THEORY OF SYSTEMS OF RAYS

By

William Rowan Hamilton

(Transactions of the Royal Irish Academy, vol. 15 (1828), pp. 69–174.)

Edited by David R. Wilkins

2001

NOTE ON THE TEXT

The *Theory of Systems of Rays* by William Rowan Hamilton was originally published in volume 15 of the *Transactions of the Royal Irish Academy*. It is included in *The Mathematical Papers of Sir William Rowan Hamilton, Volume I: Geometrical Optics*, edited for the Royal Irish Academy by A. W. Conway and J. L. Synge, and published by Cambridge University Press in 1931.

Although the table of contents describes the contents of *Part Second* and *Part Third*, neither of these were published in Hamilton's lifetime. *Part Second* was included in *The Mathematical Papers of Sir William Rowan Hamilton, Volume I: Geometrical Optics*, edited for the Royal Irish Academy by A. W. Conway and J. L. Synge, and published by Cambridge University Press in 1931. (It is not included in this edition.) The editors of this volume were unable to locate a manuscript for *Part Third*.

The printed date 'Dec. 13, 1824' in fact refers to the date on which the paper *On Caustics* was read before the Royal Irish Academy. *On Caustics: Part First* was not accepted for publication by the Academy, and was published for the first time in volume 1 of the *The Mathematical Papers of Sir William Rowan Hamilton*. The *Theory of Systems of Rays* was in fact read before the Royal Irish Academy on April 23rd, 1827. The history of the papers *On Caustics* and the *Theory of Systems of Rays* is described on page 462 of the first volume of *The Mathematical Papers of Sir William Rowan Hamilton*.

This edition corrects various errata noted by Hamilton, and listed at the end of the original publication, and at the end of the *First Supplement* in volume 16, part 2.

A small number of changes have been made to the spelling and punctuation of the original publication. Also the following discrepancies in the original publication have been corrected:

— in the Table of Contents, the heading for Part Second is 'PART SECOND: ON SYSTEMS OF REFRACTED RAYS', though elsewhere it is give as 'PART SECOND: ON ORDINARY SYSTEMS OF REFRACTED RAYS';

— in the final sentence of the article numbered 58, the first occurrence of π is denoted by ϖ, though the second occurrence in that sentence is denoted by π;

— in article 65, the formula preceding (I''') as been corrected (following Conway and Synge) by the replacement of dx by dy following d^2q'' and d^2q;

— in article 69, parentheses have been inserted into the numerators of the expression for the quantity $\mu^{(t)}$ preceding (X'''') (following Conway and Synge) in order to resolve the ambiguities which would otherwise be present in this expression;

— in article 75, equation ($\mathrm{Z}^{(8)}$), π is denoted by ϖ;

— in article 69, from equation ($\mathrm{X}^{(6)}$) onwards, the character $\mathcal{C}$ has been used in this edition to represent a character, which resembles a script C in the original publication, and which denotes a quantity defined in equation ($\mathrm{Y}^{(6)}$).

David R. Wilkins

Dublin, June 1999
Corrected October 2001

Theory of Systems of Rays. *By* W. R. Hamilton, *Professor of Astronomy in the University of Dublin.*

Read Dec. 13, 1824.*

[*Transactions of the Royal Irish Academy*, vol. 15, (1828), pp. 69–174.]

INTRODUCTION.

Those who have hitherto written upon the properties of Systems of Rays, have confined themselves for the most part to the considerations of those particular systems, which are produced by ordinary reflexion and refraction at plane surfaces and at surfaces of revolution. Malus, indeed, in his *Traité D'Optique*, has considered the subject in a more general manner, and has made some valuable remarks upon systems of rays, disposed in any manner in space, or issuing from any given surface according to any given law; but besides that those remarks are far from exhausting the subject, Malus appears to me to have committed some important errors, in the application of his theory to the systems produced by combinations of mirrors and lenses. And with the exception of this author, I am not aware that any one has hitherto sought to investigate, in all their generality, the properties of optical systems; much less to establish principles respecting systems of rays in general, which shall be applicable not only to the theory of light, but also to that of sound and of heat. To establish such principles, and to investigate such properties is the aim of the following essay. I hope that mathematicians will find its results and reasonings interesting, and that they will pardon any defects which they may perceive in the execution of so abstract and extensive a design.

Observatory,
June 1827.

* Since this paper was first read before the Academy, various delays have occurred, which postponed the printing until the present time. I have availed myself of these delays, to add some developments and applications of my Theory, which would, I thought, be useful.

CONTENTS.

PART FIRST. ON ORDINARY SYSTEMS OF REFLECTED RAYS.

I. *Analytic expressions of the Law of Ordinary Reflexion.*

II. *Theory of Focal Mirrors.*

III. *Surfaces of Constant Action.*

XII. *Aberrations.*

XIII. *Density.*

XIX. *On the determination of reflecting and refracting surfaces, by means of their caustic surfaces.*

XX. *On the caustics of a given reflecting or refracting curve.*

XXI. *On the conditions of Achromatism.*

XXII. *On systems of atmospheric rays.*

PART THIRD. ON EXTRAORDINARY SYSTEMS, AND SYSTEMS OF RAYS IN GENERAL.

XXIII. *On plane systems of rays.*

XXIV. *On developable systems.*

XXV. *On undevelopable systems.*

XXIX. *On other extraordinary systems.*

XXX. *Law of least action.*

CONCLUSION

PART FIRST.

ON ORDINARY SYSTEMS OF REFLECTED RAYS.

SECTION I.

Analytic expressions of the law of ordinary reflexion.

[1.] When a ray of light is reflected at a mirror, we know by experience, that the normal to the mirror, at the point of incidence, bisects the angle between the incident and the reflected rays. If therefore two forces, each equal to unity, were to act at the point of incidence, in the directions of the two rays, their resultant would act in the direction of the normal, and would be equal to twice the cosine of the angle of incidence. If then we denote by (ρl) $(\rho' l)$ (nl) the angles which the two rays and the normal make respectively with any assumed line (l), and by (I) the angle of incidence, we shall have the following formula,

$$\cos.\rho l + \cos.\rho' l = 2\cos.I\,.\,\cos.nl \tag{A}$$

which is the analytic representation of the known law of reflexion, and includes the whole theory of catoptrics.

[2.] It follows from (A) that if we denote by ρx, ρy, ρz, $\rho' x$, $\rho' y$, $\rho' z$, nx, ny, nz, the angles which the two rays and the normal make respectively with three rectangular axes, we shall have the three following equations,

$$\left.\begin{aligned}\cos.\rho x + \cos.\rho' x = 2\cos.I\,.\,\cos.nx\\ \cos.\rho y + \cos.\rho' y = 2\cos.I\,.\,\cos.ny\\ \cos.\rho z + \cos.\rho' z = 2\cos.I\,.\,\cos.nz\end{aligned}\right\} \tag{B}$$

which determine the direction of the reflected ray, when we know that of the incident ray, and the tangent plane to the mirror.

[3.] Let (x, y, z) be the coordinates of the point of incidence; $x+\delta x$, $y+\delta y$, $z+\delta z$, those of a point infinitely near; if this point be upon the mirror we shall have

$$\cos.nx\,.\,\delta x + \cos.ny\,.\,\delta y + \cos.nz\,.\,\delta z = 0,$$

and therefore, by (B),

$$\begin{aligned}0 = \cos.\rho x\,.\,\delta x + \cos.\rho y\,.\,\delta y + \cos.\rho z\,.\,\delta z&\\ + \cos.\rho' x\,.\,\delta x + \cos.\rho' y\,.\,\delta y + \cos.\rho' z\,.\,\delta z.&\end{aligned} \tag{C}$$

Now if we assume any point $X\ Y\ Z$ on the incident ray, at a distance ρ from the mirror, and another point $X'\ Y'\ Z'$ on the reflected ray at a distance ρ' from the mirror, the distances of those assumed points from the point $x + \delta x$, $y + \delta y$, $z + \delta z$, will be

$$\rho + \delta\rho = \rho + \frac{d\rho}{dx} \,.\, \delta x + \frac{d\rho}{dy} \,.\, \delta y + \frac{d\rho}{dz} \,.\, \delta z,$$
$$\rho' + \delta\rho' = \rho + \frac{d\rho'}{dx} \,.\, \delta x + \frac{d\rho'}{dy} \,.\, \delta y + \frac{d\rho'}{dz} \,.\, \delta z;$$

and because

$$\rho^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2,$$
$$\rho'^2 = (X' - x)^2 + (Y' - y)^2 + (Z' - z)^2,$$

we shall have

$$\frac{d\rho}{dx} = -\frac{X - x}{\rho}, \quad \frac{d\rho}{dy} = -\frac{Y - y}{\rho}, \quad \frac{d\rho}{dz} = -\frac{Z - z}{\rho},$$
$$\frac{d\rho'}{dx} = -\frac{X' - x}{\rho'}, \quad \frac{d\rho'}{dy} = -\frac{Y' - y}{\rho'}, \quad \frac{d\rho'}{dz} = -\frac{Z' - z}{\rho'},$$

that is

$$\frac{d\rho}{dx} = -\cos.\, \rho x, \quad \frac{d\rho}{dy} = -\cos.\, \rho y, \quad \frac{d\rho}{dz} = -\cos.\, \rho z,$$
$$\frac{d\rho'}{dx} = -\cos.\, \rho' x, \quad \frac{d\rho'}{dy} = -\cos.\, \rho' y, \quad \frac{d\rho'}{dz} = -\cos.\, \rho' z;$$

and finally, by (C)

$$\delta\rho + \delta\rho' = 0. \tag{D}$$

This equation (D) is called the *Principle of least Action*, because it expresses that if the coordinates of the point of incidence were to receive any infinitely small variations consistent with the nature of the mirror, the bent path $(\rho + \rho')$ would have its variation nothing; and if light be a material substance, moving with a velocity unaltered by reflection, this bent path $\rho + \rho'$ measures what in mechanics is called the *Action*, from the one assumed point to the other. Laplace has deduced the formula (D), together with analogous formulæ for ordinary and extraordinary refraction, by supposing light to consist of particles of matter, moving with certain determined velocities, and subject only to forces which are insensible at sensible distances. The manner in which I have deduced it, is independent of any hypothesis about the nature or the velocity of light; but I shall continue to call it, from analogy, the principle of least action.

[4.] The formula (D) expresses, that if we assume any two points, one on each ray, the sum of the distances of these two assumed points from the point of incidence, is equal to the sum of their distances from any infinitely near point upon the mirror. If therefore we construct an ellipsoid of revolution, having its two foci at the two assumed points, and its axis equal to the bent path traversed by the light in going from the one point to the other, this ellipsoid will touch the mirror at the point of incidence. Hence it may be inferred, that every normal to an ellipsoid of revolution bisects the angle between the lines drawn to the two foci; and therefore that rays issuing from one focus of an ellipsoid mirror, would be reflected accurately to the other.

[5.] These theorems about the ellipsoid have long been known; to deduce the known theorems corresponding, about the hyperboloid and plane, I observe that from the manner in which the formula (D) has been obtained, we must change the signs of the distances ρ, ρ', if the assumed points X, Y, Z, X', Y', Z', to which they are measured, be not upon the rays themselves, but on the rays produced. If therefore, we assume one point X, Y, Z, upon the incident ray, and the other point X', Y', Z', on the reflected ray produced behind the mirror, the equation (D) expresses that the difference of the distances of these two points from the point of incidence, is the same as the difference of their distances from any infinitely near point upon the mirror; so that if we construct a hyperboloid, having its axis equal to this difference, and having its foci at the two assumed points, this hyperboloid will touch the mirror. The normal to a hyperboloid bisects therefore the angle between the line drawn to one focus, and the produced part of the line drawn to the other focus; from which it follows, that rays issuing from one focus of a hyperboloid mirror, would after reflection diverge from the other focus. A plane is a hyperboloid whose axis is nothing, and a paraboloid is an ellipsoid whose axis is infinite; if, therefore, rays issued from the focus of a paraboloid mirror, they would be reflected parallel to its axis; and if rays issuing from a luminous point any where situated fall upon a plane mirror, they diverge after reflection from a point situated at an equal distance behind the mirror. These are the only mirrors giving accurate convergence or divergence, which have hitherto been considered by mathematicians: in the next section I shall treat the subject in a more general manner, and examine what must be the nature of a mirror, in order that it may reflect to a given point the rays of a given system.

II. *Theory of focal mirrors.*

[6.] The question, to find a mirror which shall reflect to a given focus the rays of a given system, is analytically expressed by the following differential equation,

$$(\alpha + \alpha')\,dx + (\beta + \beta')\,dy + (\gamma + \gamma')\,dz = 0, \tag{E}$$

x, y, z, being the coordinates of the mirror, and α, β, γ, α', β', γ', representing for abridgment the cosines of the angles which the incident and reflected rays make with the axes of coordinates. In this equation, which follows immediately from (C), or from (B), α, β, γ, are to be considered as given functions of x, y, z, depending on the nature of the incident system, and α', β', γ', as other given functions of x, y, z, depending on the position of the focus; and when these functions are of such a nature as to render integrable the equation (E), the integral will represent an infinite number of different mirrors, each of which will possess the property of reflecting to the given focus, the rays of the given system, and which for that reason I shall call *focal mirrors.*

[7.] To find under what circumstances the equation (E) is integrable, I observe that the part

$$\alpha'\,dx + \beta'\,dy + \gamma'\,dz$$

is always an exact differential; for if we represent by X', Y', Z' the coordinates of the given focus, and by ρ' the distance of that focus from the point of incidence, we shall have the equations

$$X' - x = \alpha'\rho', \quad Y' - y = \beta'\rho', \quad Z' - z = \gamma'\rho',$$

and therefore

$$\alpha'\,dx + \beta'\,dy + \gamma'\,dz = -d\rho'$$

because

$$\alpha'^2 + \beta'^2 + \gamma'^2 = 1, \quad \alpha'\,d\alpha' + \beta'\,d\beta' + \gamma'\,d\gamma' = 0.$$

If therefore the equation (E) be integrable, that is, if it can be satisfied by any unknown relation between x, y, z, it is necessary that in establishing this unknown relation between those three variables, the part $(\alpha\,.\,dx + \beta\,.\,dy + \gamma\,.\,dz)$ should also be an exact differential of a function of the two variables which remain independent; the condition of this circumstance is here

$$(\alpha+\alpha')\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right) + (\beta+\beta')\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right) + (\gamma+\gamma')\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right) = 0, \tag{F}$$

and I am going to shew, from the relations which exist between the functions α, β, γ, that this condition cannot be satisfied, unless we have separately

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = 0, \quad \frac{d\gamma}{dx} - \frac{d\alpha}{dz} = 0, \quad \frac{d\alpha}{dy} - \frac{d\beta}{dx} = 0, \tag{G}$$

that is, unless the formula $(\alpha\,.\,dx + \beta\,.\,dy + \gamma\,.\,dz)$ be an exact differential of a function of x, y, z, considered as three independent variables.

[8.] For this purpose I observe, that since the functions α, β, γ, are the cosines of the angles which the incident ray passing through the point (x, y, z) makes with the axes, they will not vary when the coordinates x, y, z, receive any variations δx, δy, δz proportional to those cosines α, β, γ; because then the point $x + \delta x$, $y + \delta y$, $z + \delta z$, will belong to the same incident ray as the point x, y, z. This remark gives us the following equations,

$$\begin{aligned}
\alpha\,.\,\frac{d\alpha}{dx} + \beta\,.\,\frac{d\alpha}{dy} + \gamma\,.\,\frac{d\alpha}{dz} &= 0,\\
\alpha\,.\,\frac{d\beta}{dx} + \beta\,.\,\frac{d\beta}{dy} + \gamma\,.\,\frac{d\beta}{dz} &= 0,\\
\alpha\,.\,\frac{d\gamma}{dx} + \beta\,.\,\frac{d\gamma}{dy} + \gamma\,.\,\frac{d\gamma}{dz} &= 0,
\end{aligned}$$

and combining these with the relations

$$\begin{aligned}
\alpha\,.\,\frac{d\alpha}{dx} + \beta\,.\,\frac{d\beta}{dx} + \gamma\,.\,\frac{d\gamma}{dx} &= 0,\\
\alpha\,.\,\frac{d\alpha}{dy} + \beta\,.\,\frac{d\beta}{dy} + \gamma\,.\,\frac{d\gamma}{dy} &= 0,\\
\alpha\,.\,\frac{d\alpha}{dz} + \beta\,.\,\frac{d\beta}{dz} + \gamma\,.\,\frac{d\gamma}{dz} &= 0,
\end{aligned}$$

which result from the known formula

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

we find that the three quantities

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy}, \quad \frac{d\gamma}{dx} - \frac{d\alpha}{dz}, \quad \frac{d\alpha}{dy} - \frac{d\beta}{dx},$$

are proportional to (α, β, γ), and therefore that the condition (F) resolves itself into the three equations (G).

[9.] These conditions (G) admit of a simple geometrical enunciation; they express that the rays of the incident system are cut perpendicularly by a series of surfaces, having for equation

$$\int (\alpha\, dx + \beta\, dy + \gamma\, dz) = \text{const.} \tag{H}$$

Let X, Y, Z, be the point in which an incident ray is crossed by any given surface of this series (H), and let ρ be its distance from the point of incidence (x, y, z): we shall have

$$X - x = \alpha\rho, \quad Y - y = \beta\rho, \quad Z - z = \gamma\rho,$$

and therefore,

$$\alpha \,.\, dx + \beta \,.\, dy + \gamma \,.\, dz = -d\rho,$$

because

$$\alpha \,.\, dX + \beta \,.\, dY + \gamma \,.\, dZ = 0.$$

We may therefore put the differential equation of the mirror (E) under the form

$$d\rho + d\rho' = 0,$$

of which the integral

$$\rho + \rho' = \text{const.} \tag{I}$$

expresses that the whole bent path traversed by the light in going from the perpendicular surface (H) to the mirror, and from the mirror to the focus, is of a constant length, the same for all the rays. In this interpretation of the integral (I) we have supposed the distances, ρ, ρ', positive; that is, we have supposed them measured upon the rays themselves; if on the contrary, they are measured on the rays produced behind the mirror, they are then to be considered as negative.

[10.] Then, in general, if it be required to find a mirror which shall reflect to a given focus the rays of a given system, we must try whether the rays of that system are cut perpendicularly by any series of surfaces; for unless this condition be satisfied, the problem is impossible. When we have found a surface cutting the incident rays perpendicularly, we have only to take upon each of the rays a point such that the sum or difference of its distances, from the perpendicular surface and from the given focus, may be equal to any constant quantity; the locus of the points thus determined will be a focal mirror, possessing the property required. Or, which comes to the same thing, we may make an ellipsoid or hyperboloid of revolution, having a constant axis, but a variable excentricity, move in such a manner that one focus may traverse in all directions the surface that cuts the incident rays perpendicularly, while the other focus remains fixed at the point through which all the reflected rays are to pass; the surface that envelopes the ellipsoid or hyperboloid, in all its different positions, will be the mirror required, and each ellipsoid or hyperboloid thus moving will in general have two such enveloppes. And to determine whether the reflected rays converge to the given focus, or diverge from it, it is only necessary to determine the sign of the distance ρ', which is positive in the first case, and negative in the second.

III. *Surfaces of constant action.*

[11.] We have seen, in the preceding section, that if it be possible to find a mirror, which shall reflect to a given focus the rays of a given system, those rays must be perpendicular to a series of surfaces; and that the whole bent path traversed by the light, from any one of these perpendicular surfaces to the mirror, and from the mirror to the focus, is a constant quantity, the same for all the rays. Hence it follows, reciprocally, that when rays issuing from a luminous point have been reflected at a mirror, the rays of the reflected system are cut perpendicularly by a series of surfaces; and that these surfaces may be determined, by taking upon every reflected ray a point such that the whole bent path from the luminous point to it, may be equal to any constant quantity. I am going to shew, in general, that when rays issuing from a luminous point, or from a perpendicular surface, have been any number of times reflected, by any combination of mirrors, the rays of the final system are cut perpendicularly by a series of surfaces, possessing this remarkable property, that the whole polygon path traversed by the light, in arriving at any one of them, is of a constant length, the same for all the rays.

[12.] To prove this theorem I observe, that if upon every ray of the final system we take a point, such that the whole polygon path to it, from the luminous point or perpendicular surface, may be equal to any constant quantity, the locus of the points thus determined will satisfy the differential equation

$$\frac{d\rho}{dx} \,.\, dX + \frac{d\rho}{dy} \,.\, dY + \frac{d\rho}{dz} \,.\, dZ = 0, \tag{K}$$

X, Y, Z, being the coordinates of the point, and ρ the last side of the polygon; because by hypothesis the variation of the whole path is nothing, and also that part which arises from the variation of the first point or origin of the polygon, and by the principle of least action, the part arising from the variation of the several points of incidence, is nothing; therefore the variation arising from the last point of the polygon must be nothing also, which is the condition expressed by the equation (K), and which requires either that this last point should be a fixed focus through which all the rays of the final system pass, or else that its locus should be a surface cutting these rays perpendicularly.

[13.] We see then that when rays issuing from a luminous point, or from a perpendicular surface, have been any number of times reflected, the rays of the final system are cut perpendicularly by that series of surfaces, for which

$$\Sigma(\rho) = \text{const.}, \tag{L}$$

$\Sigma(\rho)$ representing the sum of the several paths or sides of the polygon. When we come to consider the systems produced by ordinary refraction, we shall see that the rays of such a system are cut perpendicularly by a series of surfaces having for equation

$$\Sigma \,.\, (m\rho) = \text{const.},$$

$\Sigma\,.(m\rho)$ representing the sum of the several paths, multiplied each by the refractive power of the medium in which it lies. In the systems also, produced by atmospheric and by extraordinary refraction, there are analogous surfaces possessing remarkable properties, which render it desirable that we should agree upon a name by which we may denote them. Since then in mechanics the sum obtained by adding the several elements of the path of a particle, multiplied each by the velocity with which it is described, is called the *Action* of the particle; and since if light be a material substance its velocity in uncrystallized mediums is proportional to the refractive power, and is not altered by reflection: I shall call the surfaces (L) the *surfaces of constant action*; intending only to express a remarkable analogy, and not assuming any hypothesis about the nature or velocity of light.

[14.] We have hitherto supposed all the sides of the polygon positive, that is, we have supposed them all to be actually traversed by the light. This is necessarily the case for all the sides between the first and last; but if the point to which the last side of the polygon is measured were a focus from which the final rays diverge, or it it were on a perpendicular surface situated behind the last mirror, this last side would then be negative; and in like manner, if the first point, or origin of the polygon, were a focus to which the first incident rays converged, or if it were on a perpendicular surface behind the first mirror, we should have to consider the first side as negative. With these modifications the equation (L) represents all the surfaces that cut the rays perpendicularly; and to mark the analytic distinction between those which cut the rays themselves, and those which only cut the rays produced, we may call the former *positive*, and the latter *negative*: the positive surfaces of constant action lying at the front of the mirror, and the negative ones lying at the back of it.

[15.] It follows from the preceding theorems, that if with each point of the last mirror for centre, and with a radius equal to any constant quantity, increased or diminished by the sum of the sides of the polygon path, which the light has traversed in arriving at that point, we construct a sphere, the enveloppe of these spheres will be a surface cutting the final rays perpendicularly. These spheres will also have another enveloppe perpendicular to the incident rays. It follows also, that when rays, either issuing from a luminous point, or perpendicular to a given surface, have been reflected by any combination of mirrors, it is always possible to find a focal mirror which shall reflect the final rays, so as to make them all pass through any given point; namely, by choosing it so, that the sum of the sides of the whole polygon path measured to that given focus, and taken with their proper signs, may be equal to any constant quantity.

IV. *Classification of Systems of Rays.*

[16.] Before proceeding any farther in our investigations about reflected systems of rays, it will be useful to make some remarks upon systems of rays in general, and to fix upon a classification of such systems which may serve to direct our researches. By a *Ray*, in this Essay, is meant a line along which light is propagated; and by a *System of Rays* is meant an infinite number of such lines, connected by any analytic law, or any common property. Thus, for example, the rays which proceed from a luminous point in a medium of uniform density, compose one system of rays; the same rays, after being reflected or refracted, compose another system. And when we represent a ray analytically by two equations between its three coordinates, the coefficients of those equations will be connected by one or more relations depending on the nature of the system, so that they may be considered as functions of one or more arbitrary quantities. These arbitrary quantities, which enter into the equations of the ray, may be called its *Elements of Position*, because they serve to particularise its situation in the system to which it belongs. And the *number* of these arbitrary quantities, or elements of position, is what I shall take for the basis of my classification of systems of rays; calling a system with one element of position a *system of the First Class*: a system with two elements of position, a *system of the Second Class*, and so on.

[17.] Thus, if we are considering a system of rays emanating in all directions from a luminous point (a, b, c), the equations of a ray are of the form

$$\begin{aligned} x - a &= \mu(z - c) \\ y - b &= \nu(z - c), \end{aligned}$$

which involve only *two* arbitrary quantities, or elements of position, namely μ, ν, the tangents of the angles which the two projections of the ray, on the vertical planes of coordinates, make with the axis of (z); a system of this kind is therefore a system of the second class. If among the rays thus emanating in all directions from a luminous point (a, b, c), we consider those only which are contained on a given plane passing through that point, and having for equation

$$z - c = A(x - a) + B(y - b),$$

then the two quantities μ, ν, are connected by the relation

$$1 = A\mu + B\nu,$$

so that one only remains arbitrary, and the system is of the first class. In general if we consider only those rays which belong to a given cone, having the luminous point for centre, and for equation

$$\frac{y - b}{z - c} = \phi\left(\frac{x - a}{z - c}\right),$$

ϕ denoting any given function, the two quantities μ, ν, will be connected by the given relation

$$\nu = \phi(\mu),$$

and the system will be of the first class. If now we suppose a system of rays thus emanating from a luminous point, to be any number of times modified by reflection or refraction, it is evident that the class of the system will not be altered; that is, there will be the same number of arbitrary constants, or elements of position, in the final system as in the original system: provided that we do not take into account the dispersion of the differently coloured rays. But if we do take this dispersion into account, it will introduce in refracted systems a new element of position depending on the colour of the ray, and thus will raise the system to a class higher by unity.

[18.] From the preceding remarks, it is evident that optics, considered mathematically, relates for the most part, to the properties of systems of rays, of the first and second class. In the third part of this essay I shall consider the properties of those two classes in the most general point of view; but at present I shall confine myself to such as are more immediately connected with catoptrics. And I shall begin by making some remarks upon the general properties of those systems, in which the rays are cut perpendicularly by a series of surfaces; a system of this kind I shall call a *Rectangular System*. The properties of such systems are of great importance in optics; for, by what I have already proved, they include all systems of rays which after issuing from a luminous point, or from a perpendicular surface, have been any number of times reflected, by any combination of mirrors; we shall see also, in the next part, that they include also the systems produced by ordinary refraction.

