{1}{2}g t^2\right)^2; \tag{158.}$$

and if it be put under the form (U^1.),

$$C = C_1 + C_2,$$

C_1 being a first approximation, supposed to vanish with the time, then the correction C_2 must satisfy rigorously the condition

$$\left.\begin{aligned}C_2 = \int_0^t \Big\{ -\frac{\delta C_1}{\delta t} + \frac{\mu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_1} + \lambda_1 t\right)^2 + \frac{\mu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_2} + \lambda_2 t\right)^2 \\ + \frac{\nu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_3} + \lambda_3 t - \tfrac{1}{2} g t^2\right)^2 \Big\}\, dt \\ - \frac{1}{2}\int_0^t \left\{ \mu^2 \left(\frac{\delta C_2}{\delta \lambda_1}\right)^2 + \mu^2 \left(\frac{\delta C_2}{\delta \lambda_2}\right)^2 + \nu^2 \left(\frac{\delta C_2}{\delta \lambda_3}\right)^2 \right\} dt.\end{aligned}\right\} \tag{159.}$$

In passing to a second approximation we may neglect the second definite integral, and may calculate the first with the help of the approximate equations (155.); which give, in this manner,

$$\left.\begin{aligned}C_2 = &-\int_0^t \{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2\}\, dt \\ &+ \frac{\mu^2}{2}\int_0^t \{\lambda_1(\lambda_1 - p_1) + \lambda_2(\lambda_2 - p_2)\}\, t^2\, dt \\ &+ \frac{\nu^2}{2}\int_0^t (\lambda_3 - \tfrac{2}{3} g t)(\lambda_3 - p_3)\, t^2\, dt \\ = &-\frac{t}{3}\{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2\} \\ &+ \frac{t^3}{24}\{\mu^2 p_1(\lambda_1 - p_1) + \mu^2 p_2(\lambda_2 - p_2) + \nu^2 p_3(\lambda_3 - p_3)\} \\ &- \frac{t^4}{45}\nu^2 g(\lambda_3 - p_3) + \frac{t^5}{240}(\mu^4 p_1^2 + \mu^4 p_2^2 + \nu^4 p_3^2) \\ &- \frac{t^6}{240}\nu^4 g p_3 + \frac{t^7}{945}\nu^4 g^2.\end{aligned}\right\} \tag{160.}$$

We might improve this second approximation in like manner, by calculating a new definite integral C_3, with the help of the following more approximate forms for the relations between the varying elements λ_1 λ_2 λ_3 and the initial constants, deduced by our general method:

$$\left.\begin{aligned}e_1 &= -\frac{\delta C_1}{\delta p_1} - \frac{\delta C_2}{\delta p_1} = -\frac{\lambda_1 - p_1}{\mu^2 t}\left(1 + \frac{\mu^2 t^2}{6} + \frac{\mu^4 t^4}{24}\right) - \frac{t p_1}{2}\left(1 + \frac{\mu^2 t^2}{12} + \frac{\mu^4 t^4}{60}\right), \\ e_2 &= -\frac{\delta C_1}{\delta p_2} - \frac{\delta C_2}{\delta p_2} = -\frac{\lambda_2 - p_2}{\mu^2 t}\left(1 + \frac{\mu^2 t^2}{6} + \frac{\mu^4 t^4}{24}\right) - \frac{t p_2}{2}\left(1 + \frac{\mu^2 t^2}{12} + \frac{\mu^4 t^4}{60}\right), \\ e_3 &= -\frac{\delta C_1}{\delta p_3} - \frac{\delta C_2}{\delta p_3} = -\frac{\lambda_3 - p_3}{\nu^2 t}\left(1 + \frac{\nu^2 t^2}{6} + \frac{\nu^4 t^4}{24}\right) - \frac{t p_3}{2}\left(1 + \frac{\nu^2 t^2}{12} + \frac{\nu^4 t^4}{60}\right) \\ &\qquad + \frac{g t^2}{6}\left(1 + \frac{7\nu^2 t^2}{60} + \frac{\nu^4 t^4}{40}\right);\end{aligned}\right\} \tag{161.}$$

in which we can only depend on the terms as far as the second order, but which acquire a correctness of the fourth order when cleared of the small divisors, and give then

$$\left.\begin{aligned}
\lambda_1 &= p_1 - \mu^2 t(e_1 + \tfrac{1}{2}p_1 t) + \tfrac{1}{6}\mu^4 t^3 (e_1 + \tfrac{1}{4}p_1 t),\\
\lambda_2 &= p_2 - \mu^2 t(e_2 + \tfrac{1}{2}p_2 t) + \tfrac{1}{6}\mu^4 t^3 (e_2 + \tfrac{1}{4}p_2 t),\\
\lambda_3 &= p_3 - \nu^2 t(e_3 + \tfrac{1}{2}p_3 t - \tfrac{1}{6}g t^2) + \tfrac{1}{6}\nu^4 t^3 (e_3 + \tfrac{1}{4}p_3 t - \tfrac{1}{20}g t^2).
\end{aligned}\right\} \tag{162.}$$

But a little attention to the nature of this process shows that all the successive corrections to which it conducts can be only rational and integer and homogeneous functions, of the second dimension, of the quantities λ_1 λ_2 λ_3 p_1 p_2 p_3 g, and that they may all be put under the following form, which is therefore the form of their sum, or of the whole sought function C;

$$\left.\begin{aligned}
C = \ & \mu^{-2} a_\mu (\lambda_1 - p_1)^2 + b_\mu p_1 (\lambda_1 - p_1) + \mu^2 c_\mu p_1^2\\
& + \mu^{-2} a_\mu (\lambda_2 - p_2)^2 + b_\mu p_2 (\lambda_2 - p_2) + \mu^2 c_\mu p_2^2\\
& + \nu^{-2} a_\nu (\lambda_3 - p_3)^2 + b_\nu p_3 (\lambda_3 - p_3) + \nu^2 c_\nu p_3^2\\
& + f_\nu g (\lambda_3 - p_3) + \nu^2 h_\nu g p_3 + \nu^2 i_\nu g^2,
\end{aligned}\right\} \tag{163.}$$

the cofficients a_μ a_ν, &c. being functions of the small quantities μ, ν, and also of the time, of which it remains to discover the forms. Denoting therefore their differentials, taken with respect to the time, as follows,

$$da_\mu = a'_\mu \, dt, \quad da_\nu = a'_\nu dt, \ \text{\&c.}, \tag{164.}$$

and substituting the expression (163.) in the rigorous partial differential equation (158.), we are conducted to the six following equations in ordinary differentials of the first order:

$$\left.\begin{aligned}
2a'_\nu &= (2a_\nu + \nu^2 t)^2; & b'_\nu &= (2a_\nu + \nu^2 t)(b_\nu + t); & c'_\nu &= \tfrac{1}{2}(b_\nu + t)^2;\\
f'_\nu &= (2a_\nu + \nu^2 t)(f_\nu - \tfrac{1}{2}t^2); & h'_\nu &= (b_\nu + t)(f_\nu - \tfrac{1}{2}t^2); & i'_\nu &= \tfrac{1}{2}(f_\nu - \tfrac{1}{2}t^2)^2;
\end{aligned}\right\} \tag{165.}$$

along with the 6 following conditions, to determine the 6 arbitrary constants introduced by integration,

$$a_0 = -\frac{1}{2t}; \quad b_0 = -\frac{t}{2}; \quad f_0 = \frac{t^2}{6}; \quad c_0 = \frac{t^3}{24}; \quad h_0 = -\frac{t^4}{24}; \quad i_0 = \frac{t^5}{90}. \tag{166.}$$

In this manner we find, without difficulty, observing that a_μ b_μ c_μ may be formed from a_ν b_ν c_ν by changing ν to μ,

$$\left.\begin{aligned}
a_\nu &= -\tfrac{1}{2}\nu^2 t - \tfrac{1}{2}\nu \operatorname{cotan} \nu t, & a_\mu &= -\tfrac{1}{2}\mu^2 t - \tfrac{1}{2}\mu \operatorname{cotan} \mu t,\\
b_\nu &= -t + \frac{1}{\nu}\tan\frac{\nu t}{2}, & b_\mu &= -t + \frac{1}{\mu}\tan\frac{\mu t}{2},\\
c_\nu &= -\frac{t}{2\nu^2} + \frac{1}{\nu^3}\tan\frac{\nu t}{2}, & c_\mu &= -\frac{t}{2\mu^2} + \frac{1}{\mu^3}\tan\frac{\mu t}{2},\\
f_\nu &= \tfrac{1}{2}t^2 - \frac{1}{\nu^2} + \frac{t}{\nu}\operatorname{cotan}\nu t, & &\\
h_\nu &= \frac{t^2}{2\nu^2} - \frac{t}{\nu^3}\tan\frac{\nu t}{2}, & &\\
i_\nu &= \frac{t}{2\nu^4} - \frac{t^3}{6\nu^2} - \frac{t^2}{2\nu^3}\operatorname{cotan}\nu t. & &
\end{aligned}\right\} \tag{167.}$$

