ON A NEW SPECIES OF IMAGINARY QUANTITIES CONNECTED WITH A THEORY OF QUATERNIONS

By

William Rowan Hamilton

(Proceedings of the Royal Irish Academy, 2 (1844), 424–434.)

Edited by David R. Wilkins

1999

On a new Species of Imaginary Quantities connected with a theory of Quaternions. By Sir William R. Hamilton.

Read November 13, 1843.

[*Proceedings of the Royal Irish Academy*, vol. 2 (1844), pp. 424–434.]

It is known to all students of algebra that an imaginary equation of the form $i^2 = -1$ has been employed so as to conduct to very varied and important results. Sir Wm. Hamilton proposes to consider some of the consequences which result from the following system of imaginary equations, or equations between a *system of three different imaginary quantities*:

$$i^2 = j^2 = k^2 = -1 \tag{A}$$

$$ij = k, \quad jk = i, \quad ki = j; \tag{B}$$

$$ji = -k, \quad kj = -i, \quad ik = -j; \tag{C}$$

no linear relation between i, j, k being supposed to exist, so that the equation

$$\mathrm{Q} = \mathrm{Q}',$$

in which

$$\mathrm{Q} = w + ix + jy + kz,$$
$$\mathrm{Q}' = w' + ix' + jy' + kz',$$

and w, x, y, z, w', x', y', z' are real, is equivalent to the four separate equations

$$w = w', \quad x = x', \quad y = y', \quad z = z'.$$

Sir W. Hamilton calls an expression of the form Q a *quaternion*; and the four real quantities w, x, y, z he calls the *constituents* thereof. Quaternions are added or subtracted by adding or subtracting their constituents, so that

$$\mathrm{Q} + \mathrm{Q}' = w + w' + i(x + x') + j(y + y') + k(z + z').$$

Their multiplication is, in virtue of the definitions (A) (B) (C), effected by the formulae

$$\mathrm{QQ}' = \mathrm{Q}'' = w'' + ix'' + jy'' + kz'',$$

$$\left.\begin{aligned} w'' &= ww' - xx' - yy' - zz', \\ x'' &= wx' + xw' + yz' - zy', \\ y'' &= wy' + yw' + zx' - xz', \\ z'' &= wz' + zw' + xy' - yx'. \end{aligned}\right\} \tag{D}$$

which give

$$w''^2 + x''^2 + y''^2 + z''^2 = (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2),$$

and therefore

$$\mu'' = \mu\mu', \tag{E}$$

if we call the positive quantity

$$\mu = \sqrt{w^2 + x^2 + y^2 + z^2},$$

the *modulus* of the quaternion Q. The modulus of the product of any two quaternions is therefore equal to the product of the moduli. Let

$$\left.\begin{aligned} w &= \mu \cos\theta, \\ x &= \mu \sin\theta \cos\phi, \\ y &= \mu \sin\theta \sin\phi \cos\psi, \\ z &= \mu \sin\theta \sin\phi \sin\psi; \end{aligned}\right\} \tag{F}$$

then, because the equations (D) give

$$\begin{aligned} w'w'' + x'x'' + y'y'' + z'z'' &= w(w'^2 + x'^2 + y'^2 + z'^2), \\ ww'' + xx'' + yy'' + zz'' &= w'(w^2 + x^2 + y^2 + z^2), \end{aligned}$$

we have

$$\left.\begin{aligned} \cos\theta'' &= \cos\theta\cos\theta' - \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\psi - \psi')), \\ \cos\theta &= \cos\theta'\cos\theta'' + \sin\theta'\sin\theta''(\cos\phi'\cos\phi'' + \sin\phi'\sin\phi''\cos(\psi' - \psi'')), \\ \cos\theta' &= \cos\theta''\cos\theta + \sin\theta''\sin\theta(\cos\phi''\cos\phi + \sin\phi''\sin\phi\cos(\psi'' - \psi')), \end{aligned}\right\} \tag{G}$$