[19.] In any system of the second class, a ray may in general be determined by the condition of passing through an assigned point of space, for this condition furnishes two equations between the coefficients of the ray, which are in general sufficient to determine the two arbitrary elements of position. We may therefore consider the cosines (α, β, γ) of the angles which the ray makes with the axes as functions of the coordinates (x, y, z) of any point upon the ray; because, if the latter be given, the former will be determined. And if the system be rectangular, that is, if the rays be cut perpendicularly by any series of surfaces, it may be proved by the reasonings in Section II. of this essay, that these functions must be of such a nature as to render the formula

$$(\alpha \,.\, dx + \beta \,.\, dy + \gamma \,.\, dz)$$

an exact differential, independently of any relation between (x, y, z); that is, the cosines (α, β, γ) of the angles which the ray passing through any assigned point (x, y, z) makes with the axes, must be equal to the partial differential coefficients

$$\frac{dV}{dx}, \quad \frac{dV}{dy}, \quad \frac{dV}{dz},$$

of a function of (x, y, z) considered as three independent variables.

[20.] The properties of any one rectangular system, as distinguished from another, may all be deduced by analytic reasoning from the form of the function (V); and it is the method of making this deduction, together with the calculation of the form of the *characteristic function* (V) for each particular system, that appears to me to be the most complete and simple definition that can be given, of the *Application of analysis to optics*; so far as regards the systems produced by ordinary reflection and refraction, which, as I shall shew, are all rectangular. And although the systems produced by extraordinary refraction, do not in general enjoy this property; yet I shall shew that with respect to them, there exists an analogous characteristic function, from which all the circumstances of the system may be deduced: by which means optics acquires, as it seems to me, an uniformity and simplicity in which it has been hitherto deficient.

V. *On the pencils of a Reflected System.*

[21.] When a rectangular system of rays, that is a system the rays of which are cut perpendicularly by a series of surfaces, is reflected at a mirror, we have seen that the reflected system is also rectangular; the rays being cut perpendicularly by the surfaces of constant action, (III.); and that therefore the cosines of the angles which a reflected ray makes with the axes, are equal to the partial differential coefficients of a certain function (V) which I have called the *characteristic* of the system, because all the properties of the system may be deduced from it. It is this deduction which we now proceed to; and before we occupy ourselves with the entire reflected system, we are going to investigate some of the properties of the various partial systems that can be formed, by establishing any assumed relation between the rays, that is by considering only those which are reflected from any assumed curve upon the mirror.

[22.] A partial system of this kind, is a *system of the first class*; that is, the position of a ray in such a system depends only on one arbitrary element; for example, on any one coordinate of the assumed curve upon the mirror. And if we eliminate this one element, between the two equations of the ray, we shall obtain the equation of a surface, which is the locus of the rays of the partial system that we are considering. The form of this surface depends on the arbitrary curve upon the mirror, from which the rays of the partial system proceed; so that according to the infinite variety of curves which we can trace upon the given mirror, we shall have an infinite number of surfaces composed by rays of a given reflected system. And since these surfaces possess many important properties, which render it expedient that we should denote them by a name, I shall call them *pencils*: defining a pencil to be the locus of the rays of a system of the first class, that is, of a system with but one arbitrary constant.

[23.] Although, as we have seen, an infinite number of pencils may be formed by the rays of a given reflected system, yet there are certain properties common to them all, which render them susceptible of being included in one common analytic expression. For, if we denote by (V) the characteristic function [20.] of the given reflected system, so that

$$\alpha = \frac{dV}{dx}, \quad \beta = \frac{dV}{dy}, \quad \gamma = \frac{dV}{dz}, \tag{M}$$

(α, β, γ) being the cosines of the angles which the reflected ray passing through any assigned point of space (x, y, z) makes with the axes of coordinates; we shall have, for all the points of any one ray, the three equations

$$\frac{dV}{dx} = \text{const.}, \quad \frac{dV}{dy} = \text{const.}, \quad \frac{dV}{dz} = \text{const.},$$

which are equivalent to but two distinct relations, because

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \alpha^2 + \beta^2 + \gamma^2 = 1.$$

If then we consider the rays of any of the partial systems, produced by establishing an arbitrary relation between the rays of the entire reflected system; the locus of these rays, that is, the pencil of this partial system, will have for its equation

$$\frac{dV}{dy} = f\left(\frac{dV}{dx}\right), \tag{N}$$

f representing an arbitrary function, the form of which depends upon the nature of the partial system.

[24.] The form of this function (f) may be determined, if we know any curve through which the rays of the pencil pass, or any surface which they envelope. For first, the latter of these two questions may be reduced to the former, by determining upon the enveloped surface the locus of the points of contact; this is done by means of the formula

$$\frac{dV}{dx} \cdot \frac{du}{dx} + \frac{dV}{dy} \cdot \frac{du}{dy} + \frac{dV}{dz} \cdot \frac{du}{dz} = 0,$$

which expresses that the rays of the unknown pencil are tangents to the given enveloped surface $u = 0$. And when we know a curve $u = 0$, $v = 0$, through which all the rays of the pencil pass, we have only to eliminate (x, y, z) between the two equations of this curve, and the two following,

$$\alpha = \frac{dV}{dx}, \quad \beta = \frac{dV}{dy};$$

we shall thus obtain the relation between (α, β) which characterises the rays that pass through the given curve: and substituting, in this relation, the values of (α, β) considered as functions of (x, y, z) we shall have the equation of the pencil.

In this manner we can determine the shadow of any opaque body, produced by the rays of a given reflected system, if we know the equation of the body, and that of the skreen upon which the shadow is thrown; we can also determine the boundary of light and darkness upon the body, which is the curve of contact with the enveloping pencil; and if we consider visual instead of luminous rays, we can determine, on similar principles, the perspective of reflected light, that is, the apparent form and magnitude of a body seen by any combination of mirrors; at least so far as that form and magnitude depend on the shape and size of the visual cone.

[25.] Besides the general analytic expression

$$\frac{dV}{dy} = f\left(\frac{dV}{dx}\right), \tag{N}$$

which represents all the pencils of the system, by means of the arbitrary function (f), we can also find another analytic expression for those pencils, by eliminating that arbitrary function, and introducing instead of it the partial differential coefficients of the pencil of the first order.

In this manner we find, by differentiating (N) for (x) and (y) successively, and eliminating the differential coefficient of the arbitrary function (f),

$$\begin{aligned}\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx \,.\, dy}\right)^2 = \quad & \left\{\frac{d^2V}{dx \,.\, dy} \cdot \frac{d^2V}{dy \,.\, dz} - \frac{d^2V}{dy^2} \cdot \frac{d^2V}{dx \,.\, dz}\right\} \frac{dz}{dx} \\ & + \left\{\frac{d^2V}{dx \,.\, dy} \cdot \frac{d^2V}{dx \,.\, dz} - \frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy \,.\, dz}\right\} \frac{dz}{dy};\end{aligned}$$

and since the general relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, that is

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = 1,$$

gives by differentiation

$$\begin{aligned}\alpha . \frac{d^2V}{dx^2} + \beta . \frac{d^2V}{dx \,.\, dy} + \gamma . \frac{d^2V}{dx \,.\, dz} = 0, \\ \alpha . \frac{d^2V}{dx \,.\, dy} + \beta . \frac{d^2V}{dy^2} + \gamma . \frac{d^2V}{dy \,.\, dz} = 0,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{d^2V}{dx \,.\, dy} \cdot \frac{d^2V}{dy \,.\, dz} - \frac{d^2V}{dy^2} \cdot \frac{d^2V}{dx \,.\, dz} = \frac{\alpha}{\gamma} \cdot \left\{\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx \,.\, dy}\right)^2\right\}, \\ \frac{d^2V}{dx \,.\, dy} \cdot \frac{d^2V}{dx \,.\, dz} - \frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy \,.\, dz} = \frac{\beta}{\gamma} \cdot \left\{\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx \,.\, dy}\right)^2\right\},\end{aligned}$$

the partial differential equation of the pencils becomes finally

$$\frac{\alpha}{\gamma} \cdot \frac{dz}{dx} + \frac{\beta}{\gamma} \cdot \frac{dz}{dy} = 1, \tag{O}$$

which expresses that the tangent plane to the pencil contains the ray passing through the point of contact.

VI. *On the developable pencils, the two foci of a ray, and the caustic curves and surfaces.*

[26.] Among all the pencils of a given rectangular system, there is only a certain series developable; namely, those which pass through the lines of curvature on the surfaces that cut the rays perpendicularly. It follows from the known properties of normals to surfaces, that each ray has two of these developable pencils passing through it, and is therefore a common tangent to *two caustic curves*, the *arêtes de rebroussement* of these pencils; the points in which it touches those two caustic curves may be called the *two foci* of the ray; and the locus of these foci forms *two caustic surfaces*, touched by all the rays.

[27.] To determine analytically these several properties of the system, let us represent by (a, b, c) the coordinates of the point in which a ray crosses a given perpendicular surface; these coordinates will be determined, if the ray be given, so that they may be considered as functions of (α, β); we may therefore put their differentials under the form

$$\begin{aligned} da &= \frac{da}{d\alpha} \,.\, d\alpha + \frac{da}{d\beta} \,.\, d\beta, \\ db &= \frac{db}{d\alpha} \,.\, d\alpha + \frac{db}{d\beta} \,.\, d\beta, \\ dc &= \frac{dc}{d\alpha} \,.\, d\alpha + \frac{dc}{d\beta} \,.\, d\beta. \end{aligned}$$

We have also $\alpha\, da + \beta\, db + \gamma\, dc = 0$, which gives

$$\begin{aligned} \frac{dc}{d\alpha} &= -\left(\frac{\alpha}{\gamma} \,.\, \frac{da}{d\alpha} + \frac{\beta}{\gamma} \,.\, \frac{db}{d\alpha}\right), \\ \frac{dc}{d\beta} &= -\left(\frac{\alpha}{\gamma} \,.\, \frac{da}{d\beta} + \frac{\beta}{\gamma} \,.\, \frac{db}{d\beta}\right); \end{aligned}$$

with respect to the coefficients $\frac{da}{d\alpha}, \frac{da}{d\beta}, \frac{db}{d\alpha}, \frac{db}{d\beta}$, these are to be determined by differentiating the two following equations,

$$\frac{dV}{da} = \alpha, \quad \frac{dV}{db} = \beta,$$

which give

$$\begin{aligned} \frac{d^2V}{da^2} \,.\, da + \frac{d^2V}{da \,.\, db} \,.\, db + \frac{d^2V}{da \,.\, dc} \,.\, dc &= d\alpha, \\ \frac{d^2V}{da \,.\, db} \,.\, da + \frac{d^2V}{db^2} \,.\, db + \frac{d^2V}{db \,.\, dc} \,.\, dc &= d\beta, \end{aligned}$$

and therefore

$$\begin{aligned} M \,.\, da &= \left(\gamma \,.\, \frac{d^2V}{db^2} - \beta \,.\, \frac{d^2V}{db \,.\, dc}\right) . \, d\alpha + \left(\beta \,.\, \frac{d^2V}{da \,.\, dc} - \gamma \,.\, \frac{d^2V}{da \,.\, db}\right) . \, d\beta, \\ M \,.\, db &= \left(\alpha \,.\, \frac{d^2V}{db \,.\, dc} - \gamma \,.\, \frac{d^2V}{da \,.\, db}\right) . \, d\alpha + \left(\gamma \,.\, \frac{d^2V}{da^2} - \alpha \,.\, \frac{d^2V}{da \,.\, dc}\right) . \, d\beta, \end{aligned}$$

if we put for abridgment

$$\begin{aligned} M = \quad & \alpha \,.\, \left\{\frac{d^2V}{da \,.\, db} \,.\, \frac{d^2V}{db \,.\, dc} - \frac{d^2V}{da \,.\, dc} \,.\, \frac{d^2V}{db^2}\right\} \\ & + \beta \,.\, \left\{\frac{d^2V}{da \,.\, db} \,.\, \frac{d^2V}{da \,.\, dc} - \frac{d^2V}{db \,.\, dc} \,.\, \frac{d^2V}{da^2}\right\} \\ & + \gamma \,.\, \left\{\frac{d^2V}{da^2} \,.\, \frac{d^2V}{db^2} - \left(\frac{d^2V}{da \,.\, db}\right)^2\right\}. \end{aligned}$$

This being laid down, let (x, y, z) be any other point upon the ray, at a distance (ρ) from the given perpendicular surface; we shall have

$$x = a + \alpha\rho, \quad y = b + \beta\rho, \quad z = c + \gamma\rho,$$

$$d\rho = \alpha\, dx + \beta\, dy + \gamma\, dz,$$

$$dx - \alpha\, d\rho = da + \rho\, d\alpha, \quad dy - \beta\, d\rho = db + \rho\, d\beta :$$

and if the coordinates (x, y, z) belong to a caustic curve, the first members of these two last equation will vanish, so that on this hypothesis,

$$da + \rho \,.\, d\alpha = 0, \quad db + \rho \,.\, d\beta = 0;$$

eliminating (ρ) we find, for the developable pencils,

$$da \,.\, d\beta - db \,.\, d\alpha = 0, \tag{P}$$

and eliminating $\dfrac{d\beta}{d\alpha}$ we find, for the caustic surfaces,

$$\left(\rho + \frac{da}{d\alpha}\right)\left(\rho + \frac{db}{d\beta}\right) - \frac{da}{d\beta} \cdot \frac{db}{d\alpha} = 0; \tag{Q}$$

substituting for $\dfrac{da}{d\alpha}, \dfrac{da}{d\beta}, \dfrac{db}{d\alpha}, \dfrac{db}{d\beta}$, their values, and observing that by the general relation

$$\left(\frac{dV}{da}\right)^2 + \left(\frac{dV}{db}\right)^2 + \left(\frac{dV}{dc}\right)^2 = 1,$$

we have

$$\begin{aligned}
\alpha \,.\, \frac{d^2V}{da^2} + \beta \,.\, \frac{d^2V}{da \,.\, db} + \gamma \,.\, \frac{d^2V}{da \,.\, dc} &= 0, \\
\alpha \,.\, \frac{d^2V}{da \,.\, db} + \beta \,.\, \frac{d^2V}{db^2} + \gamma \,.\, \frac{d^2V}{db \,.\, dc} &= 0, \\
\alpha \,.\, \frac{d^2V}{da \,.\, dc} + \beta \,.\, \frac{d^2V}{db \,.\, dc} + \gamma \,.\, \frac{d^2V}{dc^2} &= 0,
\end{aligned}$$

$$\gamma \,.\, M = \frac{d^2V}{da^2} \cdot \frac{d^2V}{db^2} - \left(\frac{d^2V}{da \,.\, db}\right)^2,$$

we find this other form for the equation of the caustic surfaces,

$$\frac{1}{\gamma^2}\left\{\frac{d^2V}{da^2} \cdot \frac{d^2V}{db^2} - \left(\frac{d^2V}{da \,.\, db}\right)^2\right\} \,.\, \rho^2 + \left\{\frac{d^2V}{da^2} + \frac{d^2V}{db^2} + \frac{d^2V}{dc^2}\right\} \,.\, \rho + 1 = 0. \tag{R}$$

[28.] The manner in which these formulæ are to be employed is evident. We are to integrate (P) considered as a differential equation, of the first order and second degree, between α, β, or between the corresponding functions of x, y, z,

$$\alpha = \frac{dV}{dx}, \quad \beta = \frac{dV}{dy};$$

the integral will be of the form

$$\frac{dV}{dy} = f\left(\frac{dV}{dx}, C\right),$$

C being an arbitrary constant; the condition of passing through a given ray will determine the two values of this constant, corresponding to the two developable pencils: and the equations of the caustic curves, considered as the *arêtes de rebroussement* of those pencils, will follow by the known methods from the equations of the pencils themselves. The points in which a given ray touches these caustic curves, that is the two foci of the ray, are determined, without any integration, by means of (Q) or (R); and thus we can determine, by elimination alone, the equations of the two caustic surfaces, the locus of these points or foci.

[29.] In the preceding reasonings, we have supposed given the form of the characteristic function V, whose partial differential coefficients of the first order, are equal to the cosines of the angles that the reflected ray makes with the axes; let us now see how the partial differential coefficients of the second order of this function, which enter into the formulæ that we have found for the developable pencils and for the caustic surfaces, depend on the curvature of the mirror, and on the characteristic function of the incident system. Let (V') represent this latter function, so that we shall have

$$\alpha' = \frac{dV'}{dx}, \quad \beta' = \frac{dV'}{dy}, \quad \gamma' = \frac{dV'}{dz},$$

α', β', γ' being the cosines of the angles that the incident ray, measured from the mirror, makes with the axes of coordinates; and let p, q, r, s, t, be the partial differential coefficients of the mirror, of the first and second orders, so that

$$dz = p\,dx + q\,dy, \quad dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

x, y, z being the coordinates of the mirror. Then, by the first section of this essay we shall have the two equations

$$\alpha + \alpha' + p(\gamma + \gamma') = 0, \quad \beta + \beta' + q(\gamma + \gamma') = 0,$$

which give by differentiation,

$$0 = (\gamma + \gamma') \,.\, r + \frac{d^2V}{dx^2} + \frac{d^2V'}{dx^2} + 2p \,.\, \left(\frac{d^2V}{dx \,.\, dz} + \frac{d^2V'}{dx \,.\, dz}\right) + p^2 \,.\, \left(\frac{d^2V}{dz^2} + \frac{d^2V'}{dz^2}\right),$$
$$0 = (\gamma + \gamma') \,.\, t + \frac{d^2V}{dy^2} + \frac{d^2V'}{dy^2} + 2q \,.\, \left(\frac{d^2V}{dy \,.\, dz} + \frac{d^2V'}{dy \,.\, dz}\right) + q^2 \,.\, \left(\frac{d^2V}{dz^2} + \frac{d^2V'}{dz^2}\right),$$
$$0 = (\gamma + \gamma') \,.\, s + \frac{d^2(V + V')}{dx \,.\, dy} + p \,.\, \frac{d^2(V + V')}{dy \,.\, dz} + q \,.\, \frac{d^2(V + V')}{dx \,.\, dz} + pq \,.\, \frac{d^2(V + V')}{dz^2}.$$

Combining these three equations with the three which result from differentiating the equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = 1$$

we shall have the partial differential coefficients, second order, of V, when we know those of V' and of z, that is, when we know the incident system and the mirror: it will then remain to substitute them in the formulæ of the preceding paragraph, in order to find the developable pencils, and the caustic surfaces, in which we may change the partial differential coefficients of V, taken with respect to (a, b, c), to the corresponding coefficients with respect to (x, y, z).

[30.] Suppose, to give an example of the application of the preceding reasonings, that the incident rays are parallel, and that we take for the axes of (x) and (y), the tangents to the lines of curvature on the mirror at the point of incidence, so that the normal at that point shall be vertical; the partial differential coefficients of the second order of (V') will vanish, and we shall have

$$x = 0, \quad y = 0, \quad z = 0, \quad p = 0, \quad q = 0, \quad s = 0,$$

$$\alpha + \alpha' = 0, \quad \beta + \beta' = 0, \quad \gamma = \gamma' = \cos. I,$$

I being the angle of incidence; the formulæ for the partial differential coefficients of the second order of (V) become

$$\frac{d^2V}{dx^2} = -2\gamma r, \quad \frac{d^2V}{dx \,.\, dy} = 0, \quad \frac{d^2V}{dy^2} = -2\gamma t,$$

$$\frac{d^2V}{dx \,.\, dz} = 2\alpha r, \quad \frac{d^2V}{dy \,.\, dz} = 2\beta t, \quad \frac{d^2V}{dz^2} = -\frac{2(\alpha^2 r + \beta^2 t)}{\gamma},$$

and the formula (R) for the two foci, which may be thus written

$$\frac{\rho^2}{\gamma^2}\left\{\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx \,.\, dy}\right)^2\right\} + \rho \,.\left\{\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right\} + 1 = 0, \tag{R}$$

becomes

$$4rt \,.\, \rho^2 - \frac{2\rho}{\gamma} \,.\, \{(\alpha^2 + \gamma^2)r + (\beta^2 + \gamma^2)t\} + 1 = 0. \tag{S}$$

If the incident rays be perpendicular to the mirror, at the given point of incidence, then

$$\gamma = 1, \quad \alpha = 0, \quad \beta = 0,$$

and the two roots of (S) are

$$\rho = \frac{1}{2} \cdot \frac{1}{r}, \quad \rho = \frac{1}{2} \cdot \frac{1}{t},$$

that is the two focal distances are the halves of the two radii of curvature of the mirror.

If without being perpendicular to the mirror, the incident ray is contained in the plane of (xz), that is in the plane of the greatest or the least osculating circle to the mirror, we shall have $\beta = 0$, $\alpha^2 + \gamma^2 = 1$, and the two roots of (S) will be

$$\rho = \frac{1}{2} \cdot \gamma \cdot \frac{1}{r}, \quad \rho = \frac{1}{2} \cdot \frac{1}{\gamma} \cdot \frac{1}{t};$$

the first root is quarter of the chord of curvature, that is, quarter of the portion of the reflected ray intercepted within the osculating circle before mentioned; and the other root is equal to the distance of the point, where the reflected ray meets a parallel to the incident rays, passing through the centre of the other osculating circle. In general, it will appear, when we come to treat of osculating focal mirrors, that the two foci determined by the formula (S), are the foci of the greatest and least paraboloids of revolution which, having their axis parallel to the incident rays, osculate to the mirror at the point of incidence.

[31.] I shall conclude this section by remarking, that the equation of the caustic surfaces is a singular primitive of the partial differential equation (O), which we found in the preceding section to represent all the pencils of the system, and that the equations,

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma},$$

of which the complete integral represents all the rays, are also satisfied, as a singular solution, by the equations of the caustic curves: from which it may be proved, that the portion of any ray, or the arc of any caustic curve, intercepted between any two given points, is equal to the increment that the characteristic function (V) receives in passing from the one point to the other.

VII. *Lines of Reflection on a mirror.*

[32.] We have seen that the rays of a reflected system are in general tangents to two series of caustic curves, and compose two corresponding series of developable pencils; the intersections of these pencils with the mirror, form two series of remarkable curves upon that surface, which were first discovered by Malus, and which were called by him the *Lines of Reflexion.* We propose, in the present section to investigate the differential equation of these curves, and some of their principal properties; and at the same time to make some additional remarks, on the manner of calculating the foci, and the caustic surfaces.

[33.] To find the differential equation of the curves of reflexion, we may employ the formula of the preceding section,

$$\begin{aligned} d\beta \,.\, \left\{ \left(\beta \,.\, \frac{d^2V}{dx \,.\, dz} - \gamma \,.\, \frac{d^2V}{dx \,.\, dy} \right) d\beta + \left(\gamma \,.\, \frac{d^2V}{dy^2} - \beta \,.\, \frac{d^2V}{dy \,.\, dz} \right) d\alpha \right\} \\ = d\alpha \,.\, \left\{ \left(\gamma \,.\, \frac{d^2V}{dx^2} - \alpha \,.\, \frac{d^2V}{dx \,.\, dz} \right) d\beta + \left(\alpha \,.\, \frac{d^2V}{dy \,.\, dz} - \gamma \,.\, \frac{d^2V}{dx \,.\, dy} \right) d\alpha \right\}, \qquad \text{(P)} \end{aligned}$$

considering (α, β, γ) as given functions of the coordinates of the point of incidence, such that

$$d\alpha = \frac{d^2V}{dx^2} \,.\, dx + \frac{d^2V}{dx \,.\, dy} \,.\, dy + \frac{d^2V}{dx \,.\, dz} \,.\, dz,$$
$$d\beta = \frac{d^2V}{dx \,.\, dy} \,.\, dx + \frac{d^2V}{dy^2} \,.\, dy + \frac{d^2V}{dy \,.\, dz} \,.\, dz,$$

and deducing the partial differential coefficients of the characteristic function V, either immediately from the form of that function itself, if it be given, or from the equation of the mirror and from the nature of the incident system, according to the method already explained. But in this latter case, that is, when we are only given the incident system and the mirror, it will be simpler to treat the question immediately, by reasonings analogous to those by which the formula (P) was deduced.

Let, therefore, X, Y, Z, represent the coordinates of a point upon a caustic curve, at a distance (ρ) from the mirror; we shall have

$$X = x + \alpha\rho, \quad Y = y + \beta\rho, \quad Z = z + \gamma\rho,$$

$$d\rho = \alpha \,.\, d(X - x) + \beta \,.\, d(Y - y) + \gamma \,.\, d(Z - z),$$

$$dX = \alpha \,.\, (\alpha\, dX + \beta\, dY + \gamma\, dZ),$$
$$dY = \beta \,.\, (\alpha\, dX + \beta\, dY + \gamma\, dZ),$$
$$dZ = \gamma \,.\, (\alpha\, dX + \beta\, dY + \gamma\, dZ),$$

$$dx - \alpha \,.\, (\alpha\, dx + \beta\, dy + \gamma\, dz) + \rho\, d\alpha = 0,$$
$$dy - \beta \,.\, (\alpha\, dx + \beta\, dy + \gamma\, dz) + \rho\, d\beta = 0,$$
$$dz - \gamma \,.\, (\alpha\, dx + \beta\, dy + \gamma\, dz) + \rho\, d\gamma = 0;$$

eliminating $d\alpha$, $d\beta$, $d\gamma$, by these equations, from those which are obtained by differentiating the formulæ already found,

$$\alpha + \alpha' + p(\gamma + \gamma') = 0, \quad \beta + \beta' + q(\gamma + \gamma') = 0,$$

we get the two following equations,

$$\left.\begin{aligned} 0 &= \rho \,.\, \{(\gamma + \gamma')\, dp + d\alpha' + p \,.\, d\gamma'\} + (\alpha + \gamma p)(\alpha\, dx + \beta\, dy + \gamma\, dz) - (dx + p\, dz), \\ 0 &= \rho \,.\, \{(\gamma + \gamma')\, dq + d\beta' + q \,.\, d\gamma'\} + (\beta + \gamma q)(\alpha\, dx + \beta\, dy + \gamma\, dz) - (dy + q\, dz), \end{aligned}\right\} \quad \text{(T)}$$

which give by elimination of ρ, the following general equation for the lines of reflexion,

$$\frac{(\gamma + \gamma')\, dq + d\beta' + q \,.\, d\gamma'}{(\gamma + \gamma')\, dp + d\alpha' + p \,.\, d\gamma'} = \frac{(\beta + \gamma q)(\alpha\, dx + \beta\, dy + \gamma\, dz) - (dy + q\, dz)}{(\alpha + \gamma p)(\alpha\, dx + \beta\, dy + \gamma\, dz) - (dx + p\, dz)}. \quad \text{(U)}$$

[34.] Suppose, to give an example, that the incident rays are parallel, and that the axes of coordinates are chosen as in [30.], the normal at some given point of incidence for the axis of (z), and the tangents to the lines of curvature for the axes of (x) and (y); our general formula (U) will then become

$$\frac{t\,.\,dy}{r\,.\,dx} = \frac{\beta\,.\,(\alpha\,dx + \beta\,dy) - dy}{\alpha\,.\,(\alpha\,dx + \beta\,dy) - dx};$$

that is

$$\alpha\beta\,.\,(t\,.\,dy^2 - r\,.\,dx^2) - \{(\beta^2 + \gamma^2)t - (\alpha^2 + \gamma^2)r\}\,dx\,.\,dy = 0. \tag{V}$$

We shall see, in the next section, that the two directions determined by this formula, are the directions of osculation of the greatest and least paraboloids, which, having their axes parallel to the incident rays, osculate to the mirror at the point of incidence; in the mean time we may remark, that if the plane of incidence coincides with either the plane of the greatest or the least osculating circle to the mirror, or if the point of incidence be a point of spheric curvature, one of the two directions of the lines of reflexion is contained in the plane of incidence, while the other is perpendicular to that plane; and it is easy to prove, by means of the formula (V), that these are the only cases in which the lines of reflexion are perpendicular to one another, the incident rays being parallel.