The form of the function C is therefore entirely known, and we have for this *function of elements* the following rigorous expression,

$$\left.\begin{aligned}
C = & -\frac{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2}{2\mu \tan \mu t} - \frac{(\lambda_3 - p_3)^2}{2\nu \tan \nu t} \\
& - \frac{t}{2}\{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2\} \\
& - t\{p_1(\lambda_1 - p_1) + p_2(\lambda_2 - p_2) + p_3(\lambda_3 - p_3)\} \\
& + \frac{1}{\mu}\{p_1(\lambda_1 - p_1) + p_2(\lambda_2 - p_2)\} \tan \frac{\mu t}{2} + \frac{1}{\nu} p_3(\lambda_3 - p_3) \tan \frac{\nu t}{2} \\
& - \frac{t}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{\mu}(p_1^2 + p_2^2) \tan \frac{\mu t}{2} + \frac{1}{\nu} p_3^2 \tan \frac{\nu t}{2} \\
& + \left(\frac{t^2}{2} - \frac{1}{\nu^2} + \frac{t}{\nu} \operatorname{cotan} \nu t\right) g(\lambda_3 - p_3) + \left(\frac{t^2}{2} - \frac{t}{\nu} \tan \frac{\nu t}{2}\right) g p_3 \\
& + \left(\frac{t}{2\nu^2} - \frac{t^2}{6} - \frac{t^2}{2\nu} \operatorname{cotan} \nu t\right) g^2,
\end{aligned}\right\} \tag{168.}$$

which may be variously transformed, and gives by our general method the following systems of rigorous integrals of the differential equations of varying elements, (150.), (151.):

$$\left.\begin{aligned}
e_1 &= -\frac{\delta C}{\delta p_1} = -\frac{\lambda_1 - p_1}{\mu \sin \mu t} - \frac{p_1}{\mu} \tan \frac{\mu t}{2}, \\
e_2 &= -\frac{\delta C}{\delta p_2} = -\frac{\lambda_2 - p_2}{\mu \sin \mu t} - \frac{p_2}{\mu} \tan \frac{\mu t}{2}, \\
e_3 &= -\frac{\delta C}{\delta p_3} = -\frac{\lambda_3 - p_3}{\nu \sin \nu t} - \frac{p_3}{\nu} \tan \frac{\nu t}{2} + \frac{g}{\nu}\left(\frac{t}{\sin \nu t} - \frac{1}{\nu}\right),
\end{aligned}\right\} \tag{169.}$$

and

$$\left.\begin{aligned}
\kappa_1 &= \frac{\delta C}{\delta \lambda_1} = -(\lambda_1 - p_1)\left(t + \frac{1}{\mu} \operatorname{cotan} \mu t\right) + p_1\left(-t + \frac{1}{\mu} \tan \frac{\mu t}{2}\right), \\
\kappa_2 &= \frac{\delta C}{\delta \lambda_2} = -(\lambda_2 - p_2)\left(t + \frac{1}{\mu} \operatorname{cotan} \mu t\right) + p_2\left(-t + \frac{1}{\mu} \tan \frac{\mu t}{2}\right), \\
\kappa_3 &= \frac{\delta C}{\delta \lambda_3} = -(\lambda_3 - p_3)\left(t + \frac{1}{\nu} \operatorname{cotan} \nu t\right) + p_3\left(-t + \frac{1}{\nu} \tan \frac{\nu t}{2}\right) \\
& \qquad + g\left(\frac{t^2}{2} - \frac{1}{\nu^2} + \frac{t}{\nu} \operatorname{cotan} \nu t\right);
\end{aligned}\right\} \tag{170.}$$

that is,

$$\left.\begin{aligned}
\lambda_1 &= p_1 \cos \mu t - e_1 \mu \sin \mu t, \\
\lambda_2 &= p_2 \cos \mu t - e_2 \mu \sin \mu t, \\
\lambda_3 &= p_3 \cos \nu t - e_3 \nu \sin \nu t + g\left(t - \frac{1}{\nu} \sin \nu t\right),
\end{aligned}\right\} \tag{171.}$$

and

$$
\left.
\begin{aligned}
\kappa_1 &= e_1(\cos\mu t + \mu t \sin\mu t) + p_1\left(\frac{1}{\mu}\sin\mu t - t\cos\mu t\right), \\
\kappa_2 &= e_2(\cos\mu t + \mu t \sin\mu t) + p_2\left(\frac{1}{\mu}\sin\mu t - t\cos\mu t\right), \\
\kappa_3 &= e_3(\cos\nu t + \nu t \sin\nu t) + p_3\left(\frac{1}{\nu}\sin\nu t - t\cos\nu t\right) \\
&\qquad - g\left(\frac{\operatorname{vers}\nu t}{\nu^2} - \frac{t}{\nu}\sin\nu t + \frac{t^2}{2}\right).
\end{aligned}
\right\} \tag{172.}
$$

Accordingly, these rigorous expressions for the 6 varying elements, in the present dynamical question, agree with the results obtained by the ordinary methods of integration from the 6 ordinary differential equations (150.) and (151.), and with those obtained by elimination from the equations (113.), (114.), (147.).

Remarks on the foregoing Example.

30. The example which has occupied us in the last six numbers is not altogether ideal, but is realised to some extent by the motion of a projectile in a void. For if we consider the earth as a sphere, of radius R, and suppose the accelerating force of gravity to vary inversely as the square of the distance r from the centre, and to be $= g$ at the surface, this force will be represented generally by $\dfrac{gR^2}{r^2}$; and to adapt the differential equations (78.) to the motion of a projectile in a void, it will be sufficient to make

$$
U = gR^2\left(\frac{1}{r} - \frac{1}{R}\right) \tag{173.}
$$

If we place the origin of rectangular coordinates at the earth's surface, and suppose the semiaxis of $+z$ to be directed vertically upwards, we shall have

$$
r = \sqrt{(R+z)^2 + x^2 + y^2}, \tag{174.}
$$

and

$$
U = -gz + \frac{gz^2}{R} - \frac{g(x^2+y^2)}{2R}, \tag{175.}
$$

neglecting only those very small terms which have the square of the earth's radius for a divisor: neglecting therefore such terms, the force-function U in this question is of that form (110.) on which all the reasonings of the example have been founded; the small constants μ, ν, being the real and imaginary quantities $\sqrt{\dfrac{g}{R}}$, $\sqrt{\dfrac{-2g}{R}}$, respectively. We may therefore apply the results of the recent numbers to the motions of projectiles in a void, by substituting these values for the constants, and altering, where necessary, trigonometric to exponential functions. But besides the theoretical facility and the little practical importance of researches respecting such projectiles, the results would only be accurate as far as the first negative power (inclusive) of the earth's radius, because the expression (110.) for the force-function U is only accurate only so far; and therefore the rigorous and approximate investigations of the six preceding numbers, founded on that expression, are offered only as mathematical illustrations of a general *method*, extending to all problems of dynamics, at least to all those to which the law of living forces applies.

Attracting Systems resumed: Differential Equations of internal or Relative Motion; Integration by the Principal Function.

31. Returning now from this digression on the motion of a single point, to the more important study of an attracting or repelling system, let us resume the differential equations (A.), which may be thus summed up:

$$dt\,\delta H = \Sigma(d\eta\,\delta\varpi - d\varpi\delta\eta); \tag{A^2.}$$

and in order to separate the absolute motion of the whole system in space from the motions of its points among themselves, let us choose the following marks of position:

$$x_{\prime\prime} = \frac{\Sigma\,.mx}{\Sigma\,m}, \quad y_{\prime\prime} = \frac{\Sigma\,.my}{\Sigma\,m}, \quad z_{\prime\prime} = \frac{\Sigma\,.mz}{\Sigma\,m}, \tag{176.}$$

and

$$\xi_i = x_i - x_n, \quad \eta_i = y_i - y_n, \quad \zeta_i = z_i - z_n; \tag{177.}$$

that is, the 3 rectangular coordinates of the centre of gravity of the system, referred to an origin fixed in space, and the $3n-3$ rectangular coordinates of the $n-1$ masses $m_1, m_2, \ldots m_{n-1}$, referred to the nth mass m_n, as an internal and moveable origin, but to axes parallel to the former. We then find, as in the former Essay,

$$\left.\begin{aligned} T = {}& \tfrac{1}{2}(x_{\prime\prime}'^2 + y_{\prime\prime}'^2 + z_{\prime\prime}'^2)\,\Sigma\,m + \tfrac{1}{2}\,\Sigma_\prime\,.m(\xi'^2 + \eta'^2 + \zeta'^2) \\ & - \frac{1}{2\,\Sigma\,m}\left\{(\Sigma_\prime\,.m\xi')^2 + (\Sigma_\prime\,.m\eta')^2 + (\Sigma_\prime\,.m\zeta')^2\right\}, \end{aligned}\right\} \tag{178.}$$

the sign of summation $\Sigma_\prime$ referring to the first $n-1$ masses only; and therefore,

$$\left.\begin{aligned} T = {}& \frac{1}{2\,\Sigma\,m}\left\{\left(\frac{\delta T}{\delta x_{\prime\prime}'}\right)^2 + \left(\frac{\delta T}{\delta y_{\prime\prime}'}\right)^2 + \left(\frac{\delta T}{\delta z_{\prime\prime}'}\right)^2\right\} \\ & + \tfrac{1}{2}\,\Sigma_\prime\,.\frac{1}{m}\left\{\left(\frac{\delta T}{\delta \xi'}\right)^2 + \left(\frac{\delta T}{\delta \eta'}\right)^2 + \left(\frac{\delta T}{\delta \zeta'}\right)^2\right\} \\ & + \frac{1}{2m_n}\left\{\left(\Sigma_\prime\,\frac{\delta T}{\delta \xi'}\right)^2 + \left(\Sigma_\prime\,\frac{\delta T}{\delta \eta'}\right)^2 + \left(\Sigma_\prime\,\frac{\delta T}{\delta \zeta'}\right)^2\right\}. \end{aligned}\right\} \tag{179.}$$