Consider x, y, z as the rectangular coordinates of a point of space, and let R be the point where the radius vector of x, y, z (prolonged if necessary) intersects the spheric surface described about the origin with a radius equal to unity; call R the *representative point* of the quaternion Q, and let the polar coordinates ϕ and ψ, which determine R upon the sphere, be called the *co-latitude* and the *longitude* of the representative point R, or of the quaternion Q itself; let also the other angle θ be called the *amplitude* of the quaternion; so that a quaternion is completely determined by its modulus, amplitude, co-latitude and longitude. Construct the representative points R$'$ and R$''$, of the other factor Q$'$, and of the product Q$''$; and complete the spherical triangle RR$'$R$''$ by drawing the arcs RR$'$, R$'$R$''$, R$''$R. Then, the equations (G) become

$$\begin{aligned} \cos\theta'' &= \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos \mathrm{RR}', \\ \cos\theta &= \cos\theta'\cos\theta'' + \sin\theta'\sin\theta''\cos \mathrm{R}'\mathrm{R}'', \\ \cos\theta' &= \cos\theta''\cos\theta + \sin\theta''\sin\theta\cos \mathrm{R}''\mathrm{R}, \end{aligned}$$

and consequently shew that the angles of the triangle $\mathrm{RR'R''}$ are

$$\mathrm{R} = \theta, \quad \mathrm{R'} = \theta', \quad \mathrm{R''} = \pi - \theta''; \tag{H}$$

these angles are therefore respectively equal to the amplitudes of the factors, and the supplement (to two right angles) of the amplitude of the product. The equations (D) show, further, that the *product-point* $\mathrm{R''}$ is to the right or left of the *multiplicand-point* $\mathrm{R'}$, with respect to the *multipler-point* R, according as the semiaxis of $+z$ (or its intersection with the spheric surface) is to the right or left of the semiaxis of $+y$, with respect to the semiaxis of $+x$: that is, according as the positive direction of rotation in longitude is towards the right or left. A change in the order of the two quaternion-factors would throw the product-point $\mathrm{R''}$ from the right to the left, or from the left to the right of $\mathrm{RR'}$.

It results from these principles, that if $\mathrm{RR'R''}$ be any spherical triangle; if, also, $\alpha\,\beta\,\gamma$ be the rectangular coordinates of R, $\alpha'\,\beta'\,\gamma'$ those of $\mathrm{R'}$, and $\alpha''\,\beta''\,\gamma''$ of $\mathrm{R''}$, the centre of the sphere being origin, and the radius being unity; and if the rotation round $+x$ from $+y$ to $+z$ be of the same (right-handed or left-handed) character as that round R from $\mathrm{R'}$ to $\mathrm{R''}$; then the following formula of multiplication, according to the rules of quaternions, will hold good:

$$\begin{aligned} \{\cos\mathrm{R} + (i\alpha + j\beta + k\gamma)\sin\mathrm{R}\}&\{\cos\mathrm{R'} + (i\alpha' + j\beta' + k\gamma')\sin\mathrm{R'}\} \\ &= -\cos\mathrm{R''} + (i\alpha'' + j\beta'' + k\gamma'')\sin\mathrm{R''}. \end{aligned} \tag{I}$$

Developing and decomposing this imaginary or symbolic formula (I), we find that it is equivalent to the system of the four following real equations, or equations between real quantities:

$$\left.\begin{aligned} -\cos\mathrm{R''} &= \cos\mathrm{R}\cos\mathrm{R'} - (\alpha\alpha' + \beta\beta' + \gamma\gamma')\sin\mathrm{R}\sin\mathrm{R'}; \\ \alpha''\sin\mathrm{R''} &= \alpha\sin\mathrm{R}\cos\mathrm{R'} + \alpha'\sin\mathrm{R'}\cos\mathrm{R} + (\beta\gamma' - \gamma\beta')\sin\mathrm{R}\sin\mathrm{R'}; \\ \beta''\sin\mathrm{R''} &= \beta\sin\mathrm{R}\cos\mathrm{R'} + \beta'\sin\mathrm{R'}\cos\mathrm{R} + (\gamma\alpha' - \alpha\gamma')\sin\mathrm{R}\sin\mathrm{R'}; \\ \gamma''\sin\mathrm{R''} &= \gamma\sin\mathrm{R}\cos\mathrm{R'} + \gamma'\sin\mathrm{R'}\cos\mathrm{R} + (\alpha\beta' - \beta\alpha')\sin\mathrm{R}\sin\mathrm{R'}. \end{aligned}\right\} \tag{K}$$