[35.] The formulæ (T) determine not only, as we have seen, the lines of reflexion, but also the two focal distances, and therefore the caustic surfaces. For as, by elimination of (ρ), they conduct to the differential equation of the lines of reflexion, so by elimination of the differentials they conduct to a quadratic equation in (ρ), which is equivalent to the formula (R), and which determines the two focal distances. As an example of this, let us take the following general problem, to find the caustic surfaces and lines of reflexion of a mirror, when the incident rays diverge from a given luminous point X', Y', Z'. We have here

$$X' = x + \alpha'\rho', \quad Y' = y + \beta'\rho', \quad Z' = z + \gamma'\rho',$$

ρ' being the distance of the luminous point from the mirror;

$$d\rho' = -(\alpha'\,dx + \beta'\,dy + \gamma'\,dz),$$

$$-\rho'\,.\,d\alpha' = dx + \alpha'\,.\,d\rho', \quad -\rho'\,.\,d\beta' = dy + \beta'\,.\,d\rho', \quad -\rho'\,.\,d\gamma' = dz + \gamma'\,.\,d\rho',$$

and because

$$\alpha' + \gamma' p = -(\alpha + \gamma p), \quad \beta' + \gamma' q = -(\beta + \gamma q),$$

$$\alpha'\,dx + \beta'\,dy + \gamma'\,dz = -(\alpha\,dx + \beta\,dy + \gamma\,dz),$$

the equations (T) become

$$(\gamma + \gamma')\,.\,dp = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\{dx + p\,dz - (\alpha + \gamma p)(\alpha\,dx + \beta\,dy + \gamma\,dz)\},$$

$$(\gamma + \gamma')\,.\,dq = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\{dy + q\,dz - (\beta + \gamma q)(\alpha\,dx + \beta\,dy + \gamma\,dz)\}:$$

eliminating ρ, we find, for the lines of reflexion,

$$\begin{aligned} & dq\,.\,\{dx + p\,dz - (\alpha + \gamma p)(\alpha\,dx + \beta\,dy + \gamma\,dz)\} \\ & = dp\,.\,\{dy + q\,dz - (\beta + \gamma q)(\alpha\,dx + \beta\,dy + \gamma\,dz)\}, \end{aligned} \tag{W}$$

and eliminating the differentials, we find, for the focal distances,

$$\begin{aligned} & \left\{(1 + p^2 - (\alpha + \gamma p)^2)\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) - (\gamma + \gamma').r\right\} \\ \times & \left\{(1 + q^2 - (\beta + \gamma q)^2)\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) - (\gamma + \gamma').t\right\} \\ = & \left\{(pq - (\alpha + \gamma p)(\beta + \gamma q))\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) - (\gamma + \gamma').s\right\}^2. \end{aligned} \tag{X}$$

We may remark, that since (ρ') has disappeared from the equation (W) of the lines of reflexion, the direction of those lines at any given point upon the mirror depends only on the direction of the incident ray, and not on the distance of the luminous point; we see also, from the form of the equation (X), that the harmonic mean between that distance (ρ') and either of the two focal distances (ρ), does not depend on (ρ'): so that if the luminous point were to move along the incident ray, the two foci of the reflected ray would indeed change position, but the line joining each to the luminous point, would constantly pass through the same fixed point upon the normal.

VIII. *On osculating focal mirrors.*

[36.] It has long been known that a paraboloid of revolution possesses the property of reflecting to its focus, rays which are incident parallel to its axis; and that an ellipsoid in like manner will reflect to one of its two foci, rays that diverge from the other: but I do not know that any one has hitherto applied these properties of accurately reflecting mirrors, to the investigation of the caustic surfaces, and lines of reflexion of mirrors in general. There exists however a remarkable connexion between them, analogous to the connexion between the properties of spheres and of normals; and it is this connexion, not only for paraboloids and ellipsoids, but also for that general class of focal mirrors, pointed out in Section II. of this Essay, that we are now going to consider.

[37.] To begin with the simplest case, I observe that the general equation of a paraboloid of revolution may be put under the form

$$\rho = P + \alpha'\,.\,(x - X) + \beta'\,.\,(y - Y) + \gamma'\,.\,(z - Z),$$

(P) being the semiparameter, (ρ) the distance from the focus (X, Y, Z), and α', β', γ', the cosines of the angles which the axis of the paraboloid, measured from the vertex, makes with the axes of coordinates: and that the partial differential coefficients of (z), of the first and second orders, which we shall denote by (p', q', r', s', t'), are determined by the following equations,

$$\begin{aligned}
x - X + p' \,.\, (z - Z) &= \rho \,.\, (\alpha' + \gamma' p'),\\
y - Y + q' \,.\, (z - Z) &= \rho \,.\, (\beta' + \gamma' q'),\\
1 + p'^2 + r' \,.\, (z - Z) &= \rho\gamma' r' + (\alpha' + \gamma' p')^2,\\
1 + q'^2 + t' \,.\, (z - Z) &= \rho\gamma' t' + (\beta' + \gamma' q')^2,\\
p'q' + s' \,.\, (z - Z) &= \rho\gamma' s' + (\alpha' + \gamma' p')(\beta' + \gamma' q').
\end{aligned}$$

This being laid down, if we suppose that the three constants $(\alpha', \beta', \gamma')$ determined by the condition that the axis of the paraboloid shall be parallel to a given system of incident rays, we may propose to determine the other four constants (X, Y, Z, P) by the condition of osculating to a given mirror, at a given point, in a given direction. The condition of passing through the given point, will serve to determine, or rather eliminate (P), and the condition of contact produces the two equations

$$p' = p, \quad q' = q,$$

which express that the focus of the paraboloid is somewhere on the reflected ray, and which are therefore equivalent to the three following,

$$X - x = \alpha\rho, \quad Y - y = \beta\rho, \quad Z - z = \gamma\rho,$$

(α, β, γ) being the cosines of the angles which the reflected ray makes with the axes. To determine the remaining constant (ρ), by the condition that the paraboloid shall osculate to the mirror in a given direction, we are to employ the formula

$$(r' - r)\, dx^2 + 2(s' - s) dx \,.\, dy + (t' - t)\, dy^2 = 0,$$

r, s, t, being the given partial differential coefficients, second order, of the mirror, and r', s', t', the corresponding coefficients of the paraboloid, which involve the unknown distance (ρ), being determined by the equations,

$$\begin{aligned}
\rho \,.\, (\gamma + \gamma') \,.\, r' &= 1 + p^2 - (\alpha' + \gamma' p)^2,\\
\rho \,.\, (\gamma + \gamma') \,.\, s' &= pq - (\alpha' + \gamma' p)(\beta' + \gamma' q),\\
\rho \,.\, (\gamma + \gamma') \,.\, t' &= 1 + q^2 - (\beta' + \gamma' q)^2.
\end{aligned}$$

To simplify our calculations, let us, as in [30.], take the normal to the mirror for the axis of (z), and the tangents to the lines of curvature for the axes of (x) and (y); we shall then have

$$p = 0, \quad q = 0, \quad s = 0, \quad \alpha + \alpha' = 0, \quad \beta + \beta' = 0, \quad \gamma = \gamma',$$

$$2\gamma\rho . r' = \beta^2 + \gamma^2, \quad 2\gamma\rho . s' = -\alpha\beta, \quad 2\gamma\rho . t' = \alpha^2 + \gamma^2,$$

and the condition of osculation becomes

$$2\gamma\rho \,.\, (r + t\tau^2) = \beta^2 + \gamma^2 - 2\alpha\beta\tau + (\alpha^2 + \gamma^2)\tau^2, \tag{Y}$$

if we put $dy = \tau \,.\, dx$. This formula (Y) determines the osculating paraboloid for any given value of (τ), that is, for any given direction of osculation; differentiating it with respect to (τ), in order to find the greatest and least osculating paraboloids, we get

$$\begin{gathered}2\gamma\rho t \,.\, \tau = -\alpha\beta + (\alpha^2 + \gamma^2)\tau, \\ 2\gamma\rho r = \beta^2 + \gamma^2 - \alpha\beta\tau,\end{gathered}$$

equations which give, by elimination,

$$\begin{gathered}\{2\gamma\rho t - (\alpha^2 + \gamma^2)\}\{2\gamma\rho r - (\beta^2 + \gamma^2)\} - \alpha^2\beta^2 = 0, \\ \alpha\beta(t \,.\, \tau^2 - r) + \{(\alpha^2 + \gamma^2)r - (\beta^2 + \gamma^2)t\}\tau = 0 :\end{gathered}$$

and since these coincide with the formulæ (S) (V) of the two preceding sections, it follows, that when parallel rays are incident upon a mirror, the two foci of any given reflected ray, that is, the two points in which it touches the caustic surfaces, are the foci of the greatest and least paraboloids, which having their axis parallel to the incident rays, osculate to the mirror at the given point of incidence; and that the directions of the two lines of reflexion passing through that point, are the directions of osculation corresponding.

[38.] In general when the incident system is rectangular, which is always the case in nature, it follows from the principles already established that we can find an infinite number of focal mirrors, possessing the property of reflecting the rays to any given point (X, Y, Z), and having for their differential equation,

$$d\rho = \alpha'\, dx + \beta'\, dy + \gamma'\, dz = dV',$$

V' being the characteristic function of the incident system, and ρ the distance from the point of incidence (x, y, z) to the point (X, Y, Z), the focus of the focal mirror. The condition of touching the given mirror at a given point, furnishes two equations of the form

$$p' = p, \quad q' = q,$$

which expresses that the focus (X, Y, Z) is somewhere on the given reflected ray; and the condition of osculating in a given direction furnishes the equation

$$(r' - r) \,.\, dx^2 + 2(s' - s) \,.\, dx \,.\, dy + (t' - t) \,.\, dy^2 = 0,$$

(r, s, t) being given, but (r', s', t') depending on the unknown focal distance (ρ); and if we wish to make this distance a maximum or minimum, we are to satisfy the two conditions

$$(r' - r) \,.\, dx + (s' - s) \,.\, dy = 0, \quad (s' - s) \,.\, dx + (t' - t) \,.\, dy = 0,$$

which may thus be written

$$dp' = dp, \quad dq' = dq,$$

p', q', being the partial differentials, first order, of the focal mirror, and p, q, those of the given mirror. Now the general equation of focal mirrors, $d\rho = dV' = \alpha'\, dx + \beta'\, dy + \gamma'\, dz$, gives

$$x - X + p' \,.\, (z - Z) = \rho \,.\, (\alpha' + \gamma' p'),$$
$$y - Y + q' \,.\, (z - Z) = \rho \,.\, (\beta' + \gamma' q'),$$

and therefore

$$dx + p'\, dz - (\alpha' + \gamma' p')\, d\rho = \rho \,.\, (d\alpha' + p'\, d\gamma') + (Z - z + \gamma' \rho)\, dp',$$
$$dy + q'\, dz - (\beta' + \gamma' q')\, d\rho = \rho \,.\, (d\beta' + q'\, d\gamma') + (Z - z + \gamma' \rho)\, dq';$$

if then we put $p' = p$, $q' = q$, in order to express that the focal mirror touches the given mirror, we shall have, to determine dp', dq', two equations which may be thus written,

$$\left.\begin{aligned} \rho \,.\, \{(\gamma + \gamma')\, dp' + d\alpha' + p \,.\, d\gamma'\} &= dx + p\, dz - (\alpha + \gamma p)(\alpha\, dx + \beta\, dy + \gamma\, dz), \\ \rho \,.\, \{(\gamma + \gamma')\, dq' + d\beta' + q \,.\, d\gamma'\} &= dy + q\, dz - (\beta + \gamma q)(\alpha\, dx + \beta\, dy + \gamma\, dz), \end{aligned}\right\} \tag{Z}$$

and if in these equations (Z) we change (dp', dq') to (dp, dq) in order to find the greatest and least osculating focal mirrors, they become the formulæ (T) of the preceding section. Hence it follows, that in general, the foci of the greatest and least osculating mirrors, are the points in which the reflected ray touches the two caustic surfaces; and that the directions of the lines of reflexion, are the directions of osculation corresponding.

[39.] The equations (Z) determine not only the maximum and minimum values of the osculating focal distance (ρ), but also the law by which that distance varies for intermediate directions of osculation. To find this law, we are to employ the formula,

$$(r' - r) \,.\, dx^2 + 2(s' - s) \,.\, dx \,.\, dy + (t' - t) \,.\, dy^2 = 0,$$

that is

$$dp' \,.\, dx + dq' \,.\, dy = dp \,.\, dx + dq \,.\, dy.$$

Adding therefore the two equations (Z), multiplied respectively by (dx, dy), then changing $(dp'\, dx + dq'\, dy)$ to $(dp\, dx + dq\, dy)$, and reducing; we find the following general expression for the osculating focal distance

$$\rho = \frac{dx^2 + dy^2 + dz^2 - d\rho^2}{(\gamma + \gamma') \,.\, (dp \,.\, dx + dq \,.\, dy) + d\alpha' \,.\, dx + d\beta' \,.\, dy + d\gamma' \,.\, dz}. \tag{A$'$}$$

To simplify this formula, let us take the given reflected ray for the axis of (z); the numerator then reduces itself to $(dx^2 + dy^2)$, and the denominator may be put under the form

$$\epsilon\, dx^2 + \zeta \,.\, dx\, dy + \eta\, dy^2,$$

the coefficients ϵ, ζ, η, being independent of ρ, and of the differentials; if then we put

$$dy = dx \,.\, \tan.\, \psi,$$

so that (ψ) shall be the angle which the plane, passing through the ray and through the direction of osculation, makes with the plane of (xz), we shall have

$$\frac{1}{\rho} = \epsilon \,.\, \cos.^2 \psi + \zeta \,.\, \sin. \psi \,.\, \cos. \psi + \eta \,.\, \sin.^2 \psi. \tag{B$'$}$$

This formula may be still further simplified, by taking for the planes of (x, z), (y, z), the tangent planes to the developable pencils, which, by what we have proved, correspond to the maximum and minimum of (ρ). To find these planes we are to put $\dfrac{d\rho}{d\psi} = 0$, which gives,

$$\tan. 2\psi = \frac{\zeta}{\epsilon - \eta};$$

if then we take them for the planes of (x, z), (y, z) we shall have

$$\zeta = 0, \quad \epsilon = \frac{1}{\rho_1}, \quad \eta = \frac{1}{\rho_2},$$

and the formula for the osculating focal distance becomes

$$\frac{1}{\rho} = \frac{1}{\rho_1} \,.\, \cos.^2 \psi + \frac{1}{\rho_2} \,.\, \sin.^2 \psi. \tag{C$'$}$$

ρ_1, ρ_2, being the extreme values of ρ, namely the distances of the two points in which the ray touches the two caustic surfaces. The analogy of this formula (C$'$) to the known formula for the radius of an osculating sphere, is evident; and it is important to observe, that although the reciprocal of (ρ) is included between two given limits, the quantity (ρ) itself is not always included between the corresponding limits, but is on the contrary excluded from between them, when those limits are of opposite algebraic signs, that is, when the two foci of the ray are at opposite sides of the mirror: so that, in this case, there is some impropriety in the term *greatest* osculating focal distance, since there are some directions of osculation for which that distance is infinite, namely, the two directions determined by the condition

$$\frac{1}{\rho} = 0, \quad \tan.^2 \psi = -\frac{\rho_2}{\rho_1}.$$

I shall however continue to employ it, both on account of the analytic theorem which it expresses, and also on account of its analogy to the received phrase of *greatest osculating sphere*, to which the same objection may be made, when the two concavities of the surface are turned in opposite directions.

[40.] I shall conclude this section, by pointing out another remarkable property of the osculating focal mirrors; which is, that if upon the plane, passing through a given direction of osculation, we project the ray reflected from the consecutive point on that direction, the projection will cross the given ray in the osculating focus corresponding. To prove this theorem, I observe, that when the given ray is taken for axis of (z), the point where it meets the mirror for origin, and the tangent planes of the developable pencils for the planes of (x, z), (y, z), the partial differentials second order of the characteristic function (V) become, at the origin,

$$\frac{d^2V}{dx^2} = -\frac{1}{\rho_1}, \quad \frac{d^2V}{dx \,.\, dy} = 0, \quad \frac{d^2V}{dy^2} = -\frac{1}{\rho_2}, \quad \frac{d^2V}{dx \,.\, dz} = 0, \quad \frac{d^2V}{dy \,.\, dz} = 0, \quad \frac{d^2V}{dz^2} = 0,$$

and therefore the cosines of the angles which an infinitely near ray makes with the axes of (x) and (y), are

$$d\alpha = -\frac{dx}{\rho_1}, \quad d\beta = -\frac{dy}{\rho_2}.$$

Hence it follows that the equations of this infinitely near ray are of the form

$$\rho_1 x' + (z' - \rho_1)\, dx = 0, \quad \rho_2 y' + (z' - \rho_2)\, dy = 0;$$

and if we project this ray on the plane

$$x'\, dy - y'\, dx = 0,$$

which passes through the given ray and through the consecutive point on the mirror, the projecting plane will have for equation

$$\frac{\rho_1 \,.\, x' + (z' - \rho_1) \,.\, dx}{(h - \rho_1) \,.\, dx} - \frac{\rho_2 \,.\, y' + (z' - \rho_2) \,.\, dy}{(h - \rho_2) \,.\, dy} = 0,$$

(h) being the height of the point where the projection crosses the given ray, which is to be determined by the condition that the latter plane shall be perpendicular to the former, that is, by the equation,

$$\frac{\rho_1 \,.\, dy}{(h - \rho_1)\, dx} + \frac{\rho_2 \,.\, dx}{(h - \rho_2)\, dy} = 0;$$

which, when we put $dy = dx \,.\, \tan.\, \psi$, becomes

$$h = \frac{\rho_1 \,.\, \rho_2}{\rho_2 \,.\, \cos.^2 \psi + \rho_1 \,.\, \sin.^2 \psi}, \tag{D$'$}$$

a formula that evidently coincides with the one that we found before, for the height of the osculating focus.

IX. *On thin and undevelopable pencils.*

[41.] Having examined some of the most important properties of the developable pencils of a reflected system, we propose in this section to make some remarks upon pencils not developable; and we shall begin by considering *thin pencils*, that is, pencils composed of rays that are very near to a given ray; because in all the most useful applications of optical theory, it is not an entire reflected or refracted system that is employed, but only a small parcel of the rays belonging to that system.

To simplify our calculations, let us take the given ray for the axis of (z), and let us choose the coordinate planes as in the preceding paragraph; the cosines of the angles which a near ray makes with the axes of (x) and (y), will be, nearly,

$$\alpha = -\frac{x}{\rho_1}, \quad \beta = -\frac{y}{\rho_2},$$

x, y, being coordinates of the point in which it meets the mirror; and the equations of this near ray will be, nearly,

$$x' = x + \alpha z', \quad y' = y + \beta z',$$

that is

$$x' = \alpha \,.\, (z' - \rho_1), \quad y' = \beta \,.\, (z' - \rho_2), \tag{E$'$}$$

x', y', z', being the coordinates of the near ray. And if we eliminate α, β, by these equations, from the general equation (N)

$$\beta = f(\alpha),$$

which represents all the pencils of the system, we find for the general equation of thin pencils,

$$\frac{y'}{z' - \rho_2} = f\left(\frac{x'}{z' - \rho_1}\right). \tag{F$'$}$$

[42.] These equations (E′), (F′) include nearly the whole theory of thin pencils. As a first application of them, let us suppose that we are looking at a luminous point, by means of any combination of mirrors; the rays that enter the eye will not in general diverge from any one focus, and therefore will not be bounded by a cone, but by a pencil of another shape, which I shall call the *Bounding Pencil of Vision*, and the properties of which I am now going to investigate.

Suppose for this purpose, that the optic axis coincides with that given ray of the reflected system which we have taken for the axis of (z), and let (δ) represent the distance of the eye from the mirror; the circumference of the pupil will have for equations

$$z = \delta, \quad x^2 + y^2 = e^2,$$

(e) being the radius of the pupil; the rays of the bounding pencil of vision pass through this circumference, and therefore satisfy the condition

$$\alpha^2 \,.\, (\delta - \rho_1)^2 + \beta^2 \,.\, (\delta - \rho_2)^2 = e^2;$$

and eliminating α, β, from this, by means of (E′), we find the following equation for the bounding pencil of vision,

$$\left(\frac{\delta - \rho_1}{z' - \rho_1}\right)^2 . \, x'^2 + \left(\frac{\delta - \rho_2}{z' - \rho_2}\right)^2 . \, y'^2 = e^2. \tag{G$'$}$$

It is evident, from this equation, that every section of the pencil by a plane perpendicular to the optic axis, that is, to the given ray, is a little ellipse, having its centre on that ray, and its semiaxes situated in the tangent planes to the two developable pencils, that is in the planes of (x, z), (y, z). Denoting these semiaxes by (a), (b), we have

$$a = \pm e \,.\, \frac{z' - \rho_1}{\delta - \rho_1}, \quad b = \pm e \,.\, \frac{z' - \rho_2}{\delta - \rho_2}; \tag{H$'$}$$

these semiaxes become equal, that is, the little elliptic section becomes circular, first when

$$z' = \delta, \quad a = b = e,$$

that is, at the eye itself, and secondly when

$$z' = \delta - \frac{2 \,.\, (\delta - \rho_1)(\delta - \rho_2)}{(\delta - \rho_1) + (\delta - \rho_2)}, \quad a = b = \pm \frac{e \,.\, (\rho_1 - \rho_2)}{2\delta - (\rho_1 + \rho_2)},$$

that is, at a distance from the eye equal to the harmonic mean between the distances of the eye from the two foci of that reflected ray, which coincides with the optic axis. It may also be proved, that when the eye is beyond the two foci, the radius of this harmonic section, (which is to the radius of the pupil as the semi-interval between the two foci is to the distance of the eye from the middle point between them,) is less than the semiaxis major of any of the elliptic sections, that is, than the extreme aberration of the visual rays at any other distance from the eye; so that, in this case, we may consider the centre of the harmonic section as the *visible image* of the luminous point, seen by the given combination of mirrors; observing however that the *apparent distance* of the luminous point will depend on other circumstances of brightness, distinctness and magnitude, as it does in the case of direct vision with the naked eye.

[43.] One of the principal properties of thin pencils, is that the area of a perpendicular section of such a pencil is always proportional to the product of its distances from the two foci of the given ray. We may verify this theorem, in the case of the bounding pencil of vision, by means of the formulæ (H$'$) for the semiaxes of the little elliptic section; in general if we represent by Σ the area of the section of any given thin pencil, corresponding to any given value of (z'), we shall have by (E$'$)

$$2 \,.\, \Sigma = \int (y'\,dx' - x'\,dy') = (z' - \rho_1)(z' - \rho_2) \,.\, \int (\beta\,d\alpha - \alpha\,d\beta), \tag{I$'$}$$

and the definite integral $\int(\beta\,d\alpha - \alpha\,d\beta)$, depending only on the relation between α, β, is constant when the pencil is given. It follows from this theorem, that along a given ray the density of the reflected light varies inversely as the product of the distances from the two foci, and is infinite at the caustic surfaces.

[44.] The same equations (E$'$), from which we have deduced the theory of thin pencils, serve also to investigate the properties of other undevelopable surfaces, composed by the rays of the system. The most remarkable difference between an undevelopable and a developable pencil, consists in this, that the tangent plane to the latter always touches it in the whole extent of a ray; whereas in the former, when the point of contact moves along a given ray, the tangent plane changes position, and turns round that ray, like a hinge. To find the law of this rotation let the coordinate planes be chosen as before, the given ray for axis of (z), the point where it meets the mirror for origin, and the tangent planes to the two developable pencils for the planes of (xz), (yz); then by (E′), the equations of an infinitely near ray will be

$$x' = (z' - \rho_1) \,.\, d\alpha, \quad y' = (z' - \rho_2) \,.\, d\beta, \tag{K′}$$

and if it belong to a given undevelopable pencil having for equation $\beta = f(\alpha)$, we shall have

$$d\beta = f' \,.\, d\alpha,$$

f' being a given quantity; the tangent plane to this pencil, at any given distance (z') from the mirror, being obliged to contain the given ray, and to pass through a point on the consecutive, has for equation

$$\frac{y}{x} = \frac{z' - \rho_2}{z' - \rho_1} \,.\, f'; \tag{L′}$$

when z' increases, that is, when the point of contact recedes indefinitely from the mirror, this tangent plane approaches to the limiting position

$$\frac{y}{x} = f', \tag{M′}$$

which is evidently parallel to the consecutive ray (K′); and the angle (P) which it makes with this limiting position, is given by the formula

$$\tan. P = \frac{(\rho_1 - \rho_2) \,.\, f'}{z' \,.\, (1 + f'^2) - (\rho_1 + \rho_2 \,.\, f'^2)},$$

that is, if we put $f' = \tan. L$,

$$\tan. P = \frac{(\rho_1 - \rho_2) \,.\, \sin. L \,.\, \cos. L}{z' - (\rho_1 \,.\, \cos.^2 L + \rho_2 \,.\, \sin.^2 L)},$$

or, finally,

$$\tan. P = \frac{u}{\delta}, \tag{N′}$$

(u) being a constant coefficient,

$$u = (\rho_1 - \rho_2) \,.\, \sin. L \,.\, \cos. L, \tag{O′}$$

and (δ) being the distance of the point of contact from a certain fixed point upon the ray, whose distance from the mirror is

$$z' = \rho_1 \,.\, \cos.^2 L + \rho_2 \,.\, \sin.^2 L. \tag{P′}$$

[45.] The quantity (u), which thus enters as a constant coefficient into the law of rotation of the tangent plane of an undevelopable pencil, I shall call the *coefficient of undevelopability.* In the third part of this essay, I shall treat more fully of its properties, and of those of the fixed point determined by the formula (P′); in the mean time, I shall observe, that if we cut the consecutive ray (K′) by any plane perpendicular to the given ray, at a distance δ from this fixed point (P′), the interval between the two rays, corresponding to this distance (δ), is

$$\Delta = \sqrt{(u^2 + \delta^2)} \,.\, d\theta, \tag{Q$'$}$$

$(d\theta)$ being the angle between the rays; from which it follows, that the fixed point (P′) may be called the *virtual focus* of the given ray, in the given undevelopable pencil, because it is the nearest point to an infinitely near ray of that pencil; and that the coefficient of undevelopability (u), is equal to the least distance between the given ray and the consecutive ray, divided by the angle between them. We may also observe, that although a given ray has in general an infinite number of undevelopable pencils passing through it, and therefore an infinite number of virtual foci corresponding, yet these virtual foci are all included between the two points where the ray touches the two caustic surfaces, because the expression (P′)

$$z' = \rho_1 \,.\, \cos.^2 L + \rho_2 \,.\, \sin.^2 L,$$

is always included between the limits ρ_1 and ρ_2. And whenever, in this essay, the term *foci of a ray* shall occur, the two points of contact with the caustic surfaces are to be understood except when the contrary is expressed.