If then we put for abridgement,

$$\left.\begin{aligned} x_\prime' &= \frac{1}{m}\frac{\delta T}{\delta \xi'} = \xi' - \frac{\Sigma_\prime\,.m\xi'}{\Sigma\,m}, \\ y_\prime' &= \frac{1}{m}\frac{\delta T}{\delta \eta'} = \eta' - \frac{\Sigma_\prime\,.m\eta'}{\Sigma\,m}, \\ z_\prime' &= \frac{1}{m}\frac{\delta T}{\delta \zeta'} = \zeta' - \frac{\Sigma_\prime\,.m\zeta'}{\Sigma\,m}, \end{aligned}\right\} \tag{180.}$$

we shall have the expression

$$\left.\begin{aligned} H &= \tfrac{1}{2}(x_{\prime\prime}'^2 + y_{\prime\prime}^2 + z_{\prime\prime}'^2)\,\Sigma m + \tfrac{1}{2}\,\Sigma_\prime\,.m(x_\prime'^2 + y_\prime'^2 + z_\prime'^2) \\ &\quad + \frac{1}{2m_n}\left\{(\Sigma_\prime\,.mx_\prime')^2 + (\Sigma_\prime\,.my_\prime')^2 + (\Sigma_\prime\,.mz_\prime')^2\right\} - U, \end{aligned}\right\} \tag{B^2.}$$

of which the variation is to be compared with the following form of (A^2.),

$$\left.\begin{aligned} dt\,\delta H &= (dx_{\prime\prime}\,\delta x_{\prime\prime}' - dx_{\prime\prime}'\,\delta x_{\prime\prime} + dy_{\prime\prime}\,\delta y_{\prime\prime}' - dy_{\prime\prime}'\,\delta y_{\prime\prime} + dz_{\prime\prime}\,\delta z_{\prime\prime}' - dz_{\prime\prime}'\,\delta z_{\prime\prime})\,\Sigma m \\ &\quad + \Sigma_\prime\,.m(d\xi\,\delta x_\prime' - dx_\prime'\,\delta\xi + d\eta\,\delta y_\prime' - dy_\prime'\,\delta\eta + d\zeta\,\delta z_\prime' - dz_\prime'\,\delta\zeta), \end{aligned}\right\} \tag{C^2.}$$

in order to form, by our general process, $6n$ differential equations of the first order, between the $6n$ quantities $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}\ x_{\prime\prime}'\ y_{\prime\prime}'\ z_{\prime\prime}'\ \xi\ \eta\ \zeta\ x_\prime'\ y_\prime'\ z_\prime'$ and the time t. In thus taking the variation of H, we are to remember that the force-function U depends only on the $3n-3$ internal coordinates $\xi\ \eta\ \zeta$, being of the form

$$\left.\begin{aligned} U &= m_n(m_1 f_1 + m_2 f_2 + \cdots + m_{n-1} f_{n-1}) \\ &\quad + m_1 m_2 f_{1,2} + m_1 m_3 f_{1,3} + \cdots + m_{n-2} m_{n-1} f_{n-2,n-1}, \end{aligned}\right\} \tag{D^2.}$$

in which f_i is a function of the distance of m_i from m_n, and $f_{i,k}$ is a function of the distance of m_i from m_k, such that their derived functions or first differential coefficients, taken with respect to the distances, express that laws of mutual repulsion, being negative in the case of attraction; and then we obtain, as we desired, two separate groups of equations, for the motion of the whole system of points in space, and for the motions of those points among themselves; namely, first, the group

$$\left.\begin{aligned} dx_{\prime\prime} &= x_{\prime\prime}'\,dt, & dx_{\prime\prime}' &= 0, \\ dy_{\prime\prime} &= y_{\prime\prime}'\,dt, & dy_{\prime\prime}' &= 0, \\ dz_{\prime\prime} &= z_{\prime\prime}'\,dt, & dz_{\prime\prime}' &= 0, \end{aligned}\right\} \tag{181.}$$

and secondly the group

$$\left.\begin{aligned} d\xi &= \left(x_\prime' + \frac{1}{m_n}\,\Sigma_\prime\,.mx_\prime'\right)dt, & dx_\prime' &= \frac{1}{m}\frac{\delta U}{\delta\xi}dt, \\ d\eta &= \left(y_\prime' + \frac{1}{m_n}\,\Sigma_\prime\,.my_\prime'\right)dt, & dy_\prime' &= \frac{1}{m}\frac{\delta U}{\delta\eta}dt, \\ d\zeta &= \left(z_\prime' + \frac{1}{m_n}\,\Sigma_\prime\,.mz_\prime'\right)dt, & dz_\prime' &= \frac{1}{m}\frac{\delta U}{\delta\zeta}dt. \end{aligned}\right\} \tag{182.}$$

The six differential equations of the first order (181.), between $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}$, $x_{\prime\prime}'\ y_{\prime\prime}'\ z_{\prime\prime}'$ and t, contain the law of rectilinear and uniform motion of the centre of gravity of the system; and the $6n-6$ equations of the same order, (182.), between the $6n-6$ variables $\xi\ \eta\ \zeta\ x_\prime'\ y_\prime'\ z_\prime'$ and the time, are forms for the differential equations of internal or relative motion. We might eliminate the $3n-3$ auxiliary variables $x_\prime'\ y_\prime'\ z_\prime'$ between these last equations, and so obtain the following other group of $3n-3$ equations of the second order, involving only the relative coordinates and the time,

$$\left.\begin{aligned}\xi'' &= \frac{1}{m}\frac{\delta U}{\delta \xi} + \frac{1}{m_n}\,\Sigma_{\prime}\,\frac{\delta U}{\delta \xi},\\ \eta'' &= \frac{1}{m}\frac{\delta U}{\delta \eta} + \frac{1}{m_n}\,\Sigma_{\prime}\,\frac{\delta U}{\delta \eta},\\ \zeta'' &= \frac{1}{m}\frac{\delta U}{\delta \zeta} + \frac{1}{m_n}\,\Sigma_{\prime}\,\frac{\delta U}{\delta \zeta};\end{aligned}\right\} \tag{183.}$$

but it is better for many purposes to retain them under the forms (182.), omitting, however, for simplicity, the lower accents of the auxiliary variables $x'_{\prime}$ $y'_{\prime}$ $z'_{\prime}$, because it is easy to prove that these auxiliary variables (180.) are the components of centrobaric velocity, and because, in investigating the properties of internal or relative motion, we are a liberty to suppose that the centre of gravity of the system is fixed in space, at the origin of $x\ y\ z$. We may also, for simplicity, omit the lower accent of $\Sigma_{\prime}$, understanding that the summations are to fall only on the first $n-1$ masses, and denoting for greater distinctness the nth mass by a separate symbol M; and then we may comprise the differential equations of relative motion in the following simplified formula,

$$dt\,\delta H = \Sigma\,.m(d\xi\,\delta x' - dx'\,\delta\xi + d\eta\,\delta y' - dy'\,\delta\eta + d\zeta\,\delta z' - dz'\,\delta\zeta), \tag{E^2.}$$

in which

$$H = \tfrac{1}{2}\,\Sigma\,.m(x'^2 + y'^2 + z'^2) + \frac{1}{2M}\left\{(\Sigma\,.mx')^2 + (\Sigma\,.my')^2 + (\Sigma\,.mz')^2\right\} - U, \tag{F^2.}$$

And the integrals of these equations of relative motion are contained (by our general method) in the formula

$$\delta S = \Sigma\,.m(x'\,\delta\xi - a'\delta\alpha + y'\,\delta\eta - b'\delta\beta + z'\,\delta\zeta - c'\delta\gamma), \tag{G^2.}$$

in which $\alpha\ \beta\ \gamma\ a'\ b'\ c'$ denote the initial values of $\xi\ \eta\ \zeta\ x'\ y'\ z'$, and S is the *principal function of relative motion* of the system; that is, the former function S, simplified by the omission of the part which vanishes when the centre of gravity is fixed, and which gives in general the laws of motion of that centre, or the integrals of the equations (181.).