Of these equations (K), the first is only an expression of the well-known theorem, already employed in these remarks, which serves to connect a side of any spherical triangle with the three angles thereof. The three other equations (K) are an expression of another theorem (which possibly is new), namely that a force $= \sin\mathrm{R''}$, directed from the centre of the sphere to the point $\mathrm{R''}$, is statically equivalent to the system of three other forces, one directed to R, and equal to $\sin\mathrm{R}\cos\mathrm{R'}$, another directed to $\mathrm{R'}$, and equal to $\sin\mathrm{R'}\cos\mathrm{R}$, and the third equal to $\sin\mathrm{R}\sin\mathrm{R'}\sin\mathrm{RR'}$, and directed towards that pole of the arc $\mathrm{RR'}$, which lies at the same side of this arc as $\mathrm{R''}$. It is not difficult to prove this theorem otherwise; but it may be regarded as interesting to see that the four equations (K) are included so simply in the one formula (I) of multiplication of quaternions, and are obtained so easily by developing and decomposing that formula, according to the fundamental definitions (A) (B) (C). A new sort of algorithm, or calculus, for spherical trigonometry, appears to be thus given, or indicated. And by supposing the three corners of the spherical triangle $\mathrm{RR'R''}$ to tend indefinitely to close up in that one point which is the intersection of the spheric surface with the positive semiaxis of x, each coordinate α will tend to become $= 1$, and each β and γ to vanish, while the sum of the three angles will tend to become $= \pi$; so that the following well known and important equations in the usual calculus of imaginaries, as connected with plane trigonometry, namely,

$$(\cos \mathrm{R} + i \sin \mathrm{R})(\cos \mathrm{R}' + i \sin \mathrm{R}') = \cos(\mathrm{R} + \mathrm{R}') + i \sin(\mathrm{R} + \mathrm{R}'),$$

(in which $i^2 = -1$), is found to result, as a limiting case, from the more general formula (I).

In the ordinary theory there are only two different square roots of negative unity ($+i$ and $-i$), and they differ only in their signs. In the present theory, in order that a quaternion, $w + ix + jy + kz$, should have its square $= -1$, it is necessary and sufficient that we should have

$$w = 0, \qquad x^2 + y^2 + z^2 = +1;$$

we are conducted, therefore, to the extended expression

$$\sqrt{-1} = i \cos \phi + j \sin \phi \cos \psi + k \sin \phi \sin \psi, \tag{L}$$

which may be called an *imaginary unit*, because its modulus is $= 1$, and its square is negative unity. To distinguish one such imaginary unit from another, we may adopt the notation,

$$i_{\mathrm{R}} = i\alpha + j\beta + k\gamma, \text{ which gives } i_{\mathrm{R}}^2 = -1, \tag{L$'$}$$

R being still that point on the spheric surface which has α, β, γ (or $\cos \phi$, $\sin \phi \cos \psi$, $\sin \phi \sin \psi$) for its rectangular coordinates; and then the formula of multiplication (I) becomes, for any spherical triangle, in which the rotation round R, from R$'$ to R$''$, is positive,

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}') = -\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}''. \tag{I$'$}$$