X. *On the axes of a reflected system.*

[46.] We have seen that the density of light in a reflected system is greatest at the caustic surfaces; from which it is natural to infer, that this density is greatest of all at the intersection of these surfaces: a remark which has already been made by Malus, and which will be still farther confirmed, when we come to consider the aberrations. It is important therefore to investigate the nature and position of the intersection of the caustic surfaces. I am going to shew that this intersection is not in general a curve, but reduces itself to a finite number of isolated points, the foci of a finite number of rays, which are intersected in those points by all the rays infinitely near them. For this purpose I resume the formula (Q) found in Section VI.

$$\left(\rho + \frac{da}{d\alpha}\right)\left(\rho + \frac{db}{d\beta}\right) - \frac{da}{d\beta} \,.\, \frac{db}{d\alpha} = 0, \tag{Q}$$

which determines the two foci of a given ray, and in which the coefficients $\dfrac{da}{d\alpha}, \dfrac{da}{d\beta}, \dfrac{db}{d\alpha}, \dfrac{db}{d\beta}$, are connected by the following relation, deduced from the same section,

$$\alpha\beta \,.\, \left(\frac{da}{d\alpha} - \frac{db}{d\beta}\right) - (\alpha^2 + \gamma^2) \,.\, \frac{da}{d\beta} + (\beta^2 + \gamma^2) \,.\, \frac{db}{d\alpha} = 0. \tag{R$'$}$$

The condition of equal roots in (Q), is

$$\left(\frac{da}{d\alpha} - \frac{db}{d\beta}\right)^2 + 4 \,.\, \frac{da}{d\beta} \,.\, \frac{db}{d\alpha} = 0;$$

this then is the equation which determines the relation between α, β, that belongs to the rays passing through the intersection of the caustic surfaces; and it is easy to prove, by means of the formula (R′), that it resolves itself into the three following, which however, in consequence of the same formula, are equivalent to but two distinct equations:

$$\frac{da}{d\beta} = 0, \quad \frac{db}{d\alpha} = 0, \quad \frac{da}{d\alpha} - \frac{db}{d\beta} = 0. \tag{S$'$}$$

The rays determined by these equations, I shall call the *axes of the reflected system*, and their foci, for which

$$\rho = -\frac{da}{d\alpha} = -\frac{db}{d\beta}, \tag{T$'$}$$

I shall call the *principal foci*.

[47.] We have seen, that a given ray has, in general, an infinite number of virtual foci, corresponding to the undevelopable pencils, and determined by the formula (P′),

$$z' = \rho_1 \,.\, \cos.^2 L + \rho_2 \,.\, \sin.^2 L,$$

and an infinite number of osculating foci, corresponding to the osculating focal mirrors, and determined by the formula (C′)

$$\frac{1}{\rho} = \frac{1}{\rho_1} \,.\, \cos.^2 \psi + \frac{1}{\rho_2} \,.\, \sin.^2 \psi.$$

But when $\rho_2 = \rho_1$, that is, when the ray is an axis of the system, the the variable angles disappear from these formulæ, and all the virtual and all the osculating foci close up into one single point, namely, the principal focus corresponding to that axis. Hence, and from the coefficient of undevelopability vanishing, it follows, that each axis of the system is intersected, at its own focus, by all the rays infinitely near; and that this focus, is the focus of a focal mirror, which has, with the given mirror, complete contact of the second order. A point of contact of this kind, that is, a point where the given mirror is met by an axis of the reflected system, I shall call a *vertex* of the mirror.

[48.] Another remarkable property of the principal foci, is that they are the centres of spheres, which have complete contact of the second order with the surfaces that cut the rays perpendicularly; which may be proved by means of the following formulæ, deduced from (S′) and (T′), combined with the formulæ of [27.],

$$\left.\begin{aligned} \frac{d^2V}{dx^2} &= \frac{\alpha^2 - 1}{\rho}, \quad & \frac{d^2V}{dy^2} &= \frac{\beta^2 - 1}{\rho}, \quad & \frac{d^2V}{dz^2} &= \frac{\gamma^2 - 1}{\rho}, \\ \frac{d^2V}{dx \,.\, dy} &= \frac{\alpha\beta}{\rho}, & \frac{d^2V}{dx \,.\, dz} &= \frac{\alpha\gamma}{\rho}, & \frac{d^2V}{dy \,.\, dz} &= \frac{\beta\gamma}{\rho}, \end{aligned}\right\} \tag{U$'$}$$

And if we substitute these expressions (U′) in the formulæ of [29.], we find the following equations,

$$\left.\begin{aligned}(\gamma+\gamma')\,.\,r+\frac{d^2V'}{dx^2}+2p\,.\,\frac{d^2V'}{dx\,.\,dz}+p^2\,.\,\frac{d^2V'}{dz^2}&=\frac{1+p^2-(\alpha+\gamma p)^2}{\rho},\\ (\gamma+\gamma')\,.\,t+\frac{d^2V'}{dy^2}+2q\,.\,\frac{d^2V'}{dy\,.\,dz}+q^2\,.\,\frac{d^2V'}{dz^2}&=\frac{1+q^2-(\beta+\gamma q)^2}{\rho},\\ (\gamma+\gamma')\,.\,s+\frac{d^2V'}{dx\,.\,dy}+p\,.\,\frac{d^2V'}{dy\,.\,dz}+q\,.\,\frac{d^2V'}{dx\,.\,dz}+pq\,.\,\frac{d^2V'}{dz^2}&=\frac{pq-(\alpha+\gamma p)(\beta+\gamma q)}{\rho},\end{aligned}\right\}\tag{V′}$$

which determine the vertices, the axes, and the principal foci, when we know the equation of the mirror, and the characteristic function of the incident system. These formulæ (V′) may also be deduced from the equations (Z) of Section VIII. by means of the theorem that we have already established, respecting the complete contact of the second order, which exists, at a vertex, between the given mirror and the osculating focal surface corresponding: and they may be reduced to the two following; that is, to the equations (T) of Section VII.

$$\left.\begin{aligned}\rho\,.\,\{(\gamma+\gamma')\,.\,dp+d\alpha'+p\,.\,d\gamma'\}&=dx+p\,.\,dz-(\alpha+\gamma p)(\alpha\,dx+\beta\,dy+\gamma\,dz),\\ \rho\,.\,\{(\gamma+\gamma')\,.\,dq+d\beta'+q\,.\,d\gamma'\}&=dy+q\,.\,dz-(\beta+\gamma q)(\alpha\,dx+\beta\,dy+\gamma\,dz),\end{aligned}\right\}\tag{T}$$

by observing, that these equations, which in general determine the lines of reflexion on the mirror, are, at a vertex, satisfied independently of the ratio between the differentials (dx, dy), provided that we assign to (ρ) its proper value, namely the distance of the principal focus.

[49.] As an application of the preceding theory, let us suppose that the incident rays diverge from a luminous point (X', Y', Z'), and let us seek the vertices, the axes, and the principal foci of the reflected system. In this question, the equations (T) become, by [35.],

$$\begin{aligned}(\gamma+\gamma')\,.\,dp&=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\{dx+p\,dz-(\alpha+\gamma p)(\alpha\,dx+\beta\,dy+\gamma\,dz)\},\\ (\gamma+\gamma')\,.\,dq&=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\{dy+q\,dz-(\beta+\gamma q)(\alpha\,dx+\beta\,dy+\gamma\,dz)\},\end{aligned}$$

ρ' being the distance of the luminous point from the mirror; and since these equations are to be satisfied independently of the ratio between (dx, dy), they resolve themselves into the three following,

$$\left.\begin{aligned}(\gamma+\gamma')\,.\,r&=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right).\,\{1+p^2-(\alpha+\gamma p)^2\},\\ (\gamma+\gamma')\,.\,t&=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right).\,\{1+q^2-(\beta+\gamma q)^2\},\\ (\gamma+\gamma')\,.\,s&=\left(\frac{1}{\rho}+\frac{1}{\rho'}\right).\,\{pq-(\alpha+\gamma p)(\beta+\gamma q)\},\end{aligned}\right\}\tag{W′}$$

which contain the solution of the problem.

To shew the geometrical meaning of these equations (W′), let us take the vertex for origin, the normal at that point for the axis of (z), and the tangents to the lines of curvature for the axes of (x) and (y); we shall then have

$$p = 0, \quad q = 0, \quad s = 0, \quad r = \frac{1}{R}, \quad t = \frac{1}{R'}, \quad \gamma = \gamma' = \cos. I,$$

I being the angle of incidence, and R, R', being the two radii of curvature of the mirror; and the formulæ (W′) become

$$h \,.\, \cos. I = R(1 - \alpha^2) = R'(1 - \beta^2), \quad \alpha\beta = 0, \tag{X′}$$

(h) being the harmonic mean between the conjugate focal distances, so that

$$\frac{2}{h} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

The equation $\alpha\beta = 0$, shews that the plane of incidence must coincide either with the plane of the greatest or the least osculating circle to the mirror; and if we put $\beta = 0$, that is, if we choose the plane of incidence for the plane of (x, z) we shall have $R' = R(1 - \alpha^2)$, $R > R'$, so that it is with the plane of the greatest osulating circle that the plane of incidence coincides. We have also $1 - \alpha^2 = \cos.^2 I$,

$$\left.\begin{aligned} h = R \,.\, \cos. I = R' \,.\, \sec. I, \\ h^2 = RR', \quad R' = R \,.\, \cos.^2 I, \end{aligned}\right\} \tag{Y′}$$

from which it follows, that the harmonic mean between the conjugate focal distances, is equal to the geometric mean between the radii of curvature of the mirror; and that the square of the cosine of the angle of incidence is equal to the ratio of those two radii of curvature. It follows also, that the line joining the luminous point to its conjugate focus, (that is, the axis of the osculating ellipsoid) passes through the centre of the least osculating circle to the mirror; and since it is also contained in the plane of the greatest osculating circle, it is tangent to one surface of centres of curvature of the mirror.

[50.] As another application, let us take the case of parallel rays reflected by a combination of two given mirrors. Let α, β, γ, α', β', γ', be still the cosines of the angles which the last reflected and last incident ray, measured from the last mirror, make with the axes of coordinates; α'', β'', γ'', the given cosines of the angles which the first incident ray, measured towards the first mirror, makes with the same axes; x, y, z, p, q, r, s, t, the coordinates and partial differential coefficients of the last mirror, and x', y', z', p', q', r', s', t', the corresponding quantities of the first. We have then,

$$\alpha' + \alpha'' + p' \,.\, (\gamma' + \gamma'') = 0, \quad \beta' + \beta'' + q' \,.\, (\gamma' + \gamma'') = 0;$$

and therefore

$$\begin{aligned} d\alpha' + p' \,.\, d\gamma' + (\gamma' + \gamma'') \,.\, dp' = 0, \\ d\beta' + q' \,.\, d\gamma' + (\gamma' + \gamma'') \,.\, dq' = 0; \end{aligned}$$

we have also

$$x' = x + \alpha'\rho', \quad y' = y + \beta'\rho', \quad z' = z + \gamma'\rho',$$
$$dx' = dx + d\,.\,(\alpha'\rho'), \quad dy' = dy + d\,.\,(\beta'\rho'), \quad dz' = dz + d\,.\,(\gamma'\rho'),$$
$$d\rho' = \alpha'\,.\,(dx' - dx) + \beta'\,.\,(dy' - dy) + \gamma'\,.\,(dz' - dz),$$

ρ' being the path traversed by the light, in going from the one mirror to the other; by means of these equations we can find, for the quantities $(d\alpha' + p\,d\gamma')$, $(d\beta' + q\,d\gamma')$, which enter into (T), expressions which may be shewn to be of the form

$$d\alpha' + p\,d\gamma' = A\,dx + B\,dy,$$
$$d\beta' + q\,d\gamma' = B\,dx + C\,dy,$$

A, B, C, involving p, q, p', q', r', s', t', ρ': and to determine the vertices, the axes, and the principal foci, of the last reflected system, we shall have the following equations,

$$\left.\begin{aligned}\rho\,.\,\{(\gamma+\gamma')\,.\,r + A\} &= 1 + p^2 - (\alpha+\gamma p)^2,\\ \rho\,.\,\{(\gamma+\gamma')\,.\,s + B\} &= pq - (\alpha+\gamma p)(\beta+\gamma q),\\ \rho\,.\,\{(\gamma+\gamma')\,.\,t + C\} &= 1 + q^2 - (\beta+\gamma q)^2.\end{aligned}\right\} \tag{Z$'$}$$

XI. *On the images formed by mirrors.*

[51.] It appears from the preceding section, that when rays issuing from a luminous point have been reflected at a given mirror, the two caustic surfaces touched by the reflected rays intersect one another in a finite number of isolated points, at which the density of reflected light is greatest, and of which each is the conjugate focus of an ellipsoid of revolution, that has its other focus at the given luminous point, and that has contact of the second order with the given mirror. It is evident that these points of maximum of density are the images of the given luminous point, formed by the given mirror; and that in like manner, the image or images of a given point, formed by a given combination of mirrors, are the corresponding points of maximum density, to which the intersection of the last pair of caustic surfaces reduces itself, and which are the foci of focal mirrors that have contact of the second order with the last given mirror. And on similar principles are we to determine the image of a curve or of a surface, formed by any given mirror, or combination of mirrors; namely, by considering the image of the curve or surface as the locus of the images of its points.

[52.] Let us apply these principles to the investigation of the image of a planet formed by a curved mirror. The image of the planet's centre is the focus of a paraboloid of revolution, which has its axis pointed to that centre, and which has complete contact of the second order with the mirror. To find this image, together with the corresponding point of contact, or vertex on the mirror, we have the equations

$$\alpha + \alpha' + p\,.\,(\gamma+\gamma') = 0, \quad \beta + \beta' + q\,.\,(\gamma+\gamma') = 0, \tag{A$''$}$$

$$\left.\begin{aligned}(\gamma+\gamma')\,.\,\rho\,dp &= dx + p\,dz - (\alpha+\gamma p)(\alpha\,dx + \beta\,dy + \gamma\,dz),\\ (\gamma+\gamma')\,.\,\rho\,dq &= dy + q\,dz - (\beta+\gamma q)(\alpha\,dx + \beta\,dy + \gamma\,dz),\end{aligned}\right\} \tag{B$''$}$$

α, β, γ, α', β', γ', being, as before, the cosines of the angles which the reflected and incident rays make with the axes of coordinates, and (ρ) being the focal distance; the formulæ (B$''$) are satisfied by every infinitely near point upon the mirror, and therefore are equivalent to three distinct equations, which contain the conditions for the contact of the second order between the paraboloid and the mirror. Differentiating the equations (A$''$), in order to pass from the centre to the disk of the planet, and eliminating (dp, dq) by means of (B$''$), we find

$$0 = \rho \,.\, (d\alpha' + p\, d\gamma') + dx + \rho\, d\alpha + p \,.\, (dz + \rho\, d\gamma) - (\alpha + \gamma p)(\alpha\, dx + \beta\, dy + \gamma\, dz),$$
$$0 = \rho \,.\, (d\beta' + q\, d\gamma') + dy + \rho\, d\beta + q \,.\, (dz + \rho\, d\gamma) - (\beta + \gamma q)(\alpha\, dx + \beta\, dy + \gamma\, dz),$$

that is,

$$\left.\begin{aligned} 0 &= \rho \,.\, (d\alpha' + p\, d\gamma') + da + p\, dc - (\alpha + \gamma p)(\alpha\, da + \beta\, db + \gamma\, dc), \\ 0 &= \rho \,.\, (d\beta' + q\, d\gamma') + db + q\, dc - (\beta + \gamma q)(\alpha\, da + \beta\, db + \gamma\, dc), \end{aligned}\right\} \tag{C$''$}$$

if we put (a, b, c) to represent the coordinates of the image, so that

$$a - x = \alpha\rho, \quad b - y = \beta\rho, \quad c - z = \gamma\rho.$$

Differentiating also the three distinct equations which are included in (B$''$), and eliminating, we shall get a result of the form

$$\rho \,.\, d\gamma' = A\, da + B\, db + C\, dc, \tag{D$''$}$$

A, B, C, involving the partial differentials of the mirror, as high as the third order. These equations (C$''$), (D$''$), combined with the identical relation $\alpha'\, d\alpha' + \beta'\, d\beta' + \gamma'\, d\gamma' = 0$, and with the following formula,

$$d\alpha'^2 + d\beta'^2 + d\gamma'^2 = \sigma^2, \tag{E$''$}$$

in which σ is the semidiameter of the planet, contain the solution of the question; for they determine the image of any given point on the disk; and if we eliminate $d\alpha'$, $d\beta'$, $d\gamma'$, between them, we shall find the two relations between da, db, dc, which belong to the locus of those images, that is, to the image of the disk itself.

[53.] To simplify this elimination, let us take the central reflected ray for the axis of (z), that is, let us put $\alpha = 0$, $\beta = 0$, $\gamma = 1$. We shall then have by (A$''$),

$$p = \frac{-\alpha'}{1 + \gamma'}, \quad q = \frac{-\beta'}{1 + \gamma'},$$

and the formulæ (C$''$) will become

$$\rho\, d\alpha' + da = \frac{\alpha' \rho\, d\gamma'}{1 + \gamma'}, \quad \rho\, d\beta' + db = \frac{\beta' \rho\, d\gamma'}{1 + \gamma'},$$

which give, by the identical relations

$$\alpha'^2 + \beta'^2 + \gamma'^2 = 1, \quad \alpha' \,.\, d\alpha' + \beta' \,.\, d\beta' + \gamma' \,.\, d\gamma' = 0,$$

$$\left.\begin{aligned}\rho\, d\gamma' &= \alpha'\, da + \beta'\, db,\\ \rho\, d\alpha' &= -da + \frac{\alpha' \,.\, (\alpha'\, da + \beta'\, db)}{1+\gamma'},\\ \rho\, d\beta' &= -db + \frac{\beta' \,.\, (\alpha'\, da + \beta'\, db)}{1+\gamma'}.\end{aligned}\right\} \qquad (\mathrm{F}'')$$

Eliminating $d\alpha'$, $d\beta'$, $d\gamma'$, by these formulæ, from the equations (D″) and (E″), we find, for the equations of the image,

$$\left.\begin{array}{ll}\text{1st.} & (A-\alpha')\,.\,da + (B-\beta')\,.\,db + C\,.\,dc = 0,\\ \text{2d.} & da^2 + db^2 = \rho^2\,.\,\sigma^2;\end{array}\right\} \qquad (\mathrm{G}'')$$

the image is therefore, in general, an ellipse, the plane of which depends on the quantities A, B, C, which enter into the 1st of its two equations, and therefore on the partial differentials of the mirror, as high as the third order; but the 2d. of its two equations (G″), is independent of those partial differentials, and contains this remarkable theorem, that the projection of the image of the disk, on a plane perpendicular to the reflected rays, is a circle, whose radius is equal to the focal distance (ρ), multiplied by (σ) the sine of the semidiameter of the planet.

[54.] The theorem that has been just demonstrated, respecting the projection of a planet's image, is only a particular case of the following theorem, respecting reflected images in general, which easily follows from the principles of the preceding section, respecting the axes of a reflected system. This theorem is, that if we want to find the image of any small object, formed by any given combination of mirrors, and have found the image of any given point upon the object, together with the corresponding vertex upon the last mirror of the given combination; the rays which come to this given vertex, from the several points of the object, pass after reflection through the corresponding points of the image.

[55.] It follows from this theorem, that in order to form, by a single mirror, an undistorted image of any small plane object, whose plane is perpendicular to the incident rays, it is necessary and sufficient that the plane of the image be perpendicular to the reflected rays. This condition furnishes two relations between the partial differential coefficients, third order, of the mirror, which will in general determine the manner in which the object and mirror are to be placed with respect to one another, in order to produce an undistorted image. Thus, if it were required to find, how we ought to turn a given mirror, in order to produce a circular image of a planet; we should have the following condition,

$$d\rho = \alpha'\, dx + \beta'\, dy + \gamma'\, dz, \qquad (\mathrm{H}'')$$

which expresses that the reflected rays are perpendicular to the plane of the image; α', β', γ', being the cosines of the angles which the incident ray makes with the axes of coordinates; and ρ being the focal length of the mirror, which by [49.] is equal to half the geometric mean between the radii of curvature; so that it is a given function of the partial differentials, first and second orders, of the mirror,

$$\rho = \tfrac{1}{2}\,.\,\frac{1+p^2+q^2}{\sqrt{(rt-s^2)}}; \qquad (\mathrm{I}'')$$

the cosines $(\alpha', \beta', \gamma')$ may also be considered as given functions of (p, q, r, s, t), because, by [49.] the incident ray at the vertex is contained in the plane of the greatest osculating circle to the mirror, and the square of the cosine of angle of incidence is equal to the ratio of the radii of curvature. The two equations therefore, into which (H″) resolves itself, by putting separately $dy = 0$, $dx = 0$, will furnish two relations between the partial differentials of the mirror, up to the third order; these are the two relations which express the condition for the image of the planet being circular: they are identically satisfied in the case of a spheric mirror, for then the first member of (H″) vanishes, on account of the focal length being constant, and the second member on account of the incident ray coinciding with the normal; and accordingly, whatever point of a spheric mirror we choose for vertex, it will form a circular image of a planet; but when the mirror is not spheric, these two relations will in general determine a finite number of points upon it, proper to be used as vertices, in order to form an undistorted image. And when we shall have found these points, which I shall call the *Vertices of Circular Image*, it will then remain to direct towards the planet, one of the two lines which at any such vertex are contained in the plane of the greatest osculating circle to the mirror, and which make with the normal, at either side, angles, the square of whose cosine is equal to the ratio of the radii of curvature.

XII. *Aberrations.*

[56.] After the preceding investigations respecting the two foci of a reflected ray, or points of intersections with rays infinitely near; and respecting the axes of a reflected system, each of which is intersected, in one and the same point, by all the rays that are infinitely near it; we come now to consider the *Aberrations* of rays at a small but finite distance: quantities which have long been calculated for certain simple cases, but which have not, I believe, been hitherto investigated for reflected systems in general.

[57.] When rays fall on a mirror of revolution, from a luminous point in its axis, the reflected rays all intersect that axis, and the distances of those intersections from the focus, are called the longitudinal aberrations. But in general, the rays of a reflected system do not all intersect any one ray of that system; and therefore the *longitudinal* aberrations do not in general exist, in the same manner as they do for those particular cases, which have been hitherto considered. However I shall shew, in a subsequent part of this essay, that there are certain other quantities which in a manner take their place, and follow analogous laws: but at present I shall confine myself to the *lateral* aberrations measured on a plane perpendicular to a given ray, of which the theory is simpler, as well as more important.

Let therefore (x', y', z') represent the coordinates of the point in which the plane of aberration is crossed by any particular ray; these coordinates may be considered as functions of any two quantities which determine the position of that ray; for example, of the cosines of the angles which the ray makes with the axes of (x) and (y). They may therefore be developed in series of the form

$$\left.\begin{aligned}
x' &= X + \frac{dX}{d\alpha} \,.\, \alpha_{\prime} + \frac{dX}{d\beta} \,.\, \beta_{\prime} + \tfrac{1}{2}\left\{\frac{d^2X}{d\alpha^2} \,.\, \alpha_{\prime}^2 + 2\frac{d^2X}{d\alpha \,.\, d\beta} \,.\, \alpha_{\prime}\beta_{\prime} + \frac{d^2X}{d\beta^2} \,.\, \beta_{\prime}^2\right\} + \&\text{c.}\\
y' &= Y + \frac{dY}{d\alpha} \,.\, \alpha_{\prime} + \frac{dY}{d\beta} \,.\, \beta_{\prime} + \tfrac{1}{2}\left\{\frac{d^2Y}{d\alpha^2} \,.\, \alpha_{\prime}^2 + 2\frac{d^2Y}{d\alpha \,.\, d\beta} \,.\, \alpha_{\prime}\beta_{\prime} + \frac{d^2Y}{d\beta^2} \,.\, \beta_{\prime}^2\right\} + \&\text{c.}\\
z' &= Z + \frac{dZ}{d\alpha} \,.\, \alpha_{\prime} + \frac{dZ}{d\beta} \,.\, \beta_{\prime} + \tfrac{1}{2}\left\{\frac{d^2Z}{d\alpha^2} \,.\, \alpha_{\prime}^2 + 2\frac{d^2Z}{d\alpha \,.\, d\beta} \,.\, \alpha_{\prime}\beta_{\prime} + \frac{d^2Z}{d\beta^2} \,.\, \beta_{\prime}^2\right\} + \&\text{c.}
\end{aligned}\right\} \quad (\text{K}'')$$

X, Y, Z, being their values for the given ray, that is the coordinates of the point from which aberration is measured, and $(\alpha_{\prime}, \beta_{\prime})$ being the small but finite increments which the cosines (α, β) receive, in passing from the given ray to the near ray. These equations (K$''$) contain the whole theory of lateral aberration; but in order to apply them, we must shew how to calculate the partial differential coefficients of (X, Y, Z), considered as functions of (α, β). For this purpose I observe, that α, β, being themselves the partial differential coefficients of the characteristic of the system, (Section V.), may be considered as functions of the coordinates a, b, of the projection of the point in which the ray crosses any given perpendicular surface;

$$\alpha = \frac{dV}{da}, \quad \beta = \frac{dV}{db},$$

$$\begin{aligned}
\frac{d\alpha}{da} &= \frac{d^2V}{da^2} + \frac{d^2V}{da \,.\, dc} \,.\, \frac{dc}{da}, & \frac{d\alpha}{db} &= \frac{d^2V}{da \,.\, db} + \frac{d^2V}{da \,.\, dc} \,.\, \frac{dc}{db},\\
\frac{d\beta}{da} &= \frac{d^2V}{da \,.\, db} + \frac{d^2V}{db \,.\, dc} \,.\, \frac{dc}{da}, & \frac{d\beta}{db} &= \frac{d^2V}{db^2} + \frac{d^2V}{db \,.\, dc} \,.\, \frac{dc}{db};
\end{aligned}$$

(c) being the other coordinate of the perpendicular surface, connected with a, b, by the relation

$$V = \text{const.},$$

which gives

$$\frac{dV}{da} \,.\, da + \frac{dV}{db} \,.\, db + \frac{dV}{dc} \,.\, dc = 0,$$

that is

$$\alpha \, da + \beta \, db + \gamma \, dc = 0 :$$

and if we represent by (ρ) the portion of the ray, intercepted between this perpendicular surface and the plane of aberration, we shall have

$$X = a + \alpha\rho, \quad Y = b + \beta\rho, \quad Z = c + \gamma\rho.$$

By means of these formulæ, combined with the equations

$$\begin{aligned}
\alpha \,.\, dX + \beta \,.\, dY + \gamma \,.\, dZ &= 0,\\
\alpha \,.\, d^2X + \beta \,.\, d^2Y + \gamma \,.\, d^2Z &= 0,\\
\alpha \,.\, d^nX + \beta \,.\, d^nY + \gamma \,.\, d^nZ &= 0,
\end{aligned}$$

we can calculate the partial differential coefficients of the five quantities X, Y, Z, α, β, considered as functions of (a, b); and if we wish to deduce hence, their partial differential coefficients relatively to one another, we can do so by means of the following formulæ,

$$dX = \frac{dX}{d\alpha} \,.\, d\alpha + \frac{dX}{d\beta} \,.\, d\beta,$$
$$d^2X = \frac{dX}{d\alpha} \,.\, d^2\alpha + \frac{dX}{d\beta} \,.\, d^2\beta + \frac{dX}{d\alpha^2} \,.\, d\alpha^2 + 2\frac{dX}{d\alpha\,d\beta} \,.\, d\alpha \,.\, d\beta + \frac{dX}{d\beta^2} \,.\, d\beta^2, \text{ \&c.}$$

together with the corresponding formulæ for Y and Z.