Second Example: Case of a Ternary or Multiple System with one Predominant Mass; Equations of the undisturbed motions of the other masses about this, in their several Binary Systems; Differentials of all their Elements, expressed by the coefficients of one Disturbing Function.

32. Let us now suppose that the $n-1$ masses m are small in comparision with the nth mass M; and let us separate the expression (F^2.) for H into the two following parts,

$$\left.\begin{aligned}H_1 &= \Sigma\,.\frac{m}{2}\left(1 + \frac{m}{M}\right)(x'^2 + y'^2 + z'^2) - M\,\Sigma\,.mf,\\ H_2 &= \frac{m_1 m_2}{M}(x'_1x'_2 + y'_1y'_2 + z'_1z'_2 - Mf_{1,2}) + \cdots\\ &\quad + \frac{m_i m_k}{M}(x'_ix'_k + y'_iy'_k + z'_iz'_k - Mf_{i,k}) + \cdots,\end{aligned}\right\} \tag{H^2.}$$

of which the latter is small in comparision with the former, and may be neglected in a first approximation. Suppressing it accordingly, we are conducted to the following $6n-6$ differential equations of the 1st order, belonging to a simpler motion, which may be called the *undisturbed*:

$$\left.\begin{aligned}
\frac{d\xi}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta x'} = \left(1+\frac{m}{M}\right)x'; & \frac{dx'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \xi} = M\frac{\delta f}{\delta \xi};\\
\frac{d\eta}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta y'} = \left(1+\frac{m}{M}\right)y'; & \frac{dy'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \eta} = M\frac{\delta f}{\delta \eta};\\
\frac{d\zeta}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta z'} = \left(1+\frac{m}{M}\right)z'; & \frac{dz'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \zeta} = M\frac{\delta f}{\delta \zeta}.
\end{aligned}\right\} \tag{I^2.}$$

These equations arrange themselves in $n-1$ groups, corresponding to the $n-1$ binary systems (m, M); and it is easy to integrate the equations of each group separately. We may suppose, then, these integrals found, under the forms,

$$\left.\begin{aligned}
\kappa &= \chi^{(1)}(t,\xi,\eta,\zeta,x',y',z'), & \nu &= \chi^{(4)}(t,\xi,\eta,\zeta,x',y',z'),\\
\lambda &= \chi^{(2)}(t,\xi,\eta,\zeta,x',y',z'), & \tau &= \chi^{(5)}(t,\xi,\eta,\zeta,x',y',z'),\\
\mu &= \chi^{(3)}(t,\xi,\eta,\zeta,x',y',z'), & \omega &= \chi^{(6)}(t,\xi,\eta,\zeta,x',y',z'),
\end{aligned}\right\} \tag{K^2.}$$

the six quantities $\kappa\ \lambda\ \mu\ \nu\ \tau\ \omega$ being constant for the undisturbed motion of any one binary system; and therefore the six functions $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(4)}$, $\chi^{(5)}$, $\chi^{(6)}$, or κ, λ, μ, ν, τ, ω, being such as to satisfy *identically* the following equation,

$$0 = m\frac{\delta\kappa}{\delta t} + \frac{\delta\kappa}{\delta\xi}\frac{\delta H_1}{\delta x'} - \frac{\delta\kappa}{\delta x'}\frac{\delta H_1}{\delta\xi} + \frac{\delta\kappa}{\delta\eta}\frac{\delta H_1}{\delta y'} - \frac{\delta\kappa}{\delta y'}\frac{\delta H_1}{\delta\eta} + \frac{\delta\kappa}{\delta\zeta}\frac{\delta H_1}{\delta z'} - \frac{\delta\kappa}{\delta z'}\frac{\delta H_1}{\delta\zeta}, \tag{L^2.}$$

with five other analogous, for the five other elements λ, μ, ν, τ, ω, in any one binary system (m, M).

33. Returning now to the original multiple system, we may retain as definitions the equations (K^2.), but then we can no longer consider the elements $\kappa_i\ \lambda_i\ \mu_i\ \nu_i\ \tau_i\ \omega_i$ of the binary system (m_i, M) as constant, because this system is now disturbed by the other masses m_k; however, the $6n-6$ equations of disturbed relative motion, when put under the forms

$$\left.\begin{aligned}
m\frac{d\xi}{dt} &= \frac{\delta H_1}{\delta x'} + \frac{\delta H_2}{\delta x'}, & m\frac{dx'}{dt} &= -\frac{\delta H_1}{\delta\xi} - \frac{\delta H_2}{\delta\xi},\\
m\frac{d\eta}{dt} &= \frac{\delta H_1}{\delta y'} + \frac{\delta H_2}{\delta y'}, & m\frac{dy'}{dt} &= -\frac{\delta H_1}{\delta\eta} - \frac{\delta H_2}{\delta\eta},\\
m\frac{d\zeta}{dt} &= \frac{\delta H_1}{\delta z'} + \frac{\delta H_2}{\delta z'}, & m\frac{dz'}{dt} &= -\frac{\delta H_1}{\delta\zeta} - \frac{\delta H_2}{\delta\zeta},
\end{aligned}\right\} \tag{M^2.}$$

and combined with the identical equations of the kind (L^2.) give the following simple expression for the differential of the element κ, in its disturbed and variable state,

$$m\frac{d\kappa}{dt} = \frac{\delta\kappa}{\delta\xi}\frac{\delta H_2}{\delta x'} - \frac{\delta\kappa}{\delta x'}\frac{\delta H_2}{\delta\xi} + \frac{\delta\kappa}{\delta\eta}\frac{\delta H_2}{\delta y'} - \frac{\delta\kappa}{\delta y'}\frac{\delta H_2}{\delta\eta} + \frac{\delta\kappa}{\delta\zeta}\frac{\delta H_2}{\delta z'} - \frac{\delta\kappa}{\delta z'}\frac{\delta H_2}{\delta\zeta}, \tag{N^2.}$$

together with analogous expressions for the differentials of the other elements. And if we express $\xi\ \eta\ \zeta\ x'\ y'\ z'$, and therefore H_2 itself, as depending on the time and on these varying elements, we may transform the $6n-6$ differential equations of the 1st order (M^2.), between $\xi\ \eta\ \zeta\ x'\ y'\ z'\ t$, into the same number of equations of the same order between the varying elements and the time; which will be of the forms:

$$\left.\begin{aligned}
m\frac{d\kappa}{dt} &= \{\kappa,\lambda\}\frac{\delta H_2}{\delta\lambda} + \{\kappa,\mu\}\frac{\delta H_2}{\delta\mu} + \{\kappa,\nu\}\frac{\delta H_2}{\delta\nu} + \{\kappa,\tau\}\frac{\delta H_2}{\delta\tau} + \{\kappa,\omega\}\frac{\delta H_2}{\delta\omega},\\
m\frac{d\lambda}{dt} &= \{\lambda,\kappa\}\frac{\delta H_2}{\delta\kappa} + \{\lambda,\mu\}\frac{\delta H_2}{\delta\mu} + \{\lambda,\nu\}\frac{\delta H_2}{\delta\nu} + \{\lambda,\tau\}\frac{\delta H_2}{\delta\tau} + \{\lambda,\omega\}\frac{\delta H_2}{\delta\omega},\\
m\frac{d\mu}{dt} &= \{\mu,\kappa\}\frac{\delta H_2}{\delta\kappa} + \{\mu,\lambda\}\frac{\delta H_2}{\delta\lambda} + \{\mu,\nu\}\frac{\delta H_2}{\delta\nu} + \{\mu,\tau\}\frac{\delta H_2}{\delta\tau} + \{\mu,\omega\}\frac{\delta H_2}{\delta\omega},\\
m\frac{d\nu}{dt} &= \{\nu,\kappa\}\frac{\delta H_2}{\delta\kappa} + \{\nu,\lambda\}\frac{\delta H_2}{\delta\lambda} + \{\nu,\mu\}\frac{\delta H_2}{\delta\mu} + \{\nu,\tau\}\frac{\delta H_2}{\delta\tau} + \{\nu,\omega\}\frac{\delta H_2}{\delta\omega},\\
m\frac{d\tau}{dt} &= \{\tau,\kappa\}\frac{\delta H_2}{\delta\kappa} + \{\tau,\lambda\}\frac{\delta H_2}{\delta\lambda} + \{\tau,\mu\}\frac{\delta H_2}{\delta\mu} + \{\tau,\nu\}\frac{\delta H_2}{\delta\nu} + \{\tau,\omega\}\frac{\delta H_2}{\delta\omega},\\
m\frac{d\omega}{dt} &= \{\omega,\kappa\}\frac{\delta H_2}{\delta\kappa} + \{\omega,\lambda\}\frac{\delta H_2}{\delta\lambda} + \{\omega,\mu\}\frac{\delta H_2}{\delta\mu} + \{\omega,\nu\}\frac{\delta H_2}{\delta\nu} + \{\omega,\tau\}\frac{\delta H_2}{\delta\tau},
\end{aligned}\right\} \qquad (\mathrm{O}^2.)$$