If P$''$ be the *positive pole* of the arc RR$'$, or the pole to which the least rotation from R$'$ round R is positive, then the product of the two imaginary units in the first member of this formula (which may be any two such units), is the following:

$$i_{\mathrm{R}} i_{\mathrm{R}'} = -\cos \mathrm{RR}' + i_{\mathrm{P}''} \sin \mathrm{RR}'; \tag{M}$$

we have also, for the product of the same two factors, taken in the opposite order, the expression

$$i_{\mathrm{R}'} i_{\mathrm{R}} = -\cos \mathrm{RR}' - i_{\mathrm{P}''} \sin \mathrm{RR}', \tag{N}$$

which differs only in the sign of the imaginary part; and the product of these two products is unity, because, in general,

$$(w + ix + jy + kz)(w - ix - jy - kz) = w^2 + x^2 + y^2 + z^2; \tag{O}$$

we have, therefore,

$$i_{\mathrm{R}} i_{\mathrm{R}'} . i_{\mathrm{R}'} i_{\mathrm{R}} = 1, \tag{P}$$

and the products $i_{\mathrm{R}} i_{\mathrm{R}'}$ and $i_{\mathrm{R}'} i_{\mathrm{R}}$ may be said to be *reciprocals* of each other.

In general, in virtue of the fundamental equations of definition, (A), (B), (C), although the *distributive* character of the multiplication of ordinary algebraic quantities (real or imaginary) extends to the operation of the same name in the theory of quaternions, so that

$$\mathrm{Q}(\mathrm{Q}' + \mathrm{Q}'') = \mathrm{QQ}' + \mathrm{QQ}'', \text{ \&c.},$$

yet the *commutative* character is lost, and we cannot generally write for the new as for the old imaginaries,

$$\mathrm{QQ}' = \mathrm{Q}'\mathrm{Q},$$

since we have, for example, $ji = -ij$. However, in virtue of the same definitions, it will be found that another important property of the old multiplication is preserved, or extended to the new, namely, that which may be called the *associative* character of the operation, and which may have for its type the formula

$$\mathrm{Q}.\mathrm{Q}'\mathrm{Q}''.\mathrm{Q}''', \mathrm{Q}^{IV} = \mathrm{QQ}'.\mathrm{Q}''\mathrm{Q}'''\mathrm{Q}^{IV};$$

thus we have, generally,

$$\mathrm{Q}.\mathrm{Q}'\mathrm{Q}'' = \mathrm{QQ}'.\mathrm{Q}'', \tag{Q}$$

$$\mathrm{Q}.\mathrm{Q}'\mathrm{Q}''\mathrm{Q}''' = \mathrm{QQ}'.\mathrm{Q}''\mathrm{Q}''' = \mathrm{QQ}'\mathrm{Q}''.\mathrm{Q}''', \tag{Q$'$}$$

and so on for any number of factors; the notation $\mathrm{QQ}'\mathrm{Q}''$ being employed to express that one determined quaternion, which, in virtue of the theorem (Q), is obtained, whether we first multiply Q'' as a multiplicand by Q' as a multiplier, and then multiply the product $\mathrm{Q}'\mathrm{Q}''$ as a multiplicand by Q as a multiplier; or multiply first Q' by Q, and then Q'' by QQ'. With the help of this principle, we might easily prove the equation (P), by observing that its first member $= i_{\mathrm{R}} i_{\mathrm{R}'}^2 i_{\mathrm{R}} = -i_{\mathrm{R}}^2 = 1$.

In the same manner it is seen at once that

$$i_{\mathrm{R}} i_{\mathrm{R}'}.i_{\mathrm{R}'} i_{\mathrm{R}''}.i_{\mathrm{R}''} i_{\mathrm{R}'''}.\ldots.i_{\mathrm{R}^{(n-1)}} i_{\mathrm{R}} = (-1)^n, \tag{P$'$}$$

whatever n points upon the spheric surface may be denoted by R, R', R'', R''',… $\mathrm{R}^{(n-1)}$: and by combining this principle with that expressed by (M), it is not difficult to prove that for any spherical polygon $\mathrm{RR}'\ldots\mathrm{R}^{(n-1)}$, the following formula holds good:

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}')(\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}'')$$

$$\cdots(\cos \mathrm{R}^{(n-1)} + i_{\mathrm{R}^{(n-1)}} \sin \mathrm{R}^{(n-1)}) = (-1)^n, \tag{R}$$

which includes the theorem (I$'$) for the case of a spherical triangle, and in which the arrangement of the n points may be supposed, for simplicity, to be such that the rotations round R from R' to R'', round R' from R'' to R''', and so on, are all positive, and each less than two right angles, though it is easy to interpret the expression so as to include also the cases where any or all of these conditions are violated. When the polygon becomes infinitely small, and therefore plane, the imaginary units become all equal to each other, and may be denoted by the common symbol i; and the formula (R) agrees then with the known relation, that

$$\pi - \mathrm{R} + \pi - \mathrm{R}' + \pi - \mathrm{R}'' + \cdots + \pi - \mathrm{R}^{(n-1)} = 2\pi.$$

Again, let R, R', R'' be, respectively, the representative points of any three quaternions Q, Q', Q'', and let $R_{\prime}$, $R_{\prime\prime}$, $R_{\prime\prime\prime}$ be the representative points of the three other quaternions, QQ', $Q'Q''$, $QQ'Q''$, derived by multiplication from the former; then the algebraical principle expressed by the formula (Q) may be geometrically enunciated by saying that the two points $R_{\prime}$ and $R_{\prime\prime}$ are the foci of a spherical conic which touches the four sides of the spherical quadrilateral $RR'R''R_{\prime\prime\prime}$; and analogous theorems respecting spherical pentagons and other polygons may be deduced, by constructing similarly the formulæ (Q'),&c.

In general, a quaternion Q, like an ordinary imaginary quantity, may be put under the form,

$$Q = \mu(\cos\theta + (-1)^{\frac{1}{2}}\sin\theta) = w + (-1)^{\frac{1}{2}} r, \tag{S}$$

provided that we assign to $(-1)^{\frac{1}{2}}$, or $\sqrt{-1}$, the extended meaning (L), which involves two arbitrary angles; and the same general quaternion Q may be considered as a root of a quadratic equation, with real coefficients, namely

$$Q^2 - 2wQ + \mu^2 = 0, \tag{S$'$}$$

which easily conducts to the following expression for a quotient, or formula for the division of quaternions,

$$Q^{-1}Q'' = \frac{Q''}{Q} = \frac{2w - Q}{\mu^2}Q'', \tag{S$''$}$$

if we define $Q^{-1}Q''$ or $\dfrac{Q''}{Q}$ to mean that quaternion Q' which gives the product Q'', when it is multiplied as a multiplicand by Q as a multiplier. The same general formula (S$''$) of division may easily be deduced from the equation (O), by writing that equation as follows,

$$(w + ix + jy + kz)^{-1} = \frac{w - ix - jy - kz}{w^2 + x^2 + y^2 + z^2}; \tag{O$'$}$$

or it may be obtained from the four general equations of multiplication (D), by treating the four constituents of the multiplicand, namely w', x', y', z', as the four sought quantities, while w, x, y, z, and w'', x'', y'', z'', are given; or from a construction of spherical trigonometry, on principles already laid down.

The general expression (S) for a quaternion may be raised to any power with a real exponent q, in the same manner as an ordinary imaginary expression, by treating the square root of -1 which it involves as an imaginary unit i_R having (in general) a fixed direction; raising the modulus μ to the proposed real power; and multiplying the amplitude θ, increased or diminished by any whole number of circumferences, by the exponent q: thus

$$(\mu(\cos\theta + i_R \sin\theta))^q = \mu^q(\cos q(\theta + 2n\pi) + i_R \sin q(\theta + 2n\pi)), \tag{T}$$

if q be real, and if n be any whole number. For example, a quaternion has in general two, and only two, different square roots, and they differ only in their signs, being both included in the formula,