[58.] As a first application of the preceding theory, let us suppose the distance between the two rays so small, that we may neglect the squares and products of $(\alpha_{\prime}, \beta_{\prime})$; let us also suppose, that the perpendicular surface of which (a, b, c) are coordinates, crosses the given ray at the point where that ray meets the mirror, and let us take that point for origin, the given ray for the axis of (z), and the tangent planes to the two developable pencils passing through it for the planes of (x, z), (y, z): we shall then have, [40.],

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 1, \quad a = 0, \quad b = 0, \quad c = 0, \quad X = 0, \quad Y = 0, \quad Z = \rho,$$

$$dc = 0, \quad d\rho = 0, \quad dZ = 0, \quad dX = da + \rho\, d\alpha, \quad dY = db + \rho\, d\beta,$$
$$d\alpha = \frac{d^2V}{dx^2} \,.\, da + \frac{d^2V}{dx \,.\, dy} \,.\, db + \frac{d^2V}{dx \,.\, dz} \,.\, dc = -\frac{da}{\rho_1},$$
$$d\beta = \frac{d^2V}{dx \,.\, dy} \,.\, da + \frac{d^2V}{dy^2} \,.\, db + \frac{d^2V}{dy \,.\, dz} \,.\, dc = -\frac{db}{\rho_2},$$
$$\frac{dX}{d\alpha} = \rho - \rho_1, \quad \frac{dX}{d\beta} = 0, \quad \frac{dY}{d\alpha} = 0, \quad \frac{dY}{d\beta} = \rho - \rho_2,$$

ρ_1, ρ_2, being the focal lengths of the mirror; and substituting these values for the partial differential coefficients of X, Y, in the general expressions (K$''$) for the lateral aberrations, we find

$$x' = (\rho - \rho_1) \,.\, \alpha_{\prime}, \quad y' = (\rho - \rho_2) \,.\, \beta_{\prime}, \tag{L$''$}$$

ρ being the distance from the perpendicular surfaces at the mirror to the plane on which the aberration is measured. These formulæ (L$''$) are only the equations (E$'$) of the IXth section, under another form; and it follows from the principles of that section, that the whole lateral aberration may be thus expressed,

$$\surd(x'^2 + y'^2) = \theta \,.\, \surd(u^2 + \delta^2),$$

$\theta = \surd(\alpha_{\prime}^2 + \beta_{\prime}^2)$ being the angle which the near ray makes with the given ray; (u) a constant coefficient, depending on the position of the near ray, and determined by the equation

$$u = (\rho_1 - \rho_2) \,.\, \text{sin.}\, L \,.\, \text{cos.}\, L$$

(L being the angle which the plane of (x, z) makes with a plane drawn through the given ray parallel to the near ray, so that $\beta_\prime = \alpha_\prime \,.\, \tan. L$): and

$$\delta = \rho - (\rho_1 \,.\, \cos.^2 L + \rho_2 \,.\, \sin.^2 L)$$

being the distance of the point where the aberration is measured from the point at which that aberration is least. It follows also, that if we consider any small parcel of the near rays, the area on the plane of aberration over which these rays are diffused, is equal to the product of the distances of that plane from the two foci of the given ray, multiplied by a constant quantity depending on the nature of the parcel. If, for instance, we consider only those rays which make with the given ray angles not exceeding some small given angle (θ), these rays are diffused over the area of an ellipse, having for equation

$$\frac{x'^2}{(\rho - \rho_1)^2} + \frac{y'^2}{(\rho - \rho_2)^2} = \theta^2, \tag{M$''$}$$

and this area is equal to the product of the focal distances $(\rho - \rho_1)$, $(\rho - \rho_2)$, multiplied by $\pi \,.\, \theta^2$, π being the semicircumference of a circle whose radius is equal to unity.

[59.] As a second application, let us take the case where the plane of aberration passes through one of the two foci of the given ray, for example, through the second, so that $\rho = \rho_2$. In this case the formulæ (L$''$) become

$$x' = (\rho_2 - \rho_1) \,.\, \alpha_\prime, \quad y' = 0,$$

so that if we continue to neglect terms of the second order, the points in which the near rays cross the plane of aberration, are all contained on the axis of (x'), that is, on the tangent to the caustic surface. But if we take into account the aberrations of the second order, that is, if we do not neglect the squares and products of $\alpha_\prime$, $\beta_\prime$, which enter into the general expression (K$''$) for y', then the rays will cross the plane of aberration at a small but finite distance from the axis of (x'); that is, y' will have a small but finite value, which we now propose to investigate. For this purpose, that is, to calculate the coefficients in the expression

$$y' = \tfrac{1}{2} \left\{ \frac{d^2 Y}{d\alpha^2} \,.\, \alpha_\prime^2 + 2 \frac{d^2 Y}{d\alpha \,.\, d\beta} \,.\, \alpha_\prime \beta_\prime + \frac{d^2 Y}{d\beta^2} \,.\, \beta_\prime^2 \right\}, \tag{N$''$}$$

I observe that the general formula, [57.],

$$d^2 Y = \frac{dY}{d\alpha} \,.\, d^2\alpha + \frac{dY}{d\beta} \,.\, d^2\beta + \frac{d^2 Y}{d\alpha^2} \,.\, d\alpha^2 + 2 \frac{d^2 Y}{d\alpha \, d\beta} \,.\, d\alpha \,.\, d\beta + \frac{d^2 Y}{d\beta^2} \,.\, d\beta^2,$$

(in which α, β, Y, are considered as functions of the independent variables (a, b), and which is equivalent to three distinct equations) gives, in the present case,

$$\frac{d^2 Y}{d\alpha^2} = \rho_1^2 \,.\, \frac{d^2 Y}{da^2}, \quad \frac{d^2 Y}{d\alpha \,.\, d\beta} = \rho_1 \rho_2 \,.\, \frac{d^2 Y}{da \,.\, db}, \quad \frac{d^2 Y}{d\beta^2} = \rho_2^2 \,.\, \frac{d^2 Y}{db^2},$$

because

$$\frac{dY}{d\alpha} = 0, \quad \frac{dY}{d\beta} = 0, \quad d\alpha = -\frac{da}{\rho_1}, \quad d\beta = -\frac{db}{\rho_2}.$$

Again, the equation $Y = b + \beta\rho$ gives $d^2Y = \rho_2 \,.\, d^2\beta$, when we put

$$d^2b = 0, \quad \beta = 0, \quad d\rho = 0, \quad \rho = \rho_2;$$

we have therefore

$$\frac{d^2Y}{da^2} = \rho_2 \,.\, \frac{d^2\beta}{da^2}, \quad \frac{d^2Y}{da \,.\, db} = \rho_2 \,.\, \frac{d^2\beta}{da \,.\, db}, \quad \frac{d^2Y}{db^2} = \rho_2 \,.\, \frac{d^2\beta}{db^2},$$

and the question is reduced to calculating these partial differential coefficients of (β). Now, the equation

$$d\beta = \frac{d^2V}{dx \,.\, dy} \,.\, da + \frac{d^2V}{dy^2} \,.\, db + \frac{d^2V}{dy \,.\, dz} \,.\, dc,$$

gives, (when we put $d^2a = 0$, $d^2b = 0$, $dc = 0$, $\dfrac{d^2V}{dy \,.\, dz} = 0$,)

$$d^2\beta = \frac{d^3V}{dx^2 \,.\, dy} \,.\, da^2 + 2\frac{d^3V}{dx \,.\, dy^2} \,.\, da \,.\, db + \frac{d^3V}{dy^3} \,.\, db^2,$$

and therefore

$$\frac{d^2\beta}{da^2} = \frac{d^3V}{dx^2 \,.\, dy}, \quad \frac{d^2\beta}{da \,.\, db} = \frac{d^3V}{dx \,.\, dy^2}, \quad \frac{d^2\beta}{db^2} = \frac{d^3V}{dy^3},$$

(V) being the characteristic of the system; so that finally, the coefficients in the formula (N″) have for expressions

$$\frac{d^2Y}{d\alpha^2} = \rho_1^2 \,.\, \rho_2 \,.\, \frac{d^3V}{dx^2 \,.\, dy}, \quad \frac{d^2Y}{d\alpha \,.\, d\beta} = \rho_1 \,.\, \rho_2^2 \,.\, \frac{d^3V}{dx \,.\, dy^2}, \quad \frac{d^2Y}{d\beta^2} = \rho_2^3 \,.\, \frac{d^3V}{dy^3}. \qquad \text{(O}''\text{)}$$

They may therefore be calculated, either immediately from the characteristic function (V), if the form of that function be given; or from the equation of the mirror, and the characteristic of the incident system, according to the method of Section VI.

[60.] The formula (N″), which for conciseness may be written thus

$$y' = \tfrac{1}{2}(A\alpha_{\prime}^2 + 2B\alpha_{\prime}\beta_{\prime} + C\beta_{\prime}^2).$$

combined with the equation $x' = i \,.\, \alpha_{\prime}$, in which (i) denotes the interval $(\rho_2 - \rho_1)$ between the two foci of the ray, enables us to find the curve in which any thin pencil $\beta_{\prime} = f(\alpha_{\prime})$, is cut by a perpendicular plane passing through the focus of the given ray; a question for which the formulæ of Section IX. are not sufficient; since, by those formulæ, the curve would reduce itself to a right line, namely the tangent to the caustic surface. Suppose, for example, that all the rays of the thin pencil make with the given ray some given small angle (θ), in which case we have seen that an ordinary section of the pencil is a little ellipse (M″); we then have to eliminate $\alpha_{\prime}$, $\beta_{\prime}$, between the three equations

$$x' = i\alpha_{\prime}, \quad y' = \tfrac{1}{2}(A\alpha_{\prime}^2 + 2B\alpha_{\prime}\beta_{\prime} + C\beta_{\prime}^2), \quad \alpha_{\prime}^2 + \beta_{\prime}^2 = \theta^2,$$

and we find for the equation of the section

$$2i^2y' = Ax'^2 \pm 2Bx' \,.\, \sqrt{(i^2\theta^2 - x'^2)} + C(i^2\theta^2 - x'^2); \tag{P″}$$

which evidently represents a curve shaped like an hour-glass, or figure of eight, having its node on the axis of (y'), that is, on the normal to the caustic surface, at a distance $= \frac{1}{2}C \,.\, \theta^2$ from the focus, and bounded by the two tangents $x' = \pm i\theta$. The area of this curve is the double of the definite integral $\dfrac{2B}{i^2}\displaystyle\int \sqrt{(i^2\theta^2 - x'^2)} \,.\, x'\,dx'$, taken from $x' = 0$ to $x' = i\theta$; it is therefore

$$\Sigma = \frac{4}{3} \,.\, B \,.\, i \,.\, \theta^3. \tag{Q″}$$

But we must not suppose that this area, like the area of the elliptic section (M″), is the entire space over which all the intermediate rays, that is, all the rays making with the given ray angles less than (θ), are diffused upon the plane of aberration; for it is clear that these intermediate rays intersect the plane of aberration partly inside the curve (P″), and partly outside it; since the focus itself, that is, the point $x' = 0$, $y' = 0$, is outside that curve. We must therefore, in order to find the whole space occupied by the intermediate rays, investigate the enveloppe of all the curves similar to (P″), which can be formed by assigning different values to (θ), and then add to the area (Σ) of the curve (P″) itself, the area of the additional space included between it and its enveloppe. Differentiating therefore, the equation (P″) for (θ) as the only variable, we find

$$Bx' \pm C \,.\, \sqrt{(i^2\theta^2 - x'^2)} = 0, \quad x'^2 = \frac{C^2i^2\theta^2}{B^2 + C^2}, \quad y' = \frac{\frac{1}{2}C \,.\, \theta^2 \,.\, (AC - B^2)}{B^2 + C^2},$$

so that the enveloppe sought is a common parabola, having for equation,

$$2C \,.\, i^2 \,.\, y' = (AC - B^2) \,.\, x'^2, \tag{R″}$$

and the additional space (Σ'), included between it and the curve which it envelopes, being equal to the double of the definite integral

$$\frac{1}{2i^2C} \cdot \int \{Bx' - C \,.\, \sqrt{(i^2\theta^2 - x'^2)}\}^2 \,.\, dx',$$

taken from $x' = 0$, to $x' = \dfrac{i \,.\, C \,.\, \theta}{\sqrt{(B^2 + C^2)}}$, has for expression

$$\Sigma' = \frac{2}{3}i \,.\, \theta^3 \,.\, \{-B + \sqrt{(B^2 + C^2)}\};$$

so that the whole space over which the intermediate rays are diffused, has for expression

$$\Sigma + \Sigma' = \frac{2}{3}i \,.\, \theta^3 \,.\, \{B + \sqrt{(B^2 + C^2)}\}. \tag{S″}$$

In these calculations A, B, C, i, have been supposed positive: but the formula (S″) holds also when all or any of them are negative, provided that we then substitute their numeric for their algebraic values.

[61.] To find the geometric meanings of the coefficients A, B, C, which enter into the preceding expressions for the aberrations measured from a focus, let us investigate the curvatures of the caustic surface. The two focal lengths of a ray, measured from the given perpendicular surface, are determined by the formula (Q) of Section VI.

$$\left(\rho + \frac{da}{d\alpha}\right)\left(\rho + \frac{db}{d\beta}\right) - \frac{da}{d\beta} \cdot \frac{db}{d\alpha} = 0,$$

which when we make $\dfrac{da}{d\alpha} = -\rho_1$, $\dfrac{da}{d\beta} = 0$, $\dfrac{db}{d\alpha} = 0$, $\dfrac{db}{d\beta} = -\rho_2$, $\rho = \rho_2$, gives by differentiation, $d\rho = -d \cdot \left(\dfrac{db}{d\beta}\right)$. We have also, by the same section,

$$\left\{\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx \,.\, dy}\right)^2\right\} \cdot \frac{db}{d\beta} = \gamma \cdot \left(\gamma \cdot \frac{d^2V}{dx^2} - \alpha \cdot \frac{d^2V}{dx \,.\, dz}\right),$$

which when we put

$$\alpha = 0, \quad \gamma = 1, \quad d\gamma = 0, \quad \frac{d^2V}{dx \,.\, dy} = 0, \quad \frac{d^2V}{dx \,.\, dz} = 0,$$

$$\frac{d^2V}{dx^2} = -\frac{1}{\rho_1}, \quad \frac{d^2V}{dy^2} = -\frac{1}{\rho_2}, \quad \frac{db}{d\beta} = -\rho_2,$$

gives by differentiation

$$d \cdot \left(\frac{db}{d\beta}\right) = -\rho_2^2 \cdot d \cdot \left(\frac{d^2V}{dy^2}\right) = B \cdot d\alpha + C \cdot d\beta,$$

and therefore $d\rho = -(B\,d\alpha + C\,d\beta)$. If then we denote by $x_\prime$, $y_\prime$, $z_\prime$, the coordinates of the caustic surface, considered as functions of a and b, we have

$$x_\prime = a + \alpha\rho, \quad y_\prime = b + \beta\rho, \quad z_\prime = c + \gamma\rho,$$

$$dx_\prime = da + \rho\,d\alpha = i\,d\alpha, \quad dy_\prime = 0, \quad dz_\prime = d\rho = -(B\,d\alpha + C\,d\beta),$$

$$d^2y_\prime = \rho_2 \,.\, d^2\beta + 2d\beta \,.\, d\rho = A \,.\, d\alpha^2 - C \,.\, d\beta^2,$$

so that the focus of a near ray has for coordinates

$$x_\prime = i\alpha_\prime, \quad y_\prime = \tfrac{1}{2}(A\alpha_\prime^2 - C\beta_\prime^2), \quad z_\prime = \rho_2 - (B\alpha_\prime + C\beta_\prime);$$

eliminating $\alpha_\prime$, $\beta_\prime$, we find for the approximate equation of the caustic surface,

$$2i^2C \,.\, y_\prime + \{i \,.\, (z_\prime - \rho_2) + Bx_\prime\}^2 - AC \,.\, x_\prime^2 = 0,$$

which shews, that the radius of curvature R of a normal section of this surface, is given by the following equation,

$$\frac{i^2 C}{R} = i^2 \,.\, \cos.^2 \omega + 2iB \,.\, \sin.\omega \,.\, \cos.\omega + (B^2 - AC) \,.\, \sin.^2 \omega,$$

(ω) being the angle which the plane of the section makes with the plane of (yz). Making $\omega = 0$, we get $R = C$; and the maximum and minimum of R are given by the equation

$$A \,.\, R^2 + (B^2 - AC + i^2) \,.\, R - C \,.\, i^2 = 0;$$

from which it follows that C is the radius of curvature of the caustic curve, and that if we denote by (ω) the angle at which this curve crosses either line of curvature on the caustic surface, we shall have

$$C = \frac{R' R''}{R' \cos.^2 \omega + R'' \sin.^2 \omega}, \quad A = \frac{-i^2}{R' \cos.^2 \omega + R'' \sin.^2 \omega},$$

$$B = \frac{i \,.\, (R' - R'') \,.\, \sin.\omega \,.\, \cos.\omega}{R' \cos.^2 \omega + R'' \sin.^2 \omega} \tag{T''}$$

R', R'' being the two radii of curvature of the caustic surface. It appears from these formulæ (T″), that when the ray touches either line of curvature upon the caustic surface, (which is always the case when the reflected system consists of rays, which after issuing from a luminous point, have been reflected by any combination of mirrors of revolution, that have a common axis passing through the luminous point), or when the focus is a point of spheric curvature on its own caustic surface, then B vanishes, and the area (Q″) of the little hour-glass curve is equal to nothing. In fact, in this case, that curve changes shape, and becomes confounded with a little parabolic arc, which has for equation $2i^2 y' = Ax'^2 + C(i^2\theta^2 - x'^2)$, and which is comprised between the limits $x' = \pm i\theta$; this parabolic arc is crossed at its extremities by the parabola (R″), of which the equation becomes $2i^2 y' = Ax'^2$: and the whole space included between these two parabolas, that is, the whole space over which the near rays are diffused, has for expression,

$$\Sigma' = \frac{2}{3} i \,.\, C \,.\, \theta^3. \tag{U''}$$

[62.] As a third application, let us consider the case of aberrations from a principal focus. In this case we have $i = 0$, and the expressions for Σ, Σ', vanish; we must therefore have recourse to new calculations, and introduce terms of the second order, in the expression of x', as well as in that of y'. We find

$$x' = \tfrac{1}{2}\rho^3 \,.\, \left(\frac{d^3 V}{dx^3} \,.\, \alpha_{\prime}^2 + 2 \frac{d^3 V}{dx^2 \,.\, dy} \,.\, \alpha_{\prime}\beta_{\prime} + \frac{d^3 V}{dx \,.\, dy^2} \,.\, \beta_{\prime}^2 \right),$$

$$y' = \tfrac{1}{2}\rho^3 \,.\, \left(\frac{d^3 V}{dx^2 \,.\, dy} \,.\, \alpha_{\prime}^2 + 2 \frac{d^3 V}{dx \,.\, dy^2} \,.\, \alpha_{\prime}\beta_{\prime} + \frac{d^3 V}{dy^3} \,.\, \beta_{\prime}^2 \right),$$

expressions which may be thus written

$$\left.\begin{aligned} x' &= (A\alpha_\prime^2 + 2B\alpha_\prime\beta_\prime + C\beta_\prime^2), \\ y' &= (B\alpha_\prime^2 + 2C\alpha_\prime\beta_\prime + D\beta_\prime^2), \end{aligned}\right\} \tag{V$''$}$$

A, B, C, having different meanings here, from what they had in the preceding paragraphs. And if we eliminate $\alpha_\prime$, $\beta_\prime$, between those equations, by means of the relation

$$\alpha_\prime^2 + \beta_\prime^2 = \theta^2,$$

which expresses that the near rays make with the given ray an angle $= \theta$; we find, for the curve of aberration, that is, for the locus of the points in which those rays cross the perpendicular plane drawn through the principal focus, the following equation,

$$\begin{aligned} 4\{(B^2 - AC)\theta^2 - By' + Cx'\}\{(C^2 - BD)\theta^2 + By' - Cx'\} \\ = \{(A - C)y' + (D - B)x' + (BC - AD) \,.\, \theta^2\}^2, \end{aligned}$$

which may be thus written

$$A''y'^2 + 2B''x'y' + C''x'^2 - (D''y' + E''x')\theta^2 + F'' \,.\, \theta^4 = 0, \tag{W$''$}$$

if we put for abridgment

$$A'' = (A - C)^2 + 4B^2, \quad B'' = (A - C)(D - B) - 4BC, \quad C'' = (D - B)^2 + 4C^2,$$

$$D'' = (A + C) \,.\, B'' + (D + B) \,.\, A'', \quad E'' = (A + C) \,.\, C'' + (D + B) \,.\, B'',$$
$$F'' = (AD - BC)^2 - 4(B^2 - AC)(C^2 - BD).$$

These values give

$$A''C'' - B''^2 = 4\{C(A - C) + B(D - B)\}^2,$$

so that the curve (W$''$) is an ellipse; the centre of this little ellipse has for coordinates

$$a'' = \tfrac{1}{2}(A + C) \,.\, \theta^2, \quad b'' = \tfrac{1}{2}(D + B) \,.\, \theta^2,$$

and its area is

$$\Sigma = \pm\tfrac{1}{2}\pi\{C \,.\, (C - A) + B \,.\, (B - D)\} \,.\, \theta^4. \tag{X$''$}$$

If now we consider those intermediate rays, which make with the given ray some given small angle (θ'), less than (θ), the points in which these rays cut the plane of aberration will form another similar ellipse, having for equation

$$A''y'^2 + 2B''x'y' + C''x'^2 - (D''y' + E''x')\theta'^2 + F'' \,.\, \theta'^4 = 0,$$

and if (F$''$) be negative, this ellipse is entirely inside the other, and all the rays that make with the given ray angles not exceeding (θ) are diffused over the elliptic area (X$''$). But if (F$''$) be positive, that is, if the focus be outside the little ellipse of aberration (W$''$), then the intermediate rays are not all diffused over the area of the ellipse, but cut the plane of aberration partly inside that area and partly outside it. To find therefore, in this case, the whole space over which these near rays are diffused, we must seek the enveloppe of all the little ellipses similar to (W″), and then add to the area of that curve (W″) itself, the area of the space included between it and its enveloppe. This enveloppe has for equation

$$(D''y' + E''x')^2 = 4F'' \,.\, (A''y'^2 + 2B''x'y' + C''x'^2); \tag{Y''}$$

when (F″) is negative it has no existence, and when (F″) is positive it consists of two right lines passing through the focus, which are common tangents to all the little ellipses, and which may be called the *Limiting Lines of Aberration*; the space included between them and the ellipse (W″), has for expression

$$\Sigma' = \frac{\Sigma}{\pi} \,.\, (\tan.\psi - \psi), \tag{Z''}$$

Σ being the area of the ellipse, and (ψ) an angle whose cosine, multiplied by the focal distance of the centre of that ellipse, is equal to the semidiameter whose prolongation passes through the focus; we have therefore

$$\tan.\psi = \frac{\sqrt{F''}}{C(C-A) + B(B-D)}, \tag{A'''}$$

and the entire space over which the intermediate rays are diffused is

$$\Sigma + \Sigma' = \tfrac{1}{2}[\sqrt{F''} + (\pi - \psi)\{C(C-A) + B(B-D)\}] \,.\, \theta^4. \tag{B'''}$$

[63.] We have just seen, that in investigating the aberrations from a principal focus, it is necessary to distinguish two cases, essentially different from one another. In the one case, all the rays that make with the given ray angles not exceeding some given small angle (θ), are diffused over the area of a little ellipse; in the other case they are diffused over a mixtilinear space, bounded partly by an elliptic arc, and partly by two right lines, which touch that elliptic arc, and which pass through the principal focus. The analytic distinction between these two cases depends on the sign of a certain quantity F'', which is negative in the first case, and positive in the second. It is therefore interesting to examine, for any proposed system, whether this quantity be positive or negative. I am going to shew that this depends on the reality of the roots of a certain cubic equation, which determines the directions of spheric inflexion on the surfaces that cut the rays perpendicularly; I shall shew also that the sign of the same quantity, is the criterion of the reality of the roots of a certain quadratic equation, which determines the directions in which the plane of aberration is cut by the two caustic surfaces.