if we put, for abridgement,

$$\{\kappa,\lambda\} = \frac{\delta\kappa}{\delta\xi}\frac{\delta\lambda}{\delta x'} - \frac{\delta\kappa}{\delta x'}\frac{\delta\lambda}{\delta\xi} + \frac{\delta\kappa}{\delta\eta}\frac{\delta\lambda}{\delta y'} - \frac{\delta\kappa}{\delta y'}\frac{\delta\lambda}{\delta\eta} + \frac{\delta\kappa}{\delta\zeta}\frac{\delta\lambda}{\delta z'} - \frac{\delta\kappa}{\delta z'}\frac{\delta\lambda}{\delta\zeta}, \qquad (\mathrm{P}^2.)$$

and form the other symbols $\{\kappa,\mu\}$, $\{\lambda,\kappa\}$, &c., from this, by interchanging the letters. It is evident that these symbols have the properties,

$$\{\lambda,\kappa\} = -\{\kappa,\lambda\}, \quad \{\kappa,\kappa\} = 0; \qquad (184.)$$

and it results from the principles of the 15th number, that these combinations $\{\kappa,\lambda\}$, &c., when expressed as functions of the elements, do not contain the time explicitly. There are, in general, by (184.), only 15 such distinct combinations for each of the $n-1$ binary systems; but there would thus be, in all, $15n-15$, if they admitted of no further reductions: however it results from the principles of the 16th number, that $12n-12$ of these combinations may be made to vanish by a suitable choice of the elements. The following is another way of effecting as great a simplification, at least for that extensive class of cases in which the undisturbed distance between the two points of each binary system (m, M) admits of a minimum value.

Simplification of the Differential Expressions by a suitable choice of the Elements.

34. When the undisturbed distance r of m from M admists of such a minimum q, corresponding to a time τ, and satisfying at that time the conditions

$$r' = 0, \quad r'' > 0, \qquad (185.)$$

then the integrals of the group (I^2.), or the known rules of the undisturbed motion of m about M, may be presented in the following manner:

$$\left.\begin{aligned}
&\kappa = \sqrt{\{(\xi y' - \eta x')^2 + (\eta z' - \zeta y')^2 + (\zeta x' - \xi z')^2\}};\\
&\lambda = \kappa - \xi y' + \eta x';\\
&\mu = \frac{M+m}{2M}(x'^2 + y'^2 + z'^2) - Mf(r);\\
&\nu = \tan^{-1} . \frac{\eta z' - \zeta y'}{\xi z' - \zeta x'};\\
&\tau = t - \int_q^r \frac{\sqrt{\dfrac{M}{M+m}} \cdot \dfrac{dr}{\sqrt{dr^2}} \cdot dr}{\sqrt{\left\{2\mu + 2Mf(r) - \left(1 + \dfrac{m}{M}\right)\dfrac{\kappa^2}{r^2}\right\}}};\\
&\omega = \nu + \sin^{-1} . \frac{\kappa\zeta r^{-1}}{\sqrt{2\lambda\kappa - \lambda^2}} - \int_q^r \frac{\sqrt{\dfrac{M+m}{M}} \cdot \dfrac{dr}{\sqrt{dr^2}} \cdot \dfrac{\kappa}{r^2} \cdot dr}{\sqrt{\left\{2\mu + 2Mf(r) - \left(1 + \dfrac{m}{M}\right)\dfrac{\kappa^2}{r^2}\right\}}};
\end{aligned}\right\} \qquad (\mathrm{Q}^2.)$$

the minimum distance q being a function of the two elements κ, μ, which must satisfy the conditions

$$2\mu + 2Mf(q) - \left(1 + \frac{m}{M}\right)\frac{\kappa^2}{q^2} = 0, \quad Mf'(q) + \left(1 + \frac{m}{M}\right)\frac{\kappa^2}{q^3} > 0; \qquad (186.)$$

and $\sin^{-1} s$, $\tan^{-1} t$, being used (according to Sir JOHN HERSCHEL'S notation) to express, *not* the cosecant and cotangent, but the *inverse functions* corresponding to sine and cosine, or the arcs which are more commonly called arc($\sin = s$), arc($\tan = t$). It must also be observed that the factor $\dfrac{dr}{\sqrt{dr^2}}$, which we have introduced under the signs of integration, is not superfluous, but is designed to be taken as equal to positive or negative unity, according as dr is positive or negative; that is, according as r is increasing or diminishing, so as to make the element under each integral sign constantly positive. In general, it appears to be a useful rule, though not always followed by analysts, to employ the real radical symbol $\sqrt{R}$ only for positive quantities, unless the negative sign be expressly prefixed; and then $\dfrac{r}{\sqrt{r^2}}$ will denote positive or negative unity, according as r is positive or negative. The arc given by its sine, in the expression of the element ω, is supposed to be so chosen as to increase continually with the time.

35. After these remarks on the notation, let us apply the formula (P^2.) to calculate the values of the 15 combinations such as $\{\kappa, \lambda\}$, of the 6 constants or elements (Q^2.).

Since

$$r = \sqrt{\{\xi^2 + \eta^2 + \zeta^2\}}, \qquad (187.)$$

it is easy to perceive that the six combinations of the 4 first elements are as follows:

$$\{\kappa, \lambda\} = 0, \quad \{\kappa, \mu\} = 0, \quad \{\kappa, \nu\} = 0, \quad \{\lambda, \mu\} = 0, \quad \{\lambda, \nu\} = 1, \quad \{\mu, \nu\} = 0. \qquad (188.)$$

To form the 4 combinations of these first 4 elements with τ, we may observe, that this 5th element τ, as expressed in (Q^2.), involves explicitly (besides the time) the distance r, and the two elements κ, μ; but the combinations already determined show that these two elements may be treated as constant in forming the four combinations now sought; we need only attend, therefore, to the variation of r, and if we interpret by the rule (P^2.) the symbols $\{\kappa, r\}$, $\{\lambda, r\}$, $\{\mu, r\}$, $\{\nu, r\}$, and attend to the equations (I^2.), we see that

$$\{\kappa, r\} = 0, \quad \{\lambda, r\} = 0, \quad \{\mu, r\} = -\frac{dr}{dt}, \quad \{\nu, r\} = 0, \tag{189.}$$

$\dfrac{dr}{dt}$ being the total differential coefficient of r in the undisturbed motion, as determined by the equations (I^2); and, therefore, that

$$\{\kappa, \tau\} = 0, \quad \{\lambda, \tau\} = 0, \quad \{\nu, \tau\} = 0, \tag{190.}$$

and

$$\{\mu, \tau\} = -\frac{\delta\tau}{\delta r}\frac{dr}{dt} = +\frac{dt}{dr}\frac{dr}{dt} = 1: \tag{191.}$$

observing that in differentiating the expressions of the elements (Q^2), we may treat those elements as constant, if we change the differentials of ξ η ζ x' y' z' to their undisturbed values. It remains to calculate the 5 combinations of these elements with the last element ω; which is given by (Q^2.) as a function of the distance r, the coordinate ζ, and the 4 elements κ, λ, μ, ν; so that we may employ this formula,

$$\{e, \omega\} = \frac{\delta\omega}{\delta r}\{e, r\} + \frac{\delta\omega}{\delta\zeta}\{e, \zeta\} + \frac{\delta\omega}{\delta\kappa}\{e, \kappa\} + \frac{\delta\omega}{\delta\lambda}\{e, \lambda\} + \frac{\delta\omega}{\delta\mu}\{e, \mu\} + \frac{\delta\omega}{\delta\nu}\{e, \nu\}, \tag{192.}$$

in which, if e be any of the first five elements, or the distance r,

$$\{e, r\} = -\frac{1}{r}\left(\xi\frac{\delta e}{\delta x'} + \eta\frac{\delta e}{\delta y'} + \zeta\frac{\delta e}{\delta z'}\right), \quad \{e, \zeta\} = -\frac{\delta e}{\delta z'}, \quad \{e, \kappa\} = 0, \tag{193.}$$

and

$$\frac{\delta\omega}{\delta\zeta} = \left(\frac{\delta\kappa}{\delta z'}\right)^{-1}, \quad \frac{\delta\omega}{\delta r} = -\frac{d\zeta}{dr}\frac{\delta\omega}{\delta\zeta}, \quad \frac{\delta\omega}{\delta\nu} = 1; \tag{194.}$$

the formula (192.) may therefore be thus written:

$$\{e, \omega\} = \left\{\frac{z'\left(\xi\dfrac{\delta e}{\delta x'} + \eta\dfrac{\delta e}{\delta y'} + \zeta\dfrac{\delta e}{\delta z'}\right)}{\xi x' + \eta y' + \zeta z'} - \frac{\delta e}{\delta z'}\right\}\left(\frac{\delta\kappa}{\delta z'}\right)^{-1} + \{e, \nu\} + \frac{\delta\omega}{\delta\lambda}\{e, \lambda\} + \frac{\delta\omega}{\delta\mu}\{e, \mu\}. \tag{195.}$$