$$(\mu(\cos\theta + i_R \sin\theta))^{\frac{1}{2}} = \mu^{\frac{1}{2}}\left(\cos\left(\frac{\theta}{2} + n\pi\right) + i_R \sin\left(\frac{\theta}{2} + n\pi\right)\right), \tag{T$'$}$$

in which it is useless to assign to n any other values than 0 and 1; although, in the particular case where the original quaternion reduces itself to a real and negative quantity, so that $\theta = \pi$, this formula (T′) becomes

$$(-\mu)^{\frac{1}{2}} = \pm\mu^{\frac{1}{2}} i_{\mathrm{R}}, \text{ or simply } (-\mu)^{\frac{1}{2}} = \mu^{\frac{1}{2}} i_{\mathrm{R}}, \qquad (\mathrm{T}'')$$

the direction of i_{R} remaining here entirely undetermined; a result agreeing with the expression (L) or (L′) for $\sqrt{-1}$. In like manner the quaternions, which are cube roots of unity, are included in the expression

$$1^{\frac{1}{3}} = \cos\frac{2n\pi}{3} + i_{\mathrm{R}} \sin\frac{2n\pi}{3}, \qquad (\mathrm{T}''')$$

i_{R} denoting here again an imaginary unit, with a direction altogether arbitrary.

If we make, for abridgment

$$f(Q) = 1 + \frac{\mathrm{Q}}{1} + \frac{\mathrm{Q}^2}{1.2} + \frac{\mathrm{Q}^2}{1.2.3} + \&\mathrm{c.}, \qquad (\mathrm{U})$$

the series here indicated will be always convergent, whatever quaternion Q may be; and we can always separate its real and imaginary parts by the formula

$$f(w + i_{\mathrm{R}} r) = f(w)(\cos r + i_{\mathrm{R}} \sin r); \qquad (\mathrm{U}')$$

which gives, reciprocally, for the inverse function f^{-1}, the expression

$$f^{-1}(\mu(\cos\theta + i_{\mathrm{R}} \sin\theta)) = \log\mu + i_{\mathrm{R}}(\theta + 2n\pi), \qquad (\mathrm{U}'')$$

u being any whole number, and $\log\mu$ being the natural, or Napierian, logarithm of μ, or, in other words, that real quantity, positive or negative, of which the function f is equal to the given real and positive modulus μ. And although the ordinary property of exponential functions, namely

$$f(\mathrm{Q}).f(\mathrm{Q}') = f(\mathrm{Q} + \mathrm{Q}'),$$

does not in general hold good, in the present theory, unless the two quaternions Q and Q′ be codirectional, yet we may raise the function f to any real power by the formula

$$(f(w + i_{\mathrm{R}} r))^q = f(q(w + i_{\mathrm{R}}\overline{r + 2n\pi})), \qquad (\mathrm{U}''')$$

which it is natural to extend, by definition, to the case where the exponent q becomes itself a quaternion. The general equation,

$$\mathrm{Q}_{\prime}^{q} = \mathrm{Q}_{\prime}', \qquad (\mathrm{V})$$

when put under the form

$$(f(w + i_{\mathrm{R}} r))^q = f(w' + i_{\mathrm{R}'} r'), \qquad (\mathrm{V}')$$

will then give

$$q = \frac{\{w' + i_{\mathrm{R}'}(r' + 2n'\pi)\}\{w - i_{\mathrm{R}}(r + 2n\pi)\}}{w^2 + (r + 2n\pi)^2}; \qquad (\mathrm{V}'')$$

and thus the general expression for a quaternion q, which is one of the logarithms of a given quaternion $\mathrm{Q}_{\prime}'$ to a given base $\mathrm{Q}_{\prime}$, is found to involve two independent whole numbers n and n', as in the theories of Graves and Ohm, respecting the general logarithms of ordinary imaginary quantities to ordinary imaginary bases.

For other developments and applications of the new theory, it is necessary to refer to the original paper from which this abstract is taken, and which will probably appear in the twenty-first volume of the Transactions of the Academy.