First then, with respect to the caustic surfaces, it may be proved, by reasonings similar to those of [61.], that the two foci of a near ray have for coordinates

$$x_{\prime} = x' + \rho_{\prime}\alpha_{\prime}, \quad y_{\prime} = y' + \rho_{\prime}\beta_{\prime}, \quad z_{\prime} = \rho + \rho_{\prime},$$

x', y', being the coordinates of the point in which the near ray crosses the plane of aberration, determined by the formulæ (V$''$), and $(\rho_\prime)$ having a double value determined by the following quadratic equation

$$(\tfrac{1}{2}\rho_\prime + A\alpha_\prime + B\beta_\prime)(\tfrac{1}{2}\rho_\prime + C\alpha_\prime + D\beta_\prime) - (B\alpha_\prime + C\beta_\prime)^2 = 0,$$

in which A, B, C, D, have the same meanings as in the preceding paragraph. The intersection therefore of the caustic surfaces with the plane of aberration, is to be found by putting $\rho_\prime = 0$, which gives $z' = \rho$, $x_\prime = x'$, $y_\prime = y'$,

$$(A\alpha_\prime + B\beta_\prime)(C\alpha_\prime + D\beta_\prime) - (B\alpha_\prime + C\beta_\prime)^2 = 0; \tag{C$'''$}$$

the condition for the roots being real, in this quadratic (C$'''$), is

$$(AD - BC)^2 - 4(B^2 - AC)(C^2 - BD) > 0, \tag{D$'''$}$$

that is, $F'' > 0$, so that unless this condition be satisfied, the caustic surfaces do not intersect the plane of aberration; and when this condition is satisfied, the intersection consists of two right lines, which are determined by the equation

$$(Ay' - Bx')(Cy' - Dx') = (By' - Cx')^2, \tag{E$'''$}$$

and which may easily be shewn to be the same with those limiting lines which we have already considered.

[64.] Secondly, respecting the surfaces that cut the rays perpendicularly, and which are given by the differential equation

$$\alpha\, da + \beta\, db + \gamma\, dc = 0;$$

we have seen in a former section that the principal foci are the centres of spheres that have contact of the second order with these perpendicular surfaces; and if we wish to find the directions in which they are cut by those osculating spheres, we must express that the sum of the terms of the third order in the development of the ordinate of the sphere, is equal to the corresponding sum, in the development of the perpendicular surface. This condition, when the ray is taken for the axis of (z), gives $d^3c = 0$, that is $d^2\alpha \,.\, da + d^2\beta \,.\, db = 0$, which produces the following cubic equation, (see [59.])

$$0 = \frac{d^3V}{dx^3} \,.\, da^3 + 3 \,.\, \frac{d^3V}{dx^2 \,.\, dy} \,.\, da^2 \,.\, db + 3 \,.\, \frac{d^3V}{dx \,.\, dy^2} \,.\, da \,.\, db^2 + \frac{d^3V}{dy^3} \,.\, db^3. \tag{F$'''$}$$

This equation determines the directions of *spheric inflexion* upon the perpendicular surface, that is, the directions in which it is cut by its osculating sphere; and the condition for there being three such directions, that is, for the three roots of this cubic equation being real, is

$$\begin{aligned}&\left\{\frac{d^3V}{dx^3}\cdot\frac{d^3V}{dy^3}-\frac{d^3V}{dx^2\,.\,dy}\cdot\frac{d^3V}{dx\,.\,dy^2}\right\}^2\\ &-4\,.\left\{\left(\frac{d^3V}{dx^2\,.\,dy}\right)^2-\frac{d^3V}{dx^3}\cdot\frac{d^3V}{dx\,.\,dy^2}\right\}\left\{\left(\frac{d^3V}{dx\,.\,dy^2}\right)^2-\frac{d^3V}{dx^2\,.\,dy}\cdot\frac{d^3V}{dy^3}\right\}\\ <\;&0,\end{aligned}\tag{G$'''$}$$

that is $F'' < 0$. When, therefore, the principal focus is inside the little ellipses of aberration, there are three directions of spheric inflexion on the surfaces that cut the rays perpendicularly; and when it is outside those little ellipses, there is but one such direction. It appears also, from the formula (F$'''$), that the aberrations of the second order do not vanish, unless the surfaces that cut the rays perpendicularly have contact of the third order with the osculating spheres that have their centre at the principal focus; this condition is expressed by four equations which are not in general satisfied: and for this reason I shall dispense with considering the aberrations of the third order, because they only present themselves in some particular cases; for example, in spheric mirrors, the theory of which has perhaps been sufficiently studied by others.

[65.] I shall conclude this section by shewing that the conditions for contact of the third order between the perpendicular surface and its osculating sphere, which, as we have just seen, are the conditions for the aberrations of the second order vanishing, are also the conditions for contact of the third order, between the mirror and the osculating focal surface (Section VIII.); and that the sign of that quantity (F'') which distinguishes between the two different kinds of aberration from a principal focus, and which, as we have seen, depends on the number of directions in which the perpendicular surface is cut by the osculating sphere, depends also on the number of directions in which the mirror is cut by its osculating focal surface.

To prove these theorems, I observe that if we denote by (p'', q'') the partial differential coefficients, first order, of the focal surface, that is, of the surface which would reflect accurately the rays of the given incident system to the focus (X, Y, Z), the condition that determines the directions, in which this surface cuts the given mirror, with which (by Section X.) it has complete contact of the second order, is

$$d^2p''\,.\,dx + d^2q''\,.\,dy = d^2p\,.\,dx + d^2q\,.\,dy, \tag{H$'''$}$$

and that this same equation, when it is to be satisfied independently of the ratio between dx, dy resolves itself into four distinct equations, which are the conditions for contact of the third order, between the given mirror and its osculating focal surface. Now, if we represent by α'', β'', γ'', the cosines of the angles which the reflected ray would make with the axes, if it came from the focal surface, and not from the given mirror, we shall have (Section II.)

$$\alpha'' + \alpha' + (\gamma'' + \gamma')p'' = 0, \quad \beta'' + \beta' + (\gamma'' + \gamma')q'' = 0,$$

and therefore

$$d\alpha'' + d\alpha' + (d\gamma'' + d\gamma') \,.\, p'' + (\gamma'' + \gamma') \,.\, dp'' = 0,$$
$$d\beta'' + d\beta' + (d\gamma'' + d\gamma') \,.\, q'' + (\gamma'' + \gamma') \,.\, dq'' = 0,$$

$$d^2\alpha'' + d^2\alpha' + (d^2\gamma'' + d^2\gamma') \,.\, p'' + 2(d\gamma'' + d\gamma') \,.\, dp'' + (\gamma'' + \gamma') \,.\, d^2p'' = 0,$$
$$d^2\beta'' + d^2\beta' + (d^2\gamma'' + d^2\gamma') \,.\, q'' + 2(d\gamma'' + d\gamma') \,.\, dq'' + (\gamma'' + \gamma') \,.\, d^2q'' = 0,$$

α', β', γ' being the cosines of the angles which the incident ray makes with the axes; in the same manner, we have for the given mirror,

$$\alpha + \alpha' + (\gamma + \gamma') \,.\, p = 0, \quad \beta + \beta' + (\gamma + \gamma') \,.\, q = 0,$$

$$d\alpha + d\alpha' + (d\gamma + d\gamma') \,.\, p + (\gamma + \gamma') \,.\, dp = 0,$$
$$d\beta + d\beta' + (d\gamma + d\gamma') \,.\, q + (\gamma + \gamma') \,.\, dq = 0,$$

$$d^2\alpha + d^2\alpha' + (d^2\gamma + d^2\gamma') \,.\, p + 2(d\gamma + d\gamma') \,.\, dp + (\gamma + \gamma') \,.\, d^2p = 0,$$
$$d^2\beta + d^2\beta' + (d^2\gamma + d^2\gamma') \,.\, q + 2(d\gamma + d\gamma') \,.\, dq + (\gamma + \gamma') \,.\, d^2q = 0,$$

and because of the contact of the second order, which exists between the two surfaces, we have

$$p'' = p, \quad q'' = q, \quad \alpha'' = \alpha, \quad \beta'' = \beta, \quad \gamma'' = \gamma,$$

$$dp'' = dp, \quad dq'' = dq, \quad d\alpha'' = d\alpha, \quad d\beta'' = d\beta, \quad d\gamma'' = d\gamma,$$

$$(\gamma + \gamma')(d^2p'' \,.\, dx + d^2q'' \,.\, dy) + 2(d\gamma + d\gamma')(dp \,.\, dx + dq \,.\, dy)$$
$$+ d^2(\alpha'' + \alpha') \,.\, dx + d^2(\beta'' + \beta') \,.\, dy + d^2(\gamma'' + \gamma') \,.\, dz = 0,$$
$$(\gamma + \gamma')(d^2p \,.\, dx + d^2q \,.\, dy) + 2(d\gamma + d\gamma')(dp \,.\, dx + dq \,.\, dy)$$
$$+ d^2(\alpha + \alpha') \,.\, dx + d^2(\beta + \beta') \,.\, dy + d^2(\gamma + \gamma') \,.\, dz = 0;$$

the condition (H$'''$) may therefore be thus written,

$$d^2\alpha'' \,.\, dx + d^2\beta'' \,.\, dy + d^2\gamma'' \,.\, dz = d^2\alpha \,.\, dx + d^2\beta \,.\, dy + d^2\gamma \,.\, dz : \qquad (\mathrm{I}''')$$

besides, when the given ray, or axis of the system, is made the axis of (z), and when we take for the two independent variables the two quantities (a, b), that is, the coordinates of the projection of the point in which the ray crosses the perpendicular surface, [57.], we have, from [59.], and from the nature of the functions α'', β'', γ'',

$$d^2\alpha'' = 0, \quad d^2\beta'' = 0, \quad d^2\gamma'' = d^2\gamma, \quad da = dx, \quad db = dy,$$

$$d^2\alpha = \frac{d^3V}{dx^3} \,.\, dx^2 + 2 \,.\, \frac{d^3V}{dx^2 \,.\, dy} \,.\, dx \,.\, dy + \frac{d^3V}{dx \,.\, dy^2} \,.\, dy^2,$$
$$d^2\beta = \frac{d^3V}{dx^2 \,.\, dy} \,.\, dx^2 + 2 \,.\, \frac{d^3V}{dx \,.\, dy^2} \,.\, dx \,.\, dy + \frac{d^3V}{dy^3} \,.\, dy^2,$$

so that (I‴) becomes

$$\frac{d^3V}{dx^3} \,.\, dx^3 + 3 \,.\, \frac{d^3V}{dx^2 \,.\, dy} \,.\, dx^2 \,.\, dy + 3 \,.\, \frac{d^3V}{dx \,.\, dy^2} \,.\, dx \,.\, dy^2 + \frac{d^3V}{dy^3} \,.\, dy^3 = 0; \qquad (\mathrm{K}''')$$

this then is the cubic equation which determines on the given mirror, the directions of *focal inflection*, that is, the directions in which it is cut by the osculating focal mirror; and comparing this with the cubic equation (F‴) which determines the directions of spheric inflexion on the perpendicular surfaces, we see that the planes which pass through these directions of spheric inflexion, and through the axes of the system, pass also through the directions of focal inflexion on the mirror; so that the number of the latter directions is the same as the number of the former. If then there be but one direction of focal inflexion on the mirror, that is, if the cubic equation (K‴) have two of its roots imaginary, the principal focus is outside the little ellipses of aberration, and the caustic surfaces cross the plane of aberration, in those two limiting lines, or tangents to the little ellipses, which we have considered in [62.]; but if there be three directions of focal inflexion, that is, if the three roots of (K‴) be real, then the limiting lines of aberration become imaginary, and the principal focus is inside the little ellipses. And if the equation (K‴) be identically satisfied, that is, if the mirror have contact of the third order with its osculating focal surface, then the little ellipses themselves disappear, and the aberrations of the second order vanish.

XIII. *Density.*

[66.] Malus, who first discovered that the rays of a reflected system are in general tangents to two caustic surfaces, has given in his *Traité D'Optique*, (published among the *Mémoirs des Savans Étrangers*) the following method for investigating the density of the reflected light at any given point of the system. He considers two infinitely near pairs of developable surfaces formed by the rays; and as he believed himself to have demonstrated that the two surfaces of such a pair are not in general perpendicular to one another, when the rays have been more than once reflected, he concludes that the perpendicular section of the parcel of rays comprised between the four developable surfaces, will be in general shaped as an oblique angled parallelogram, whose area is equal to the product of the two focal distances of the section, multiplied by the sine of the angle formed by the two developable surfaces of each pair. He then compares this area with the area over which the same rays would be diffused, if they had proceeded without interruption to an equal distance from the luminous point; and he takes the reciprocal ratio of these areas for the measure of the density of the reflected light, compared with that of the direct light. The calculations required in this method are of considerable intricacy, and the most remarkable result to which they lead, is that along a given ray the density varies inversely as the product of the focal distances, being infinite at the caustic surfaces, and greatest at their intersection. The same result follows from the theorem which I have pointed out in [43.] respecting small parcels of rays bounded by any thin pencil, of whatever shape; and that theorem furnishes a general method for investigating the density of the reflected light, at points not upon the caustic surfaces, which appears to me simpler than that of Malus, and which for that reason I am going here to explain.

Suppose then that rays issuing from a luminous point have been any number of times reflected by any combination of mirrors; let us put Δ to represent the density of the direct light at the distance unity from the luminous point, and let us put (s) to represent the space over which any given small parcel of that light, bounded by any thin cone, is perpendicularly diffused at that distance. Then, if we represent by (ρ) the first side of the polygon, that is, the portion of any given incident ray comprised between the luminous point and the first mirror, the perpendicular section of the incident parcel, at that distance from the luminous point, will have its area $\Sigma = \rho^2 \,.\, s$; and the space over which the parcel is diffused upon the mirror, has for expression $\dfrac{\rho^2 \,.\, s}{\cos. I}$, I being the angle of incidence. Immediately after reflexion, the parcel will again have its perpendicular section $\rho^2 \,.\, s = \Sigma$; and if we represent by F_1', F_2', the two focal lengths of the first mirror, that is, the distances from the point of incidence to the two points where the first reflected ray touches the first pair of caustic surfaces, we shall have by [43.] the following expression for the perpendicular section of the reflected parcel, at any distance (ρ') from the first mirror:

$$\Sigma' = \frac{\Sigma \,.\, (\rho' - F_1')(\rho' - F_2')}{F_1' \,.\, F_2'}, \text{ in which } \Sigma = \rho^2 \,.\, s. \tag{L$'''$}$$

Now let ρ' be the second side of the polygon, that is, the path run over by the light in going from the first mirror to the second, and let (F_1'', F_2'') be the two focal lengths of the second mirror; we shall have, in a similar manner, for the area of the perpendicular section of the parcel, after the second reflexion, at a distance ρ'' from the second mirror,

$$\Sigma'' = \frac{\Sigma' \,.\, (\rho'' - F_1'')(\rho'' - F_2'')}{F_1'' \,.\, F_2''}; \tag{M$'''$}$$

and so on, for any number of reflexions. Having thus got the space over which the reflected rays are perpendicularly diffused, the density is obtained by this formula

$$\Delta^{(n)} = \frac{s \,.\, \Delta}{\Sigma^{(n)}}. \tag{N$'''$}$$

For instance, if the rays have been but once reflected, then the density is

$$\Delta' = \frac{s \,.\, \Delta}{\Sigma'} = \frac{\Delta}{\rho^2} \cdot \frac{F_1' \,.\, F_2'}{(\rho' - F_1')(\rho' - F_2')}, \tag{O$'''$}$$

a formula which agrees with that of Malus; after two reflections, the density is

$$\Delta'' = \frac{s \,.\, \Delta}{\Sigma''} = \frac{\Delta' \,.\, F_1'' \,.\, F_2''}{(\rho'' - F_1'')(\rho'' - F_2'')}, \tag{P$'''$}$$

Δ' being the density immediately before the second reflexion: a formula which is different from that of Malus, and which appears by me to be simpler.

[67.] The two preceding methods, namely, that of Malus, and that of the preceding paragraph, fail when the density is to be measured at the caustic surfaces; for they only shew that the density at those surfaces is infinitely greater than at other points of the system, without shewing by what law the density varies in passing from one point of a caustic surface to another. This question, which has not been treated by Malus, appears to me too important to be passed over, although the discussion of it is more difficult than the investigation of the density at ordinary points of the system.

Let us then, as a first approximation, resume the formulæ of [60.]

$$x' = i\alpha_{\prime}, \quad y' = \tfrac{1}{2}(A\alpha_{\prime}^2 + 2B\alpha_{\prime}\beta_{\prime} + C\beta_{\prime}^2),$$

x', y' being the coordinates of the point in which a near ray crosses the plane of aberration, that is, a plane perpendicular to the given ray, passing through the focus of that ray; $\alpha_{\prime}$, $\beta_{\prime}$, small but finite quantities, namely, the cosines of the angles which the near ray makes with the axes of (x') and (y'), the former of which axes is a tangent and the other a normal to the caustic surface; A, B, C, coefficients depending on the curvatures of that surface, and on the interval (i) between the two foci of the ray. To find by these equations the space over which the rays, that pass through any given small area on the plane of aberration, are diffused upon another perpendicular plane, which crosses the given reflected ray at the point where that ray meets the mirror, we are to employ these other formulæ (see [58.])

$$a = -\rho_1 \,.\, \alpha_{\prime}, \quad b = -\rho_2 \,.\, \beta_{\prime};$$

a, b, being the coordinates of the point in which a near ray crosses this latter plane, and ρ_1, ρ_2, the distances of that point from the two caustic surfaces, that is, the two focal lengths of the mirror. In this manner we find, that to any given point (x', y') on the plane of aberration, correspond two other points on the other perpendicular plane, determined by the equations

$$a = -\frac{\rho_1 \,.\, x'}{i}, \quad b = \frac{\rho_2}{Ci} \,.\, (Bx' \mp \surd\{2Ci^2y' + (B^2 - AC)x'^2\}); \qquad (\text{Q}''')$$

understanding however that these two points become imaginary, when the quantity under the radical sign is negative, that is, when the point (x', y') is at the wrong side of the enveloping parabola (R″), [60.]; which parabola, within the small extent in which it is taken, may be considered as confounded with the normal section of the caustic surface made by the plane of aberration. Now, if we consider any infinitely little rectangle upon this latter plane, having for the coordinates of its four corners

$$\text{1st. } x',\, y', \quad \text{2d. } x' + dx',\, y', \quad \text{3d. } x',\, y' + dy', \quad \text{4th. } x' + dx',\, y' + dy',$$

the rays which pass inside this little rectangle are diffused over two little parallelograms on the other perpendicular plane; the four corners of the one having for coordinates,

$$\text{1st. } a,\, b, \quad \text{2d. } a + da,\, b + \frac{db}{dx'} \,.\, dx',$$

$$\text{3d. } a,\, b + \frac{db}{dy'} \,.\, dy', \quad \text{4th. } a + da,\, b + \frac{db}{dx'} \,.\, dx' + \frac{db}{dy'} \,.\, dy',$$

and the four corners of the other having for coordinates,

$$1\text{st.}\ a,\ b',\quad 2\text{d.}\ a+da,\ b'+\frac{db'}{dx'}\,.\,dx',$$

$$3\text{d.}\ a,\ b'+\frac{db'}{dy'}\,.\,dy',\quad 4\text{th.}\ a+da,\ b'+\frac{db'}{dx'}\,.\,dx'+\frac{db'}{dy'}\,.\,dy',$$

b, b' being the two values of (b) given by the formulæ (Q‴). The areas of these two parallelograms are each equal to $\left(da\,.\,\dfrac{db}{dy'}\,.\,dy'\right)$, that is to

$$\frac{\rho_1\,.\,\rho_2\,.\,dx'\,.\,dy'}{\sqrt{\{2Ci^2y'+(B^2-AC)x'^2\}}};$$

and the area of the little rectangle on the plane of aberration is $dx'\,.\,dy'$; if then we denote by $\Delta^{(\mu)}$ the density at the mirror, we shall have for the density at the point x', y', on the plane of aberration, the following approximate expression

$$\Delta^{(\alpha)}=\frac{2\,.\,\rho_1\,.\,\rho_2\,.\,\Delta^{(\mu)}}{\sqrt{\{2Ci^2y'+(B^2-AC)x'^2\}}};\qquad (\text{R}''')$$

an expression which shews that at the caustic surface the density is infinitely greater than at the mirror; and that near the caustic surface the density is not uniform, but varies nearly inversely as the square root of the perpendicular distance from that surface; so that we may consider this density as constant in any one of the little parabolic bands comprised between two infinitely near parallels to the enveloping curve (R″) [60.].

[68.] To treat this question, respecting the variation of density upon the plane of aberration, in a more accurate manner, let us take into account the remaining terms of the developments of x' und y', as given by the general theory, which we have explained at the beginning of the preceding section. For although we were at liberty to neglect these terms, when we were only in quest of approximate formulæ to represent the manner in which certain of the near rays are diffused over the plane of aberration; yet, when we are returning from this latter plane to the perpendicular plane at the mirror, it is not safe to neglect any term on account of its smallness, until we have examined whether, in thus returning, its influence may not be magnified in such a manner as to become sensible.

Let us then resume the general series (K″) [57.]

$$x'=X+\frac{dX}{d\alpha}\,.\,\alpha_,+\frac{dX}{d\beta}\,.\,\beta_,+\tfrac{1}{2}\,.\left\{\frac{d^2X}{d\alpha^2}\,.\,\alpha_,^2+2\frac{d^2X}{d\alpha\,.\,d\beta}\,.\,\alpha_,\beta_,+\frac{d^2X}{d\beta^2}\,.\,\beta_,^2\right\}+\&\text{c.},$$
$$y'=Y+\frac{dY}{d\alpha}\,.\,\alpha_,+\frac{dY}{d\beta}\,.\,\beta_,+\tfrac{1}{2}\,.\left\{\frac{d^2Y}{d\alpha^2}\,.\,\alpha_,^2+2\frac{d^2Y}{d\alpha\,.\,d\beta}\,.\,\alpha_,\beta_,+\frac{d^2Y}{d\beta^2}\,.\,\beta_,^2\right\}+\&\text{c.},$$

in which we have at present

$$X=0,\quad Y=0,\quad \frac{dX}{d\alpha}=i,\quad \frac{dX}{d\beta}=0,\quad \frac{dY}{d\alpha}=0,\quad \frac{dY}{d\beta}=0,$$

$$\frac{d^2X}{d\alpha^2} = \rho_1^3 \cdot \frac{d^3V}{dx^3}, \quad \frac{d^2X}{d\alpha\,.\,d\beta} = \rho_1^2 \cdot \rho_2 \cdot \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2X}{d\beta^2} = \rho_1 \cdot \rho_2^2 \cdot \frac{d^3V}{dx\,.\,dy^2},$$
$$\frac{d^2Y}{d\alpha^2} = \rho_1^2 \cdot \rho_2 \cdot \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2Y}{d\alpha\,.\,d\beta} = \rho_1 \cdot \rho_2^2 \cdot \frac{d^3V}{dx\,.\,dy^2}, \quad \frac{d^2Y}{d\beta^2} = \rho_2^3 \cdot \frac{d^3V}{dy^3},$$

V being the characteristic function; so that

$$\frac{d^2X}{d\alpha\,.\,d\beta} = \frac{d^2Y}{d\alpha^2}, \quad \text{and} \quad \frac{d^2X}{d\beta^2} = \frac{d^2Y}{d\alpha\,.\,d\beta}.$$

We have in like manner,

$$\alpha_{\prime} = \frac{d\alpha}{da} \,.\, a + \frac{d\alpha}{db} \,.\, b + \tfrac{1}{2} \,.\, \left\{ \frac{d^2\alpha}{da^2} \,.\, a^2 + 2\frac{d^2\alpha}{da\,.\,db} \,.\, ab + \frac{d^2\alpha}{db^2} \,.\, b^2 \right\} + \&\text{c.},$$
$$\beta_{\prime} = \frac{d\beta}{da} \,.\, a + \frac{d\beta}{db} \,.\, b + \tfrac{1}{2} \,.\, \left\{ \frac{d^2\beta}{da^2} \,.\, a^2 + 2\frac{d^2\beta}{da\,.\,db} \,.\, ab + \frac{d^2\beta}{db^2} \,.\, b^2 \right\} + \&\text{c.},$$

in which, at present,

$$\frac{d\alpha}{da} = -\frac{1}{\rho_1}, \quad \frac{d\alpha}{db} = 0, \quad \frac{d\beta}{da} = 0, \quad \frac{d\beta}{db} = -\frac{1}{\rho_2},$$
$$\frac{d^2\alpha}{da^2} = \frac{d^3V}{dx^3}, \quad \frac{d^2\alpha}{da\,.\,db} = \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2\alpha}{db^2} = \frac{d^3V}{dx\,.\,dy^2};$$
$$\frac{d^2\beta}{da^2} = \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2\beta}{da\,.\,db} = \frac{d^3V}{dx\,.\,dy^2}, \quad \frac{d^2\beta}{db^2} = \frac{d^3V}{dy^3}.$$

And if we substitute these expressions for $\alpha_{\prime}$, $\beta_{\prime}$, in the two series for x' and y', we shall get two other series of the form

$$\left. \begin{aligned} x' &= \frac{dx'}{da} \,.\, a + \frac{dx'}{db} \,.\, b + \tfrac{1}{2} \,.\, \left\{ \frac{d^2x'}{da^2} \,.\, a^2 + 2\frac{d^2x'}{da\,.\,db} \,.\, ab + \frac{d^2x'}{db^2} \,.\, b^2 \right\} + \&\text{c.}, \\ y' &= \frac{dy'}{da} \,.\, a + \frac{dy'}{db} \,.\, b + \tfrac{1}{2} \,.\, \left\{ \frac{d^2y'}{da^2} \,.\, a^2 + 2\frac{d^2y'}{da\,.\,db} \,.\, ab + \frac{d^2y'}{db^2} \,.\, b^2 \right\} + \&\text{c.}, \end{aligned} \right\} \quad (\text{S}''')$$

in which, at present,

$$\frac{dx'}{da} = -\frac{i}{\rho_1}, \quad \frac{dx'}{db} = 0, \quad \frac{dy'}{da} = 0, \quad \frac{dy'}{db} = 0,$$
$$\frac{d^2x'}{da^2} = \rho_2 \cdot \frac{d^3V}{dx^3}, \quad \frac{d^2x'}{da\,.\,db} = \rho_2 \cdot \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2x'}{db^2} = \rho_2 \cdot \frac{d^3V}{dx\,.\,dy^2},$$
$$\frac{d^2y'}{da^2} = \rho_2 \cdot \frac{d^3V}{dx^2\,.\,dy}, \quad \frac{d^2y'}{da\,.\,db} = \rho_2 \cdot \frac{d^3V}{dx\,.\,dy^2}, \quad \frac{d^2y'}{db^2} = \rho_2 \cdot \frac{d^3V}{dy^3},$$

and in which the other coefficients can also be calculated by means of the characteristic function.