We easily find, by this formula, that

$$\{\kappa, \omega\} = -1; \quad \{\lambda, \omega\} = 0; \quad \{\mu, \omega\} = 0; \quad \{r, \omega\} = \frac{dr}{dt}\frac{\delta\omega}{\delta\mu}; \tag{196.}$$

and

$$\{\nu, \omega\} = -\frac{\delta\nu}{\delta z'}\frac{\delta\omega}{\delta\zeta} - \frac{\delta\omega}{\delta\lambda} = 0. \tag{197.}$$

The formula (195.) extends to the combination $\{\tau, \omega\}$ also; but in calculating this last combination we are to remember that τ is given by (Q^2.) as a function of κ, μ, r, such that

$$\frac{\delta\tau}{\delta r} = -\frac{dt}{dr}; \tag{198.}$$

and thus we see, with the help of the combinations (196.) already determined, that

$$\{\tau, \omega\} = -\frac{\delta\tau}{\delta\kappa} - \frac{\delta\omega}{\delta\mu} = \frac{\delta}{\delta\kappa}\int_q^r \Theta_r\, dr + \frac{\delta}{\delta\mu}\int_q^r \Omega_r\, dr, \tag{199.}$$

if we represent for abridgement by Θ_r and Ω_r the coefficients of dr under the interal signs in (Q^2.), namely,

$$\left.\begin{aligned} \Theta_r &= \sqrt{\frac{M}{M+m}} \cdot \frac{dr}{\sqrt{dr^2}}\left\{2\mu + 2Mf(r) - \frac{M+m}{M}\cdot\frac{\kappa^2}{r^2}\right\}^{-\frac{1}{2}}, \\ \Omega_r &= \frac{\kappa}{r^2}\sqrt{\frac{M+m}{M}} \cdot \frac{dr}{\sqrt{dr^2}}\left\{2\mu + 2Mf(r) - \frac{M+m}{M}\cdot\frac{\kappa^2}{r^2}\right\}^{-\frac{1}{2}}, \end{aligned}\right\} \tag{200.}$$

These coefficients are evidently connected by the relation

$$\frac{\delta\Theta_r}{\delta\kappa} + \frac{\delta\Omega_r}{\delta\mu} = 0, \tag{201.}$$

which gives

$$\frac{\delta}{\delta\kappa}\int_{r_{\prime}}^r \Theta_r\, dr + \frac{\delta}{\delta\mu}\int_{r_{\prime}}^r \Omega_r\, dr = 0, \tag{202.}$$

$r_{\prime}$ being any quantity which does not vary with the elements κ and μ; we might therefore at once conclude by (199.) that the combination $\{\tau, \omega\}$ vanishes, if a difficulty were not occasioned by the necessity of varying the lower limit q, which depends on those two elements, and by the circumstance that at this lower limit the coefficients Θ_r Ω_r become infinite. However, the relation (202.) shows that we may express the combination $\{\tau, \omega\}$ as follows:

$$\{\tau, \omega\} = \frac{\delta}{\delta\kappa}\int_q^{r_{\prime}} \Theta_r\, dr + \frac{\delta}{\delta\mu}\int_q^{r_{\prime}} \Omega_r\, dr, \tag{203.}$$

$r_{\prime}$ being an auxiliary and arbitrary quantity, which cannot really affect the result, but may be made to facilitate the calculation; or in other words, we may assign to the distance r any arbitrary value, not varying for infinitesimal variations of κ, μ, which may assist in calculating the value of the expression (199.). We may therefore suppose that the increase of distance $r - q$ is small, and corresponds to a small positive interval of time $t - \tau$, during which the distance r and its differential coefficient r' are constantly increasing; and then after the first moment τ, the quantity

$$\Theta_r = \frac{1}{r'} \tag{204.}$$

will be constantly finite, positive, and decreasing, during the same interval, so that its integral must be greater than if it had constantly its final value; that is,

$$t - \tau = \int_q^r \Theta_r \, dr > (r - q)\Theta_r. \tag{205.}$$

Hence, although Θ_r tends to infinity, yet $(r - q)\Theta_r$ tends to zero, when by diminishing the interval we make r tend to q; and therefore the following difference

$$\int_q^r \Omega_r \, dr - \frac{M+m}{M} \frac{\kappa}{q^2} \int_q^r \Theta_r \, dr = \frac{M+m}{M} \int_q^r \left(\frac{\kappa}{r^2} - \frac{\kappa}{q^2} \right) \Theta_r \, dr \tag{206.}$$

will also tend to 0, and so will also its partial differential coefficient of the first order, taken with respect to μ. We find therefore the following formula for $\{\tau, \omega\}$, (remembering that this combination has been shown to be independent of r,)

$$\{\tau, \omega\} = \mathop{\Lambda}_{r=q} \left\{ \frac{\delta}{\delta\kappa} \int_q^r \Theta_r \, dr + \frac{M+m}{M} \frac{\kappa}{q^2} \frac{\delta}{\delta\mu} \int_q^r \Omega_r \, dr \right\}; \tag{207.}$$

the sign $\mathop{\Lambda}_{r=q}$ implying that the limit is to be taken to which the expression tends when r tends to q. In this last formula, as in (199.), the integral $\int_q^r \Theta_r \, dr$ may be considered as a known function of r, q, κ, μ, or simply of r, q, κ, if μ be eliminated by the first condition (186.); and since it vanishes independently of κ when $r = q$, it may thus be denoted:

$$\int_q^r \Theta_r \, dr = \phi(r, q, \kappa) - \phi(q, q, \kappa), \tag{208.}$$

the form of the function ϕ depending on the law of attraction or repulsion. This integral therefore, when considered as depending on κ and μ, by depending on κ and q, need not be varied with respect to κ, in calculating $\{\tau, \omega\}$ by (207.), because its partial differential coefficient $\left(\frac{\delta}{\delta\kappa} \int_q^r \Theta_r \, dr \right)$, obtained by treating q as constant, vanishes at the limit $r = q$; nor need it be varied with respect to q, because, by (186.),

$$\frac{\delta q}{\delta\kappa} + \frac{M+m}{M} \frac{\kappa}{q^2} \frac{\delta q}{\delta\mu} = 0; \tag{209.}$$

it may therefore be treated as constant, and we find at last

$$\{\tau, \omega\} = 0, \tag{210.}$$

the two terms (199.) or (203.) both tending to infinity when r tends to q, but always destroying each other.

36. Collecting now our results, and presenting for greater clearness each combination under the two forms in which it occurs when the order of the elements is changed, we have, for each binary system, the following thirty expressions:

$$\left.\begin{array}{lllll}
\{\kappa,\lambda\}=0, & \{\kappa,\mu\}=0, & \{\kappa,\nu\}=0, & \{\kappa,\tau\}=0, & \{\kappa,\omega\}=-1,\\
\{\lambda,\kappa\}=0, & \{\lambda,\mu\}=0, & \{\lambda,\nu\}=1, & \{\lambda,\tau\}=0, & \{\lambda,\omega\}=0,\\
\{\mu,\kappa\}=0, & \{\mu,\lambda\}=0, & \{\mu,\nu\}=0, & \{\mu,\tau\}=1, & \{\mu,\omega\}=0,\\
\{\nu,\kappa\}=0, & \{\nu,\lambda\}=-1, & \{\nu,\mu\}=0, & \{\nu,\tau\}=0, & \{\nu,\omega\}=0,\\
\{\tau,\kappa\}=0, & \{\tau,\lambda\}=0, & \{\tau,\mu\}=-1, & \{\tau,\nu\}=0, & \{\tau,\omega\}=0,\\
\{\omega,\kappa\}=1, & \{\omega,\lambda\}=0, & \{\omega,\mu\}=0, & \{\omega,\nu\}=0, & \{\omega,\tau\}=0;
\end{array}\right\} \qquad (\mathrm{R}^2.)$$

so that the three combinations

$$\{\mu,\tau\}, \quad \{\omega,\kappa\}, \quad \{\lambda,\nu\}$$

are each equal to positive unity; the three inverse combinations

$$\{\tau,\mu\}, \quad \{\kappa,\omega\}, \quad \{\nu,\lambda\}$$

are each equal to negative unity; and all the others vanish. The six differential equations of the first order, for the 6 varying elements of any one binary system (m, M), are therefore, by $(\mathrm{O}^2.)$,

$$\left.\begin{array}{ll}
m\dfrac{d\mu}{dt}=\dfrac{\delta H_2}{\delta\tau}, & m\dfrac{d\tau}{dt}=-\dfrac{\delta H_2}{\delta\mu},\\
m\dfrac{d\omega}{dt}=\dfrac{\delta H_2}{\delta\kappa}, & m\dfrac{d\kappa}{dt}=-\dfrac{\delta H_2}{\delta\omega},\\
m\dfrac{d\lambda}{dt}=\dfrac{\delta H_2}{\delta\nu}, & m\dfrac{d\nu}{dt}=-\dfrac{\delta H_2}{\delta\lambda},
\end{array}\right\} \qquad (\mathrm{S}^2.)$$

and if we still omit the variation of t, they may all be summed up in this form for the variation of H_2,

$$\delta H_2 = \Sigma\,.m(\mu'\delta\tau - \tau'\delta\mu + \omega'\delta\kappa - \kappa'\delta\omega + \lambda'\delta\nu - \nu'\delta\lambda), \qquad (\mathrm{T}^2.)$$

which single formula enables us to derive all the $6n-6$ differential equations of the first order, for all the varying elements of all the binary systems, from the variation or from the partial differential coefficients of a single quantity H_2, expressed as a function of those elements.