This being laid down, let us put $x' = r\,.\cos. v$, $y' = r\,.\sin. v$, and let us develope (a) and (b) according to the powers of (r). The developments will be of the form

$$\left.\begin{aligned} a &= r^{m}\,.\,u + r^{m'}\,.\,u' + r^{m''}\,.\,u'' + \ldots, \\ b &= r^{n}\,.\,w + r^{n'}\,.\,w' + r^{n''}\,.\,w'' + \ldots, \end{aligned}\right\} \tag{T$'''$}$$

$m, m', m'' \ldots n, n', n'' \ldots$ being positive and increasing exponents, which may or may not be fractional, and $u, u', u'' \ldots w, w', w'' \ldots$ being functions of the angle (v): which functions, as well as the exponents of the terms that multiply them, we have now to determine. Substituting therefore the values (T$'''$) in the series (S$'''$) we find the following equations:

$$\begin{aligned} \text{1st.}\ldots\ldots\ 0 &= -r\,.\cos. v + \frac{dx'}{da}\,.\,(r^{m}\,.\,u + \ldots) + \tfrac{1}{2}\,.\,\frac{d^2x'}{da^2}\,.\,(r^{m}\,.\,u + \ldots)^2 \\ &\quad + \frac{d^2x'}{da\,.\,db}\,.\,(r^{m}\,.\,u + \ldots)(r^{n}\,.\,w + \ldots) + \tfrac{1}{2}\,.\,\frac{d^2x'}{db^2}\,.\,(r^{n}\,.\,w + \ldots)^2 + \&\text{c.}, \\ \text{2d.}\ldots\ldots\ 0 &= -r\,.\sin. v + \tfrac{1}{2}\,.\,\frac{d^2y'}{da^2}\,.\,(r^{m}\,.\,u + \ldots)^2 + \frac{d^2y'}{da\,.\,db}\,.\,(r^{m}\,.\,u + \ldots)(r^{n}\,.\,w + \ldots) \\ &\quad + \tfrac{1}{2}\,.\,\frac{d^2y'}{db^2}\,.\,(r^{n}\,.\,w + \ldots)^2 + \&\text{c.} \end{aligned}$$

In order that these two equations should be identically satisfied, we must have, in the first place, for the exponents of the lowest powers of r in the developments (T$'''$)

$$m = 1, \quad n = \tfrac{1}{2};$$

and for the corresponding coefficients, (u, w),

$$\begin{aligned} (1)\ldots\ 0 &= -\cos. v + \frac{dx'}{da}\,.\,u + \tfrac{1}{2}\,.\,\frac{d^2x'}{db^2}\,.\,w^2, \\ (2)\ldots\ 0 &= -\sin. v + \tfrac{1}{2}\,.\,\frac{d^2y'}{db^2}\,.\,w^2, \end{aligned}$$

that is, in the notation of the preceding paragraph,

$$\left.\begin{aligned} 2\rho_1\rho_2\,.\cos. v + 2i\rho_2\,.\,u &= B\,.\,w^2, \\ 2\rho_2^2\,.\sin. v &= C\,.\,w^2. \end{aligned}\right\} \tag{U$'''$}$$

In a similar manner we find for the next greater exponents $m' = \frac{3}{2}$, $n' = 1$; and for the corresponding coefficients u', w',

$$\begin{aligned} (1)'\ldots\ 0 &= \frac{dx'}{da}\,.\,u' + \frac{d^2x'}{da\,.\,db}\,.\,uw + \frac{d^2x'}{db^2}\,.\,ww' + \tfrac{1}{6}\,.\,\frac{d^3x'}{db^3}\,.\,w^3, \\ (2)'\ldots\ 0 &= \frac{d^2y'}{da\,.\,db}\,.\,u + \frac{d^2y'}{db^2}\,.\,w' + \tfrac{1}{6}\,.\,\frac{d^3y'}{db^3}\,.\,w^2. \end{aligned}$$

And so proceeding, we can find as many of the exponents and coefficients of the developments (T‴), as may be necessary; the exponents forming the two following series,

$$m = 1, \quad m' = \tfrac{3}{2}, \quad m'' = 2, \quad \ldots \quad m^{(t)} = \frac{t+2}{2},$$

$$n = \tfrac{1}{2}, \quad n' = 1, \quad n'' = \tfrac{3}{2} \quad \ldots \quad n^{(t)} = \frac{t+1}{2};$$

and the coefficients being successively determined by equations of the following form,

$$\left.\begin{aligned} (1)^{(t)} \ldots 0 &= \frac{dx'}{da} \,.\, u^{(t)} + \frac{d^2x'}{db^2} \,.\, w \,.\, w^{(t)} + k_1^{(t)}, \\ (2)^{(t)} \ldots 0 &= \frac{d^2y'}{db^2} \,.\, w \,.\, w^{(t)} + k_2^{(t)}, \end{aligned}\right\} \qquad (\mathrm{V}''')$$

$k_1^{(t)}.r^{\frac{t+2}{2}}$, $k_2^{(t)}.r^{\frac{t+2}{2}}$, representing for abridgment the sums of the known terms of the dimension $\left(\dfrac{t+2}{2}\right)$ in the expansions of x' and y', according to the powers of r, obtained by substituting in (S‴) the assumed developments (T‴) in place of a and b. The quantities $k_1^{(t)}$, $k_2^{(t)}$, are therefore rational functions of the preceding coefficients $u, u', \ldots u^{(t-1)}$, $w, w', \ldots w^{(t-1)}$, and therefore finally of u and w; and these functions do, or do not, change sign along with w, according as t is an odd or an even number. Hence it follows, that the developments (T‴), which represent the coordinates of the points, where the near rays passing through any assigned point upon the plane of aberration are intersected by the perpendicular plane at the mirror, are of the form

$$\begin{aligned} a &= r \,.\, (u + r \,.\, u'' + r^2 \,.\, u'''' + \ldots) \pm r^{\frac{3}{2}} \,.\, (u' + r \,.\, u''' + \ldots), \\ b &= \pm r^{\frac{1}{2}} \,.\, (w + r \,.\, w'' + r^2 \,.\, w'''' + \ldots) + r \,.\, (w' + r \,.\, w''' + \ldots), \end{aligned}$$

the coefficients of the fractional powers being real or imaginary, according as w is real or imaginary, that is, by (U‴), according as $\left(\dfrac{\sin. v}{C}\right)$ is positive or negative; or finally, by [61.], according as the assigned point (r, v) upon the plane of aberration, is, or is not, situated at that side of the tangent plane of the caustic surface, towards which is turned the convexity of the caustic curve. However, when the polar angle (v) approaches to (0) or $(180°)$, that is when the right line joining the focus of the given ray to the assigned point upon the plane of aberration, tends to become a tangent to the caustic surface, the numeric value of $(\sin. v)$, and therefore of (w), diminishes indefinitely; and consequently the coefficients which contain negative powers of that quantity, increase without limit, so that the series (T‴) become at length illusory. In this case, therefore, it becomes necessary to have recourse to new developments, which will be indicated in the succeeding paragraph. But abstracting for the present from this case, which, in examining the variation of the density of the reflected light upon the plane of aberration, may usually be avoided by a proper choice of the focus from which the aberrations are to be measured: it may easily be shewn, by reasonings similar to those of the preceding paragraph, that if we consider any infinitely small polar rectangle upon the plane of aberration, having its base $= r \,.\, dv$, and its altitude $= dr$; the rays which pass inside this little rectangle, are, at the mirror, diffused nearly perpendicularly over two little parallelograms, whose areas are

$$\left.\begin{aligned} \Sigma_1^{(\mu)} &= \left(\frac{da}{dv} \,.\, \frac{db}{dr} - \frac{da}{dr} \,.\, \frac{db}{dv}\right) .\, dr \,.\, dv, \\ \Sigma_2^{(\mu)} &= \left(\frac{da'}{dr} \,.\, \frac{db'}{dv} - \frac{da'}{dv} \,.\, \frac{db'}{dr}\right) .\, dr \,.\, dv, \end{aligned}\right\} \tag{W$'''$}$$

a, b, a', b', being the coordinates of the two points, determined by the series (T$'''$). Substituting for these coordinates their values, we find that the two areas (W$'''$) are the two values of the following expression,

$$\begin{aligned} \Sigma^{(\mu)} = {} & \tfrac{1}{2} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv \,.\, \left\{\frac{du}{dv} + r \,.\, \frac{du''}{dv} + r^2 \,.\, \frac{du''''}{dv} + \ldots \pm r^{\frac{1}{2}} \,.\, \left(\frac{du'}{dv} + r \,.\, \frac{du'''}{dv} + \ldots\right)\right\} \\ & \times \{w + 3 \,.\, r \,.\, w'' + 5 \,.\, r^2 \,.\, w'''' + \ldots \pm 2\, r^{\frac{1}{2}} \,.(w' + 2 \,.\, r \,.\, w''' + \ldots)\} \\ & - \tfrac{1}{2} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv \,.\, \left\{\frac{dw}{dv} + r \,.\, \frac{dw''}{dv} + r^2 \,.\, \frac{dw''''}{dv} + \ldots \pm r^{\frac{1}{2}} \,.\, \left(\frac{dw'}{dv} + r \,.\, \frac{dw'''}{dv} + \ldots\right)\right\} \\ & \times \{2 \,.\, (u + 2 \,.\, r \,.\, u'' + 3 \,.\, r^2 \,.\, u'''' + \ldots) \pm r^{\frac{1}{2}} \,.\, (3 \,.\, u' + 5 \,.\, r \,.\, u''' + \ldots)\} \\ = {} & \tfrac{1}{2} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv \,.\, \left\{\left(\frac{du}{dv} + r \,.\, \frac{du''}{dv} + \ldots\right)(w + 3 \,.\, r \,.\, w'' + \ldots)\right. \\ & \left. - 2 \,.\, \left(\frac{dw}{dv} + r \,.\, \frac{dw''}{dv} + \ldots\right)(u + 2 \,.\, r \,.\, u'' + \ldots)\right\} \\ & + \tfrac{1}{2} \,.\, r^{\frac{3}{2}} \,.\, dr \,.\, dv \,.\, \left\{2 \,.\, \left(\frac{du'}{dv} + r \,.\, \frac{du'''}{dv} + \ldots\right)(w' + 2 \,.\, r \,.\, w''' + \ldots)\right. \\ & \left. - \left(\frac{dw'}{dv} + r \,.\, \frac{dw'''}{dv} + \ldots\right)(3 \,.\, u' + 5 \,.\, r \,.\, u''' + \ldots)\right\} \\ & \pm \tfrac{1}{2} \,.\, r \,.\, dr \,.\, dv \,.\, \left\{\left(\frac{du'}{dv} + r \,.\, \frac{du'''}{dv} + \ldots\right)(w + 3 \,.\, r \,.\, w'' + \ldots)\right. \\ & \left. - \left(\frac{dw}{dv} + r \,.\, \frac{dw''}{dv} + \ldots\right)(3 \,.\, u' + 5 \,.\, r \,.\, u''' + \ldots)\right\} \\ & \pm r \,.\, dr \,.\, dv \,.\, \left\{\left(\frac{du}{dv} + r \,.\, \frac{du''}{dv} + \ldots\right)(w' + 2 \,.\, r \,.\, w''' + \ldots)\right. \\ & \left. - \left(\frac{dw'}{dv} + r \,.\, \frac{dw'''}{dv} + \ldots\right)(u + 2 \,.\, r \,.\, u'' + \ldots)\right\}, \end{aligned}$$

which is of the form

$$\begin{aligned} \Sigma^{(\mu)} = {} & \tfrac{1}{2} \,.\, r^{\frac{1}{2}} \,.\, (U^{(0)} + U^{(1)} \,.\, r + U^{(2)} \,.\, r^2 + \ldots) \,.\, dr \,.\, dv \\ & \pm \tfrac{1}{2} \,.\, r \,.\, (U_{\prime}^{(0)} + U_{\prime}^{(1)} \,.\, r + U_{\prime}^{(2)} \,.\, r^2 + \ldots) \,.\, dr \,.\, dv, \end{aligned} \tag{X$'''$}$$

the coefficients $U^{(0)}, U^{(1)}, \dots\ U_{\prime}^{(0)}, U_{\prime}^{(1)}, \dots$ being functions of the polar angles (v). The densities of the reflected light, at these little parallelograms (W‴), have for developments

$$\left.\begin{aligned} \Delta_1 &= \Delta^{(\mu)} + \frac{d\,.\,\Delta^{(\mu)}}{da}\,.\,a + \frac{d\,.\,\Delta^{(\mu)}}{db}\,.\,b + \dots, \\ \Delta_2 &= \Delta^{(\mu)} + \frac{d\,.\,\Delta^{(\mu)}}{da}\,.\,a' + \frac{d\,.\,\Delta^{(\mu)}}{db}\,.\,b' + \dots, \end{aligned}\right\} \tag{Y‴}$$

$\Delta^{(\mu)}$ being, as in the preceding paragraph, the density at the point $a = 0$, $b = 0$, that is, at the point where the given ray meets the mirror: and substituting in these developments (Y‴), the values of a, b, a', b', given by the series (T‴), we find that the two densities Δ_1, Δ_2, are the two values of the following expression:

$$\begin{aligned} \Delta = \Delta^{(\mu)} &+ \frac{d\,.\,\Delta^{(\mu)}}{da}\{r\,.\,(u + r\,.\,u'' + \dots) \pm r^{\frac{3}{2}}\,.\,(u' + r\,.\,u''' + \dots)\} \\ &+ \frac{d\,.\,\Delta^{(\mu)}}{db}\{\pm r^{\frac{1}{2}}\,.\,(w + r\,.\,w'' + \dots) + r\,.\,(w' + r\,.\,w''' + \dots)\} \\ &+ \&\text{c.}, \end{aligned}$$

which is of the form

$$\begin{aligned} \Delta = \Delta^{(\mu)} &+ \Delta_1^{(\mu)}\,.\,r + \Delta_2^{(\mu)}\,.\,r^2 + \dots \\ &\pm r^{\frac{1}{2}}\,.\,(\Delta_{(\frac{1}{2})}^{(\mu)} + \Delta_{(\frac{3}{2})}^{(\mu)}\,.\,r + \dots), \end{aligned} \tag{Z‴}$$

the coefficients being functions of the polar angle (v).

Similarly, if we denote by γ_1, γ_2, the cosines of the angles which the two near rays, passing through the two points (a, b), (a', b'), make with the axis of (z), that is, with the given ray, these cosines will have developments of the form

$$\begin{aligned} \gamma &= 1 + \tfrac{1}{2}\left\{\frac{d^2\gamma}{da^2}\,.\,a^2 + 2\frac{d^2\gamma}{da\,.\,db}\,.\,ab + \frac{d^2\gamma}{db^2}\,.\,b^2\right\} + \dots \\ &= 1 + \Gamma^{(1)}\,.\,r + \Gamma^{(2)}\,.\,r^2 \dots \\ &\quad \pm r^{\frac{1}{2}}\,.\,(\Gamma_{\prime}^{(1)}\,.\,r + \Gamma_{\prime}^{(2)}\,.\,r^2 + \dots), \end{aligned} \tag{A⁗}$$

the coefficients being also functions of the polar angle (v); and the whole number of the near reflected rays, which pass within the little rectangle $(r\,.\,dr\,.\,dv)$ upon the plane of aberration, being equal to the sum of the two values of the product $\gamma\,.\,\Delta\,.\,\Sigma^{(\mu)}$, will be expressed by a development of the form

$$Q^{(\alpha)} = r^{\frac{1}{2}}\,.\,dr\,.\,dv\,.\,(Q^{(0)} + Q^{(1)}\,.\,r + Q^{(2)}\,.\,r^2 + \dots), \tag{B⁗}$$

where $Q^{(0)} = \Delta^{(\mu)}\,.\,U^{(0)}$, and the other coefficients $Q^{(1)}, Q^{(2)}, \dots$ are other functions of the polar angle (v), which may be determined by the formulæ (X‴), (Z‴), (A⁗). Confining ourselves to the first term of this development, and dividing by $r\,.\,dr\,.\,dv$, that is by the area of the little polar rectangle upon the plane of aberration, we find the following approximate expression for the density of the reflected light at the point (r, v) upon this latter plane,

$$\Delta^{(\alpha)} = \frac{Q^{(\alpha)}}{r \,.\, dr \,.\, dv} = Q^{(0)} \,.\, r^{-\frac{1}{2}}; \tag{C$''''$}$$

which nearly agrees with the formula (R$'''$) of the preceding paragraph, because, as we have seen,

$$Q^{(0)} = \Delta^{(\mu)} \,.\, U^{(0)} = \Delta^{(\mu)} \,.\, \left(w \,.\, \frac{du}{dv} - 2 \,.\, u \,.\, \frac{dw}{dv} \right) = \frac{\Delta^{(\mu)} \,.\, \rho_1 \,.\, \rho_2}{i \,.\, \sqrt{(\frac{1}{2}C \,.\, \sin. v)}},$$

and therefore

$$Q^{(0)} \,.\, r^{-\frac{1}{2}} = \frac{\Delta^{(\mu)} \,.\, \rho_1 \,.\, \rho_2}{i \,.\, \sqrt{\frac{1}{2}C \,.\, y'}}.$$

More accurately, the density $\Delta^{(\mu)}$ being equal to the sum of the two quotients obtained by dividing the quantity of light corresponding to each of the little parallelograms (W$'''$), by the space over which that quantity is perpendicularly diffused at the point (r, v), has for expression

$$\begin{aligned} \Delta^{(\alpha)} &= \frac{\gamma_1 \,.\, \Delta_1 \,.\, \Sigma_1^{(\mu)}}{\gamma_1 \,.\, r \,.\, dr \,.\, dv} + \frac{\gamma_2 \,.\, \Delta_2 \,.\, \Sigma_2^{(\mu)}}{\gamma_2 \,.\, r \,.\, dr \,.\, dv} = \frac{\Delta_1 \,.\, \Sigma_1^{(\mu)} + \Delta_2 \,.\, \Sigma_2^{(\mu)}}{r \,.\, dr \,.\, dv} \\ &= r^{-\frac{1}{2}} \,.\, (U^{(0)} + U^{(1)} \,.\, r + \ldots)(\Delta^{(\mu)} + \Delta_1^{(\mu)} \,.\, r + \ldots) \\ &\quad + r^{\frac{1}{2}} \,.\, (U_{\prime}^{(0)} + U_{\prime}^{(1)} \,.\, r + \ldots)(\Delta^{(\mu)}_{(\frac{1}{2})} + \Delta^{(\mu)}_{(\frac{3}{2})} \,.\, r + \ldots). \end{aligned} \tag{D$''''$}$$

The first term of this development being the same as the approximate expression (C$''''$), and therefore agreeing nearly with the formula (R$'''$) of [67.], we see, by this method, as well as by the less accurate one of the 67th paragraph, that the density upon the plane of aberration varies nearly inversely as the square root of the perpendicular distance from the caustic surface: a conclusion which might also be deduced from the general theorem [43.], that along a given ray the density varies inversely as the product of the distances from its two foci. But the present method has the advantage of enabling us to take into account as many of the remaining terms of the density as may be necessary, by means of the formula (D$''''$); it gives also, by integration of the formulæ (B$''''$) and (X$'''$), the whole number of the near reflected rays which pass within any small assigned space $\iint r \,.\, dr \,.\, dv$, upon the plane of aberration, and the whole corresponding space on the perpendicular plane at the mirror; since this latter space is expressed by the sum of the following integrals:

$$S^{(\mu)} = \iint (\Sigma_1^{(\mu)} + \Sigma_2^{(\mu)}) = \iint U^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv + \iint U^{(1)} \,.\, r^{\frac{3}{2}} \,.\, dr \,.\, dv + \&\text{c.}, \tag{E$''''$}$$

and the corresponding quantity of light is expressed by this other sum,

$$Q^{(s)} = \iint Q^{(\alpha)} = \iint Q^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv + \iint Q^{(1)} \,.\, r^{\frac{3}{2}} \,.\, dr \,.\, dv + \&\text{c.}, \tag{F$''''$}$$

the integrals in these developments being taken within the same limits as the given integral $\iint r \,.\, dr \,.\, dv$, which represents the assigned space upon the plane of aberration, and the extreme values of (v) being supposed such as not to render the series (T‴) illusory. These series (T‴) serve also to correct the approximate expression of the preceding paragraph, for the first term of (a); which first term was there taken as

$$a = -\frac{\rho_1 \,.\, x'}{i}, \tag{Q‴ [67.]}$$

whereas by employing the remaining terms in the development of x' and y', we have now found it to be

$$a = u \,.\, r = \left(\frac{B\rho_2 \,.\, \sin. v - C\rho_1 \,.\, \cos. v}{C \,.\, i} \right) r = \frac{B\rho_2 \,.\, y' - C\rho_1 \,.\, x'}{C \,.\, i},$$

a value which differs from the preceding, by the addition of $\left(\frac{B \,.\, \rho_2 \; y'}{C \,.\, i} \right)$. And if, by means of this corrected value, and by using as many of the remaining terms of (T‴), as the question may render necessary, we eliminate (r) and (v) from the polar equation of any given curve upon the plane of aberration; for example, from that of the boundary of the space $\iint r.dr.dv$, for which we have already determined the corresponding quantity of light, and the area over which that quantity is diffused on the perpendicular plane at the mirror; we shall find the approximate equation of the boundary of this latter area, and thus resolve a new and extensive class of questions respecting thin pencils, for which the formulae of Section IX. and those of the 60th paragraph would be either inadequate or inconvenient.

As an example of application of the reasonings of the present paragraph, let us conceive a small circular sector, upon the plane of aberration, having its centre at the focus of the given ray, and having its radius (r) so small, that we may confine ourselves, in each development, to the lowest powers of that radius. Let (ψ) denote the semiangle of this sector, and let (v'') be the polar angle which the bisecting radius makes with the axis of (x'); then $v'' - \psi$, $v'' + \psi$, will be the extreme values of the polar coordinate (v), while the corresponding limits of the radius vector will be (0) and (r). Denoting by $(S^{(c)})$ the whole space occupied on the perpendicular plane at the mirror, by the rays which pass within the given little circular sector, and by $(Q^{(c)})$ the number of these near reflected rays; the formulæ (E⁗) (F⁗) give, for these quantities

$$S^{(c)} = \iint U^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv = \tfrac{2}{3} \,.\, r^{\frac{3}{2}} \,.\, \iint U^{(0)} \,.\, dv,$$
$$Q^{(c)} = \iint Q^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv = \tfrac{2}{3} \,.\, r^{\frac{3}{2}} \,.\, \iint Q^{(0)} \,.\, dv,$$

or, substituting for $U^{(0)}$, $Q^{(0)}$ their values,

$$\left.\begin{aligned}
S^{(c)} &= \frac{2\,.\,\rho_1\,.\,\rho_2\,.\,r^{\frac{3}{2}}}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C}}\,.\int\frac{dv}{\sqrt{\sin.\,v}},\\
Q^{(c)} &= \frac{2\,.\,\rho_1\,.\,\rho_2\,.\,\Delta^{(\mu)}\,.\,r^{\frac{3}{2}}}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C}}\,.\int\frac{dv}{\sqrt{\sin.\,v}},
\end{aligned}\right\}\qquad(\mathrm{G}'''')$$

the integral in each expression being taken from $v = v'' - \psi$, to $v = v'' + \psi$; so that we have the relation

$$Q^{(c)} = \Delta^{(\mu)}\,.\,S^{(c)}.\qquad(\mathrm{H}'''')$$

If the semiangle of the sector be so small that we may neglect its cube and higher powers, the definite integral $U^{(0)}\,.\,dv$, being the difference of the developments

$$\begin{aligned}
&U^{(0)}\,.\,\psi + \frac{dU^{(0)}}{dv''}\,.\,\frac{\psi^2}{2} + \frac{d^2U^{(0)}}{dv''^2}\,.\,\frac{\psi^3}{2\,.\,3} + \dots,\\
-&U^{(0)}\,.\,\psi + \frac{dU^{(0)}}{dv''}\,.\,\frac{\psi^2}{2} - \frac{d^2U^{(0)}}{dv''^2}\,.\,\frac{\psi^3}{2\,.\,3} + \dots,
\end{aligned}$$

is nearly equal to $2\,.\,U^{(0)}\,.\,\psi$; and the quantities $S^{(c)}$, $Q^{(c)}$, may be thus expressed,

$$\left.\begin{aligned}
S^{(c)} &= \frac{4\,.\,\rho_1\,.\,\rho_2}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C}}\,.\,\frac{r^{\frac{3}{2}}\psi}{\sqrt{\sin.\,v''}} = \frac{4\,.\,\rho_1\,.\,\rho_2\,.\,s''}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C\,.\,y''}},\\
Q^{(c)} &= \frac{4\,.\,\rho_1\,.\,\rho_2\,.\,\Delta^{(\mu)}}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C}}\,.\,\frac{r^{\frac{3}{2}}\psi}{\sqrt{\sin.\,v''}} = \frac{4\,.\,\rho_1\,.\,\rho_2\,.\,\Delta^{(\mu)}\,.\,s''}{3\,.\,i\,.\,\sqrt{\frac{1}{2}C\,.\,y''}},
\end{aligned}\right\}\qquad(\mathrm{I}'''')$$

s'' being the area of the little circular sector, and y'' being the projection of its bisecting radius upon the normal to the caustic surface; so that if the sector were to receive a rotation in its own plane round its own centre, that is, in the plane of aberration round the focus of the given ray, the area at the mirror ($S^{(c)}$) and the quantity of light ($Q^{(c)}$) would vary nearly inversely as the square root of the cosine of the angle, which the bisecting radius of the sector made with the normal to the caustic surface. If, on the contrary, without supposing the angles (v'') or (ψ) to vary, we alter the length of the radius, or transport the centre of the sector to any other point on either of the two caustic surfaces, so as to produce another sector, similar and similarly situated; it follows from (G'''') that the quantities $S^{(c)}$ and $Q^{(c)}$ will vary as the following expressions,

$$\rho_1\,.\,\rho_2\,.\,r^{\frac{3}{2}}\,.\,i^{-1}\,.\,C^{-\frac{1}{2}},\quad \rho_1\,.\,\rho_2\,.\,\Delta^{(\mu)}\,.\,r^{\frac{3}{2}}\,.\,i^{-1}\,.\,C^{-\frac{1}{2}};$$

so that if the centre of the sector be fixed, they vary as the sesquiplicate power of its radius; and if the radius be given, but not the centre, then they vary, the one as the product of the two focal lengths of the mirror, divided by the difference of those two focal lengths and by the square root of the radius of curvature of the caustic curve; and the other, as this latter quotient, multiplied by the density of the reflected light at the corresponding point of the mirror. These latter theorems, being founded on the formulæ (G′′′′), do not require that we should neglect any of the powers of ψ, that is of the semiangle of the circular sector; they may even be extended, by means of the equations (E′′′′) (F′′′′), to the case of similar and similarly situated sectors, bounded by lines of any other form. If, for instance, we suppose any small isosceles triangle, having its height $= h$, and its base $= 2 \,.\, h \,.\, \tan.\, \theta$, to move in such a manner that its summit is always situated on one of the two caustic surfaces, while the ray passing through that point is perpendicular to its plane, and the bisector of its vertical angle is perpendicular to the caustic surface; and if we put $Q^{(i)}$, $S^{(i)}$, to denote, respectively, the number of the near reflected rays that pass inside this little triangle, and the space over which those rays are diffused, on the perpendicular plane at the mirror; we shall have the approximate equations,

$$Q^{(i)} = \iint Q^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv = \frac{\rho_1 \,.\, \rho_2 \,.\, \Delta^{(\mu)}}{i \,.\, \sqrt{\frac{1}{2}C}} \,.\, \iint \frac{r^{\frac{1}{2}} \,.\, dr \,.\, dv}{\sqrt{\sin.\, v}},$$

$$S^{(i)} = \iint U^{(0)} \,.\, r^{\frac{1}{2}} \,.\, dr \,.\, dv = \frac{\rho_1 \,.\, \rho_2}{i \,.\, \sqrt{\frac{1}{2}C}} \,.\, \iint \frac{r^{\frac{1}{2}} \,.\, dr \,.\, dv}{\sqrt{\sin.\, v}},$$

in which the integrals are to be taken from $r = 0$ to $r = \dfrac{h}{\sin.\, v}$, and from $v = \frac{1}{2} \,.\, \pi - \theta$, to $v = \frac{1}{2} \,.\, \pi + \theta$. Performing these integrations, we find

$$\left.\begin{aligned} Q^{(i)} &= \frac{4 \,.\, \rho_1 \,.\, \rho_2 \,.\, \Delta^{(\mu)} \,.\, h^{\frac{3}{2}}}{3 \,.\, i \,.\, \sqrt{\frac{1}{2}C}} \,.\, \tan.\, \theta = \frac{4 \,.\, \rho_1 \,.\, \rho_2 \,.\, \Delta^{(\mu)} \,.\, s^{(i)}}{3 \,.\, i \,.\, \sqrt{\frac{1}{2}C \,.\, h}}, \\ S^{(i)} &= \frac{4 \,.\, \rho_1 \,.\, \rho_2 \,.\, h^{\frac{3}{2}}}{3 \,.\, i \,.\, \sqrt{\frac{1}{2}C}} \,.\, \tan.\, \theta = \frac{4 \,.\, \rho_1 \,.\, \rho_2 \,.\, s^{(i)}}{3 \,.\, i \,.\, \sqrt{\frac{1}{2}C \,.\, h}}, \end{aligned}\right\} \qquad (\mathrm{K}'''')$$

$s^{(i)}$ being the area of the little isosceles triangle; expressions analogous to those which we found before, for the case of the circular sector, and leading to similar results.