If we choose to introduce into the expression $(\mathrm{T}^2.)$, for δH_2, the variation of the time t, we have only to change $\delta\tau$ to $\delta\tau - \delta t$, because, by $(\mathrm{Q}^2.)$, δt enters only so accompanied; that is, t enters only under the form $t - \tau_i$, in the expressions of ξ_i η_i ζ_i x'_i y'_i z'_i as functions of the time and of the elements; we have, therefore

$$\frac{\delta H_2}{\delta t} = -\,\Sigma\,\frac{\delta H_2}{\delta\tau} = -\,\Sigma\,.m\mu'; \qquad (211.)$$

and since, by (H^2.), (Q^2),

$$H_1 = \Sigma\,.m\mu, \tag{212.}$$

we find, finally,

$$\frac{dH_1}{dt} = -\frac{\delta H_2}{\delta t}. \tag{U^2.}$$

This remarkable form for the differential of H_1, considered as a varying element, is general for all problems of dynamics. It may be deduced by the general method from the formulæ of the 13th and 14th numbers, which give

$$\left.\begin{aligned}\frac{dH_1}{dt} &= \frac{\delta H_2}{\delta\kappa_1}\Sigma\left(\frac{\delta H_1}{\delta\eta}\frac{\delta\kappa_1}{\delta\varpi} - \frac{\delta H_1}{\delta\varpi}\frac{\delta\kappa_1}{\delta\eta}\right) + \cdots + \frac{\delta H_2}{\delta\kappa_{6n}}\Sigma\left(\frac{\delta H_1}{\delta\eta}\frac{\delta\kappa_{6n}}{\delta\varpi} - \frac{\delta H_1}{\delta\varpi}\frac{\delta\kappa_{6n}}{\delta\eta}\right)\\ &= \frac{\delta H_2}{\delta\kappa_1}\frac{\delta\kappa_1}{\delta t} + \frac{\delta H_2}{\delta\kappa_2}\frac{\delta\kappa_2}{\delta t} + \cdots + \frac{\delta H_2}{\delta\kappa_{6n}}\frac{\delta\kappa_{6n}}{\delta t} = -\frac{\delta H_2}{\delta t},\end{aligned}\right\} \tag{213.}$$

$\kappa_1\,\kappa_2\,\ldots\,\kappa_{6n}$ being any $6n$ elements of a system expressed as functions of the time and of the quantities $\eta\ \omega$; or concisely by this special consideration, that $H_1 + H_2$ is constant in the disturbed motion, and that in taking the first total differential coefficient of H_2 with respect to the time, the elements may by (F^1.) be treated as constant. It is also a remarkable corollary of the general principles just referred to, but not one difficult to verify, that the first partial differential coefficient $\frac{\delta\kappa_s}{\delta t}$, of any element κ_s, taken with respect to the time, may be expressed as a function of the elements alone, not involving the time explicitly.

On the essential distinction between the Systems of Varying Elements considered in this Essay and those hitherto employed by mathematicians.

37. When we shall have integrated the differential equations of varying elements (S^2.), we can then calculate the varying relative coordinates $\xi\ \eta\ \zeta$, for any binary system (m, M), by the rules of undisturbed motion, as expressed by the equations (I^2.), (Q^2.), or by the following connected formulæ:

$$\left.\begin{aligned}\xi &= r\left(\cos\theta + \frac{\lambda}{\kappa}\sin(\theta-\nu)\sin\nu\right),\\ \eta &= r\left(\sin\theta - \frac{\lambda}{\kappa}\sin(\theta-\nu)\cos\nu\right),\\ \zeta &= \frac{r}{\kappa}\sqrt{2\lambda\kappa-\lambda^2}\sin(\theta-\nu);\end{aligned}\right\} \tag{V^2.}$$

in which the distance r is determined as a function of the time t and of the elements τ, κ, μ, by the 5th equation (Q^2.), and in which

$$\theta = \omega + \int_q^r \frac{\sqrt{\dfrac{M+m}{M}}\cdot\dfrac{dr}{\sqrt{dr^2}}\cdot\dfrac{\kappa}{r^2}dr}{\sqrt{\left\{2\mu + 2Mf(r) - \left(1+\dfrac{m}{M}\right)\dfrac{\kappa^2}{r^2}\right\}}}, \tag{W^2.}$$

q being still the minimum of r, when the orbit is treated as constant, and being still connected with the elements κ, μ, by the first equation of condition (186.). In astronomical language, M is the sun, m a planet, ξ η ζ are the heliocentric rectangular coordinates, r is the radius vector, θ the longitude in the orbit, ω the longitude of the perihelion, ν of the node, $\theta - \omega$ is the true anomaly, $\theta - \nu$ the argument of latitude, μ the constant part of the half square of undisturbed heliocentric velocity, diminished in the ratio of the sun's mass (M) to the sum ($M + m$) of masses of sun and planet, κ is the double of the areal velocity diminished in the same ratio, $\dfrac{\lambda}{\kappa}$ is the versed sine of the inclination of the orbit, q the perihelion distance, and τ the time of perihelion passage. The law of attraction or repulsion is here left undetermined; for Newton's law, μ is the sun's mass divided by the axis major of the orbit taken negatively, and κ is the square root of the semiparameter, multiplied by the sun's mass, and divided by the square root of the sum of the masses of sun and planet. But the varying ellipse or other orbit, which the foregoing formulæ require, differs essentially (though little) from that hitherto employed by astronomers: because it gives correctly the heliocentric coordinates, but *not* the heliocentric components of velocity, without differentiating the elements in the calculation; and therefore does *not touch* but *cuts*, (though under a very small angle,) the actual heliocentric orbit, described under the influence of all the disturbing forces.

38. For it results from the foregoing theory, that if we differentiate the expressions (V^2.) for the heliocentric coordinates, without differentiating the elements, and then assign to those new varying elements their values as functions of the time, obtained from the equations (S^2), and deduce the centrobaric components of velocity by the formulæ (I^2.), or by the following:

$$x' = \frac{M\xi'}{M+m}, \quad y' = \frac{M\eta'}{M+m}, \quad z' = \frac{M\zeta'}{M+m}; \tag{214.}$$

then these centrobaric components will be the same functions of the time and of the new varying elements which might be otherwise deduced by elimination from the integrals (Q^2.), and will represent rigorously (by the extension given in the theory to those last-mentioned integrals) the components of velocity of the disturbed planet m, relatively to the centre of gravity of the whole solar system. We chose, as more suitable to the general course of our method, that these centrobaric components of velocity should be the auxiliary variables to be combined with the heliocentric coordinates, and to have their disturbed values rigorously expressed by the formulæ of undisturbed motion; but in making this choice it became necessary to modify these latter formulæ, and to determine a varying orbit essentially distinct in theory (though little differing in practice) from that conceived so beautifully by Lagrange. The orbit which he imagined was more simply connected with the heliocentric motion of a *single planet*, since it gave, for such heliocentric motion, the velocity as well as the position; the orbit which we have chosen is perhaps more closely combined with the conception of a *multiple system*, moving about its common centre of gravity, and influenced in every part by the actions of all the rest. Whichever orbit shall be hereafter adopted by astronomers, they will remember that both are equally fit to represent the celestial appearances, if the numeric elements of either set be suitable determined by observation, and the elements of the other set of orbits be deduced from these by calculation. Meantime mathematicians will judge, whether in sacrificing a part of the simplicity of that geometrical conception on which the theories of LAGRANGE and POISSON are founded, a simplicity of another kind has not been introduced, which was wanting in those admirable theories; by our having succeeded in expressing rigorously the differentials of *all* our own new varying elements through the coefficients of a *single* function: whereas it has seemed necessary hitherto to employ one function for the Earth disturbed by Venus, and another function for Venus disturbed by the Earth.

Integration of the Simplified Equations, which determine the new varying Elements.

39. The simplified differential equations of varying elements, (S^2.), are of the same form as the equations (A.), and may be integrated in a similar manner. If we put, for abridgement,

$$(\tau, \kappa, \nu) = \int_0^t \left\{ \Sigma \left(\tau \frac{\delta H_2}{\delta \tau} + \kappa \frac{\delta H_2}{\delta \kappa} + \nu \frac{\delta H_2}{\delta \nu} \right) - H_2 \right\} dt, \qquad (X^2.)$$

and interpret similarly the symbols (μ, ω, λ), &c., we can easily assign the variations of the following 8 combinations, (τ, κ, ν), (μ, ω, λ), (μ, κ, ν), (τ, ω, λ), (τ, ω, ν), (μ, κ, λ), (τ, κ, λ), (μ, ω, ν); namely

$$\left.\begin{aligned}
\delta(\tau, \kappa, \nu) &= \Sigma\,.m(\tau\,\delta\mu - \tau_0\,\delta\mu_0 + \kappa\,\delta\omega - \kappa_0\,\delta\omega_0 + \nu\,\delta\lambda - \nu_0\,\delta\lambda_0) - H_2\,\delta t,\\
\delta(\mu, \omega, \lambda) &= \Sigma\,.m(\mu_0\,\delta\tau_0 - \mu\,\delta\tau + \omega_0\,\delta\kappa_0 - \omega\,\delta\kappa + \lambda_0\,\delta\nu_0 - \lambda\,\delta\nu) - H_2\,\delta t,\\
\delta(\mu, \kappa, \nu) &= \Sigma\,.m(\mu_0\,\delta\tau_0 - \mu\,\delta\tau + \kappa\,\delta\omega - \kappa_0\,\delta\omega_0 + \nu\,\delta\lambda - \nu_0\,\delta\lambda_0) - H_2\,\delta t,\\
\delta(\tau, \omega, \lambda) &= \Sigma\,.m(\tau\,\delta\mu - \tau_0\,\delta\mu_0 + \omega_0\,\delta\kappa_0 - \omega\,\delta\kappa + \lambda_0\,\delta\nu_0 - \lambda\,\delta\nu) - H_2\,\delta t,\\
\delta(\tau, \omega, \nu) &= \Sigma\,.m(\tau\,\delta\mu - \tau_0\,\delta\mu_0 + \omega_0\,\delta\kappa_0 - \omega\,\delta\kappa + \nu\,\delta\lambda - \nu_0\,\delta\lambda_0) - H_2\,\delta t,\\
\delta(\mu, \kappa, \lambda) &= \Sigma\,.m(\mu_0\,\delta\tau_0 - \mu\,\delta\tau + \kappa\,\delta\omega - \kappa_0\,\delta\omega_0 + \lambda_0\,\delta\nu_0 - \lambda\,\delta\nu) - H_2\,\delta t,\\
\delta(\tau, \kappa, \lambda) &= \Sigma\,.m(\tau\,\delta\mu - \tau_0\,\delta\mu_0 + \kappa\,\delta\omega - \kappa_0\,\delta\omega_0 + \lambda_0\,\delta\nu_0 - \lambda\,\delta\nu) - H_2\,\delta t,\\
\delta(\mu, \omega, \nu) &= \Sigma\,.m(\mu_0\,\delta\tau_0 - \mu\,\delta\tau + \omega_0\,\delta\kappa_0 - \omega\,\delta\kappa + \nu\,\delta\lambda - \nu_0\,\delta\lambda_0) - H_2\,\delta t,
\end{aligned}\right\} \qquad (Y^2.)$$

$\kappa_0\ \lambda_0\ \mu_0\ \nu_0\ \tau_0\ \omega_0$ being the initial values of the varying elements $\kappa\ \lambda\ \mu\ \nu\ \tau\ \omega$. If, then, we consider, for example, the first of these 8 combinations (τ, κ, ν), as a function of all of the $3n - 3$ elements $\mu_i\ \omega_i\ \lambda_i$, and of their initial values $\mu_{0,i}\ \omega_{0,i}\ \lambda_{0,i}$, involving also in general the time explicitly, we shall have the following forms for the $6n - 6$ rigorous integrals of the $6n - 6$ equations (S^2.):

$$\left.\begin{aligned}
m_i\tau_i &= \frac{\delta}{\delta\mu_i}(\tau, \kappa, \nu); &\quad m_i\tau_{0,i} &= -\frac{\delta}{\delta\mu_{0,i}}(\tau, \kappa, \nu);\\
m_i\kappa_i &= \frac{\delta}{\delta\omega_i}(\tau, \kappa, \nu); &\quad m_i\kappa_{0,i} &= -\frac{\delta}{\delta\omega_{0,i}}(\tau, \kappa, \nu);\\
m_i\nu_i &= \frac{\delta}{\delta\lambda_i}(\tau, \kappa, \nu); &\quad m_i\nu_{0,i} &= -\frac{\delta}{\delta\lambda_{0,i}}(\tau, \kappa, \nu);
\end{aligned}\right\} \qquad (Z^2.)$$

and in like manner we can deduce forms for the same rigorous integrals, from any one of the eight combinations (Y^2.). The determination of all the varying elements would therefore be fully accomplished, if we could find the complete expression for any one of these 8 combinations.

40. A first approximate expression for any one of them can be found from the form under which we have supposed H_2 to be put, namely, as a function of the elements and of the time, which may be thus denoted:

$$H_2 = H_2(t, \kappa_1, \lambda_1, \mu_1, \nu_1, \tau_1, \omega_1, \cdots \kappa_{n-1}, \lambda_{n-1}, \mu_{n-1}, \nu_{n-1}, \tau_{n-1}, \omega_{n-1}); \qquad (\mathrm{A}^3.)$$

by changing in this function the varying elements to their initial values, and employing the following approximate integrals of the equations (S^2.),

$$\left.\begin{aligned} \mu &= \mu_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \tau_0}\,dt, & \tau &= \tau_0 - \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \mu_0}\,dt, \\ \omega &= \omega_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \kappa_0}\,dt, & \kappa &= \kappa_0 - \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \omega_0}\,dt, \\ \lambda &= \lambda_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \nu_0}\,dt, & \nu &= \nu_0 - \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta \lambda_0}\,dt. \end{aligned}\right\} \qquad (\mathrm{B}^3.)$$

For if we denote, for example, the first of the 8 combinations (Y^2.) by G, so that

$$G = (\tau, \kappa, \nu), \qquad (\mathrm{C}^3.)$$

we shall have, as a first approximate value,

$$G_1 = \int_0^t \left\{ \Sigma \left(\tau_0 \frac{\delta H_2}{\delta \tau_0} + \kappa_0 \frac{\delta H_2}{\delta \kappa_0} + \nu_0 \frac{\delta H_2}{\delta \nu_0} \right) - H_2 \right\} dt; \qquad (\mathrm{D}^3.)$$

and after thus expressing G_1 as a function of the time, and of the initial elements, we can eliminate the initial quantities of the forms τ_0 κ_0 ν_0, and introduce in their stead the final quantities μ ω λ, so as to obtain an expression for G_1 of the kind supposed in (Z^2.), namely, a function of the time t, the varying elements μ ω λ, and their initial values μ_0 ω_0 λ_0. An approximate expression thus found may be corrected by a process of that kind, which has often been employed in this Essay for other similar purposes. For the function G, or the combination (τ, κ, ν), must satisfy rigorously, by (Y^2.), (A^3.), the following partial differential equation:

$$0 = \frac{\delta G}{\delta t} + H_2 \left(t, \frac{1}{m_1}\frac{\delta G}{\delta \omega_1}, \lambda_1, \mu_1, \frac{1}{m_1}\frac{\delta G}{\delta \lambda_1}, \frac{1}{m_1}\frac{\delta G}{\delta \mu_1}, \omega_1, \frac{1}{m_2}\frac{\delta G}{\delta \omega_2}, \cdots \omega_{n-1} \right); \qquad (\mathrm{E}^3.)$$

and each of the other analogous functions or combinations (Y^2.) must satisfy an analogous equation: if then we change G to $G_1 + G_2$, and neglect the squares and products of the coefficients of the small correction G_2, G_1 being a first approximation such as that already found, we are conducted, as a second approximation on principles already explained, to the following expression for this correction G_2:

$$G_2 = -\int_0^t \left\{ \frac{\delta G_1}{\delta t} + H_2 \left(\frac{1}{m_1}\frac{\delta G_1}{\delta \omega_1}, \lambda_1, \mu_1, \frac{1}{m_1}\frac{\delta G_1}{\delta \lambda_1}, \frac{1}{m_1}\frac{\delta G_1}{\delta \mu_1}, \omega_1, \cdots \right) \right\} dt: \qquad (\mathrm{F}^3.)$$

which may be continually and indefinitely improved by a repetition of the same process of correction. We may therefore, theoretically, consider the problem as solved; but it must remain for future consideration, and perhaps for actual trial, to determine which of all these various processes of successive and indefinite approximation, deduced in the present Essay and in the former, as corollaries of one general Method, and as consequences of one central Idea, is best adapted for numeric application, and for the mathematical study of phenomena.