Returning to the case of the sector, we have yet to determine the boundary of the space ($S^{(c)}$) on the perpendicular plane at the mirror. For this purpose, we are to eliminate (r) and (v) by means of the following expressions, (T′′′),

$$a = u \,.\, r = (B\rho_2 \,.\, \sin.\, v - C\rho_1 \,.\, \cos.\, v) \,.\, i^{-1} \,.\, C^{-1} \,.\, r,$$
$$b = \pm w \,.\, r^{\frac{1}{2}} = \pm \rho_2 \,.\, \sqrt{2} \,.\, C^{-\frac{1}{2}} \,.\, \sqrt{r \,.\, \sin.\, v},$$

from the polar equations of the boundaries of the sector, namely

Ist. $\quad v = v'' - \psi = v_1,$

IId. $\quad v = v'' + \psi = v_2,$

IIId. $\quad r = r,$

of which the two first represent the bounding radii, and the third the circular arc. Putting for abridgment,

$$\rho_2 \,.\, \sqrt{2} \,.\, C^{-\frac{1}{2}} = \sqrt{\epsilon}, \quad B\rho_2 \,.\, i^{-1} \,.\, C^{-1} = \epsilon \,.\, P^{-1}, \quad C\rho_1 = B\rho_2 \,.\, \tan.\, v',$$

conditions which give $\epsilon = 2\rho_2^2 \,.\, C^{-1}$, $P = 2i \,.\, \rho_2 \,.\, B^{-1}$; and supposing, for simplicity, that $\sqrt{\epsilon}$ is real, and that $v' < \dfrac{\pi}{2}$, that is, supposing C and $\tan.\, v'$ positive, a condition which may always be satisfied by a proper direction of the positive portions of the axes of y' and x'; our expressions for a, b, become

$$a = \epsilon \,.\, P^{-1} \,.\, r \,.\, \sec.\, v' \,.\, \sin.(v - v'), \quad b = \pm\sqrt{(\epsilon \,.\, r \,.\, \sin.\, v)}, \tag{L''''}$$

and we find the following equations for the boundary of the space $S^{(c)}$,

$$\left.\begin{array}{ll} \text{1st.} & P \,.\, a = b^2 \,.\, \sec.\, v' \,.\, \text{cosec.}\, v_1 \,.\, \sin.(v_1 - v'), \\ \text{2d.} & P \,.\, a = b^2 \,.\, \sec.\, v' \,.\, \text{cosec.}\, v_2 \,.\, \sin.(v_2 - v'), \\ \text{3d.} & P \,.\, a = b^2 \mp \tan.\, v' \,.\, \sqrt{(\epsilon^2 \,.\, r^2 - b^4)}. \end{array}\right\} \tag{M''''}$$

The two first of these equations represent parabolic arcs, having their common vertex at the origin of (a) and (b), that is at the point where the given ray meets the mirror, and having their common axis coincident with the axis of (x) or of (a), and therefore parallel to the tangent of the curve in which the caustic surface is cut by the plane of aberration. It is, then, in the points of these little parabolic arcs, that the rays which pass through the bounding radii of the little circular sector, are intersected by the perpendicular plane at the mirror; and from the manner in which their parameters depend on the inclination of those bounding radii to the tangent of the caustic surface, it is evident that any intermediate radius of the sector has an intermediate parabola corresponding. The ends of these little parabolic arcs are contained on two equal and opposite portions of a curve of the fourth degree represented by the third of the three equations (M''''), it is then in these two opposite portions of this third curve, that the rays which pass through the bounding arc of the sector are crossed by the perpendicular plane at the mirror. With respect to the form of this third curve, considered in its whole extent, it is easy to see that it is in general shaped like a *heart*, being bisected, first by the axis of (a), which we may call the *diameter* of the curve, and secondly by a parabola

$$b^2 = P \,.\, a, \tag{N''''}$$

which we may call its *diametral parabola*, and bounded by the four following tangents,

$$\text{1st. } b = +\sqrt{\epsilon \,.\, r}; \quad \text{2d. } b = -\sqrt{\epsilon \,.\, r};$$

$$\text{3d. } a = -\epsilon \,.\, r \,.\, P^{-1} \,.\, \tan.\, v'; \quad \text{4th. } a = \epsilon \,.\, r \,.\, P^{-1} \,.\, \sec.\, v'; \tag{O''''}$$

of which the two first are parallel to the diameter, and the two last perpendicular thereto. We may remark that the diametral parabola, (N''''), corresponds to the rays that pass through the axis of y', that is, through the normal to the caustic surface; and that the two points where it meets the curve, are the points of contact corresponding to the two first of the four tangents (O''''). The point of contact corresponding to the third of these tangents, is situated at what may be called the negative end of the diameter; and the fourth touches the curve in two distinct points, whose common distance from the diameter is $b = \pm\sqrt{(\epsilon \,.\, r \,.\, \cos. v')}$, and which may be called the two *summits* of the heart. The curve has also another tangent parallel to the axis of (b), which touches it at the point

$$a = +\epsilon \,.\, r \,.\, P^{-1} \,.\, \tan. v', \quad b = 0, \tag{P''''}$$

that is, at the positive end of the diameter; and which crosses the curve in two other points, equally distant from the diameter, and having for coordinates,

$$a = \epsilon \,.\, r \,.\, P^{-1} \,.\, \tan. v', \quad b = \pm\sqrt{(\epsilon \,.\, r \,.\, \sin. 2v')},$$

And the whole area of this heartlike curve is equal to the following definite integral,

$$\Pi = 4P^{-1} \,.\, \tan. v' \,.\, \int \sqrt{(\epsilon^2 \,.\, r^2 - b^4)} \,.\, db, \tag{Q''''}$$

the integral being taken from $b = 0$, to $b = \sqrt{\epsilon \,.\, r}$. In the next paragraph we shall return to this definite integral, and shew its optical value.

[69.] But the preceding calculations only shew how the density varies *near* the caustic surface; to find the law of the variation *at* that surface, we must reason in a different manner. For if the infinitely small rectangle on the plane of aberration, which we have considered in the preceding paragraph, have one of its corners on the caustic surface, we can no longer consider the density as uniform, even in the infinitely small extent of that rectangle. But if we consider the rays that pass within a given infinitely small distance (dr) from a given point upon the caustic surface, for example, from the focus of the given ray, we can find the space over which these rays are diffused upon the perpendicular plane at the mirror; and this space, multiplied by the density at the mirror, may be taken for the measure of the density at the given focus, not as compared with the density at the mirror, but with the density at other points upon the caustic surface.

To calculate this measure, let us consider the following more general question, to find the whole number ($Q^{(r)}$) of the near reflected rays which pass within any small but finite distance (r) from the focus of the given ray, and the space ($S^{(r)}$) over which these rays are diffused, on the perpendicular plane at the mirror. This question evidently comes to supposing the little circular sector ($r^2 \,.\, \psi$) of the preceding paragraph completed into an entire circle, and consequently may be solved by integrating the formulæ (E'''') (F'''') of that paragraph, within the double limits afforded by the equation of the circle on the one hand, and by that of the section of the caustic surface on the other; since it is easy to see that only a part of the little circular area ($\pi \,.\, r^2$) is illumined, namely, the part which lies at that side of the caustic surface, towards which is turned the convexity of the caustic curve.

But as the formulæ of the preceding paragraph were founded on the developments (T''') which, as we have before remarked, become illusory when the polar radius (r) approaches to become a tangent to the caustic surface, (a position of that radius which we are not now at liberty to neglect,) it becomes necessary to investigate other developments, and to transform the double integrals (E'''') (F'''') of [68.] into others better suited for the question that we are now upon. And to effect this the more clearly, it seems convenient to consider separately the four following problems: 1st, to find general expressions for the coefficients $u^{(t)}$, $w^{(t)}$, which enter into the developments (T'''), and to examine what negative powers they contain of the sine of the polar angle (v); 2d., to eliminate these negative powers, and so transform the two series (T''') into others which shall contain none but ascending powers of any variable quantity; 3d., to effect corresponding transformations on the integral formulæ (E'''') (F'''') of the preceding paragraph; and 4th, to perform the double integrations within the limits of the question.

In this manner we shall obtain developments proceeding according to the ascending powers of the little circular radius (r), to represent the optical quantities which we have denoted by $Q^{(r)}$, $S^{(r)}$; it will then remain to suppose (r) infinitely small, and the resulting expressions $Q^{(dr)}$, $S^{(dr)}$, which must evidently satisfy the relation

$$Q^{(dr)} = \Delta^{(\mu)} \,.\, S^{(dr)}, \tag{R''''}$$

$\Delta^{(\mu)}$ being the density at the mirror, will each serve to measure the density at the caustic surface in the sense that we have already explained.

(I.) First then, with respect to the coefficients $u^{(t)}$, $w^{(t)}$, of the series (T'''),

$$a = ur + u'r^{\frac{3}{2}} + \dots u^{(t)} \,.\, r^{\frac{t+2}{2}} + \dots,$$
$$b = wr^{\frac{1}{2}} + w'r + \dots w^{(t)} \,.\, r^{\frac{t+1}{2}} + \dots;$$

it is evident that if we differentiate these series with respect to $\sqrt{r}$, we shall have, supposing $\sqrt{r}$ to vanish after the differentiations,

$$\frac{da}{d\sqrt{r}} = 0, \quad \frac{d^2a}{d\sqrt{r^2}} = 2 \,.\, u, \quad \frac{d^3a}{d\sqrt{r^3}} = 2 \,.\, 3 \,.\, u', \dots,$$

$$\frac{db}{d\sqrt{r}} = w, \quad \frac{d^2b}{d\sqrt{r^2}} = 2 \,.\, w', \quad \frac{d^3b}{d\sqrt{r^3}} = 2 \,.\, 3 \,.\, w'', \dots,$$

and in general

$$\frac{d^s a}{d\sqrt{r^s}} = [s]^s \,.\, u^{(s-2)}, \quad \frac{d^s b}{d\sqrt{r^s}} = [s]^s \,.\, w^{(s-1)} : \tag{S''''}$$

$[s]^s$ expressing, according to the notation of Vandermonde, the factorial quantity $1 \,.\, 2 \,.\, 3 \,.\dots.\, (s-1) \,.\, s$. If then we differentiate, with respect to $\sqrt{r}$, the equations

$$r \,.\, \cos.\, v = x', \quad r \,.\, \sin.\, v = y',$$

considering x', y', as functions of a and b, and these as functions of $\sqrt{r}$; the resulting equations,

$$\frac{d^s \,.\, r \,.\, \cos.\, v}{d\sqrt{r^s}} = \frac{d^s \,.\, x'}{d\sqrt{r^s}}, \quad \frac{d^s \,.\, r \,.\, \sin.\, v}{d\sqrt{r^s}} = \frac{d^s \,.\, y'}{d\sqrt{r^s}}, \tag{T''''}$$

wll serve, with the help of the formulæ (S''''), to determine successively the coefficients u', w', u'', w'', ... $u^{(t)}$, $w^{(t)}$, as functions of those which precede them; observing that the partial differentials $\dfrac{dx'}{da}$, $\dfrac{dx'}{db}$, $\dfrac{d^2x'}{da^2}$, ... $\dfrac{dy'}{da}$, $\dfrac{dy'}{db}$, $\dfrac{d^2y'}{da^2}$, ... are the coefficients of the series (S''') of [68.], and are to be deduced from the characteristic function of the system in the manner there described. To develope these equations (T''''), we have, for the first members,

$$\frac{d\,.\,r\,.\cos.v}{d\sqrt{r}} = 2\sqrt{r}\,.\cos.v, \qquad \frac{d\,.\,r\,.\sin.v}{d\sqrt{r}} = 2\sqrt{r}\,.\sin.v;$$
$$\frac{d^2\,.\,r\,.\cos.v}{d\sqrt{r^2}} = 2\,.\cos.v, \qquad \frac{d^2\,.\,r\,.\sin.v}{d\sqrt{r^2}} = 2\,.\sin.v;$$
$$\frac{d^3\,.\,r\,.\cos.v}{d\sqrt{r^3}} = 0, \qquad \frac{d^3\,.\,r\,.\sin.v}{d\sqrt{r^3}} = 0:$$

and in general, when $s > 2$,

$$\frac{d^s\,.\,r\,.\cos.v}{d\sqrt{r^s}} = 0, \quad \frac{d^s\,.\,r\,.\sin.v}{d\sqrt{r^s}} = 0:$$

and for the second members,

$$\left.\begin{aligned} \frac{d^s\,.\,x'}{d\sqrt{r^s}} &= [s]^s\,.\,\Sigma\,.\frac{d^{\Sigma\alpha+\Sigma\beta}x'}{da^{\Sigma\alpha}\;db^{\Sigma\beta}}\,.\,\lambda^{(s)},\\ \frac{d^s\,.\,y'}{d\sqrt{r^s}} &= [s]^s\,.\,\Sigma\,.\frac{d^{\Sigma\alpha+\Sigma\beta}y'}{da^{\Sigma\alpha}\;db^{\Sigma\beta}}\,.\,\lambda^{(s)}, \end{aligned}\right\} \tag{U''''}$$

if we put for abridgment,

$$\begin{aligned} \lambda^{(s)} = {} & \left(\frac{da}{d\sqrt{r}}\right)^{\alpha_1}.\left(\frac{d^2a}{d\sqrt{r^2}}\right)^{\alpha_2}\cdots\left(\frac{d^sa}{d\sqrt{r^s}}\right)^{\alpha_s}\times\left(\frac{db}{d\sqrt{r}}\right)^{\beta_1}.\left(\frac{d^2b}{d\sqrt{r^2}}\right)^{\beta_2}\cdots\left(\frac{d^sb}{d\sqrt{r^s}}\right)^{\beta_s}\\ & \times([0]^{-1})^{\alpha_1}\,.\,([0]^{-2})^{\alpha_2}\ldots([0]^{-s})^{\alpha_s}\times([0]^{-1})^{\beta_1}\,.\,([0]^{-2})^{\beta_2}\ldots([0]^{-s})^{\beta_s}\\ & \times[0]^{-\alpha_1}\,.\,[0]^{-\alpha_2}\ldots[0]^{-\alpha_s}\times[0]^{-\beta_1}\,.\,[0]^{-\beta_2}\ldots[0]^{-\beta_s}; \end{aligned} \tag{V''''}$$

$\alpha_1, \alpha_2, \ldots \alpha_s$, $\beta_1, \beta_2, \ldots \beta_s$, being any positive integers which satisfy the following relation,

$$\begin{aligned} s = {} & \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots s\,.\,\alpha_s\\ & + \beta_1 + 2\beta_2 + 3\beta_3 + \ldots s\,.\,\beta_s; \end{aligned} \tag{W''''}$$

and Σ being the symbol of a sum, so that

$$\Sigma\,\alpha = \alpha_1 + \alpha_2 + \ldots \alpha_s, \quad \Sigma\,\beta = \beta_1 + \beta_2 + \ldots \beta_s.$$

Developing in this manner the equations (T''''), and observing by our present choice of the coordinate planes, we have, [68.],

$$\frac{dx'}{db} = 0; \quad \frac{dy'}{da} = 0; \quad \frac{dy'}{db} = 0; \quad \frac{da}{d\sqrt{r}} = 0:$$

we arrive again at the same equations which in the preceding paragraph we deduced by substituting in (S'''), for the components a, b, of aberration at the mirror, their assumed developments (T'''), and by then comparing the corresponding powers of r, that is, of the aberration at the caustic surface. Thus, if we make $s = 1$, the equations (T'''') become identical; if $s = 2$, they become

$$\begin{aligned}
2\cos. v = {} & \frac{dx'}{da} \cdot \frac{d^2 a}{d\sqrt{r}^2} + \frac{dx'}{db} \cdot \frac{d^2 b}{d\sqrt{r}^2} \\
& + \frac{d^2 x'}{da^2} \cdot \left(\frac{da}{d\sqrt{r}}\right)^2 + 2 \, . \, \frac{d^2 x'}{da \, . \, db} \cdot \frac{da}{d\sqrt{r}} \cdot \frac{db}{d\sqrt{r}} + \frac{d^2 x'}{db^2} \cdot \left(\frac{db}{d\sqrt{r}}\right)^2, \\
2\sin. v = {} & \frac{dy'}{da} \cdot \frac{d^2 a}{d\sqrt{r}^2} + \frac{dy'}{db} \cdot \frac{d^2 b}{d\sqrt{r}^2} \\
& + \frac{d^2 y'}{da^2} \cdot \left(\frac{da}{d\sqrt{r}}\right)^2 + 2 \, . \, \frac{d^2 y'}{da \, . \, db} \cdot \frac{da}{d\sqrt{r}} \cdot \frac{db}{d\sqrt{r}} + \frac{d^2 y'}{db^2} \cdot \left(\frac{db}{d\sqrt{r}}\right)^2,
\end{aligned}$$

that is,

$$\begin{aligned}
2\cos. v &= \frac{dx'}{da} \, . \, 2u + \frac{d^2 x'}{db^2} w^2, \\
2\sin. v &= \qquad\qquad\quad \frac{d^2 y'}{db^2} w^2,
\end{aligned}$$

as in the formulæ (1), (2), or (U''') of [68.]; if $s = 3$, they become after reductions,

$$\begin{aligned}
0 &= \frac{dx'}{da} \cdot \frac{d^3 a}{d\sqrt{r}^3} + 3 \, . \left(\frac{d^2 x'}{da \, . \, db} \cdot \frac{d^2 a}{d\sqrt{r}^2} + \frac{d^2 x'}{db^2} \cdot \frac{d^2 b}{d\sqrt{r}^2} \right) \cdot \frac{db}{d\sqrt{r}} + \frac{d^3 x'}{db^3} \cdot \left(\frac{db}{d\sqrt{r}}\right)^3, \\
0 &= \qquad\qquad\quad\;\; 3 \, . \left(\frac{d^2 y'}{da \, . \, db} \cdot \frac{d^2 a}{d\sqrt{r}^2} + \frac{d^2 y'}{db^2} \cdot \frac{d^2 b}{d\sqrt{r}^2} \right) \cdot \frac{db}{d\sqrt{r}} + \frac{d^3 y'}{db^3} \cdot \left(\frac{db}{d\sqrt{r}}\right)^3,
\end{aligned}$$

that is,

$$\begin{aligned}
0 &= 6 \, . \, \frac{dx'}{da} \, . \, u' + 3 \, . \left(\frac{d^2 x'}{da \, . \, db} \, . \, 2u + \frac{d^2 x'}{db^2} \, . \, 2w' \right) . \, w + \frac{d^3 x'}{db^3} \, . \, w^3, \\
0 &= \qquad\qquad\quad\; 3 \, . \left(\frac{d^2 y'}{da \, . \, db} \, . \, 2u + \frac{d^2 y'}{db^2} \, . \, 2w' \right) . \, w + \frac{d^3 y'}{db^3} \, . \, w^3,
\end{aligned}$$

as in the formulæ (1)', (2)' of the same paragraph; $s = 4$ gives

$$
\begin{aligned}
0 = {} & \frac{dx'}{da} \cdot \frac{d^4 a}{d\sqrt{r^4}} + 4 \,.\left(\frac{d^2 x'}{da \,.\, db} \cdot \frac{d^3 a}{d\sqrt{r^3}} + \frac{d^2 x'}{db^2} \cdot \frac{d^3 b}{d\sqrt{r^3}} \right) \cdot \frac{db}{d\sqrt{r}} \\
& + 3 \,.\left(\frac{d^2 x'}{da^2} \cdot \left(\frac{d^2 a}{d\sqrt{r^2}} \right)^2 + 2 \,.\, \frac{d^2 x'}{da \,.\, db} \cdot \frac{d^2 a}{d\sqrt{r^2}} \cdot \frac{d^2 b}{d\sqrt{r^2}} + \frac{d^2 x'}{db^2} \cdot \left(\frac{d^2 b}{d\sqrt{r^2}} \right)^2 \right) \\
& + 6 \,.\left(\frac{d^3 x'}{da \,.\, db^2} \cdot \frac{d^2 a}{d\sqrt{r^2}} + \frac{d^3 x'}{db^3} \cdot \frac{d^2 b}{d\sqrt{r^2}} \right) \cdot \left(\frac{db}{d\sqrt{r}} \right)^2 + \frac{d^4 x'}{db^4} \cdot \left(\frac{db}{d\sqrt{r}} \right)^4, \\
0 = {} & 4 \,.\left(\frac{d^2 y'}{da \,.\, db} \cdot \frac{d^3 a}{d\sqrt{r^3}} + \frac{d^2 y'}{db^2} \cdot \frac{d^3 b}{d\sqrt{r^3}} \right) \cdot \frac{db}{d\sqrt{r}} \\
& + 3 \,.\left(\frac{d^2 y'}{da^2} \cdot \left(\frac{d^2 a}{d\sqrt{r^2}} \right)^2 + 2 \,.\, \frac{d^2 y'}{da \,.\, db} \cdot \frac{d^2 a}{d\sqrt{r^2}} \cdot \frac{d^2 b}{d\sqrt{r^2}} + \frac{d^2 y'}{db^2} \cdot \left(\frac{d^2 b}{d\sqrt{r^2}} \right)^2 \right) \\
& + 6 \,.\left(\frac{d^3 y'}{da \,.\, db^2} \cdot \frac{d^2 a}{d\sqrt{r^2}} + \frac{d^3 y'}{db^3} \cdot \frac{d^2 b}{d\sqrt{r^2}} \right) \cdot \left(\frac{db}{d\sqrt{r}} \right)^2 + \frac{d^4 y'}{db^4} \cdot \left(\frac{db}{d\sqrt{r}} \right)^4,
\end{aligned}
$$

that is,

$$
\begin{aligned}
(1)'' \ldots \quad 0 = {} & 24 \,.\, \frac{dx'}{da} \,.\, u'' + 24 \,.\left(\frac{d^2 x'}{da \,.\, db} u' + \frac{d^2 x'}{db^2} \,.\, w'' \right) .\, w \\
& + 12 \,.\left(\frac{d^2 x'}{da^2} \,.\, u^2 + 2 \,.\, \frac{d^2 x'}{da \,.\, db} \,.\, u \,.\, w' + \frac{d^2 x'}{db^2} \,.\, w'^2 \right) \\
& + 12 \,.\left(\frac{d^3 x'}{da \,.\, db^2} \,.\, u + \frac{d^3 x'}{db^3} \,.\, w' \right) .\, w^2 + \frac{d^4 x'}{db^4} \,.\, w^4, \\
(2)'' \ldots \quad 0 = {} & 24 \,.\left(\frac{d^2 y'}{da \,.\, db} u' + \frac{d^2 y'}{db^2} \,.\, w'' \right) .\, w \\
& + 12 \,.\left(\frac{d^2 y'}{da^2} \,.\, u^2 + 2 \,.\, \frac{d^2 y'}{da \,.\, db} \,.\, u \,.\, w' + \frac{d^2 y'}{db^2} \,.\, w'^2 \right) \\
& + 12 \,.\left(\frac{d^3 y'}{da \,.\, db^2} \,.\, u + \frac{d^3 y'}{db^3} \,.\, w' \right) .\, w^2 + \frac{d^4 y'}{db^4} \,.\, w^4;
\end{aligned}
$$

and generally, if we make the differential index $s = t + 2$, and divide by the factorial quantity $[t+2]^{t+2}$, we shall reproduce the equations (V‴) [68.] under the form

$$
\begin{aligned}
(1)^{(t)} \ldots \quad 0 &= \frac{dx'}{da} \,.\, u^{(t)} + \frac{d^2 x'}{db^2} \,.\, w \,.\, w^{(t)} + \Sigma \,.\, \frac{d^{\Sigma\alpha + \Sigma\beta} x'}{da^{\Sigma\alpha} db^{\Sigma\beta}} \,.\, \mu^{(t)}, \\
(2)^{(t)} \ldots \quad 0 &= \phantom{\frac{dx'}{da} \,.\, u^{(t)} +{}} \frac{d^2 y'}{db^2} \,.\, w \,.\, w^{(t)} + \Sigma \,.\, \frac{d^{\Sigma\alpha + \Sigma\beta} y'}{da^{\Sigma\alpha} db^{\Sigma\beta}} \,.\, \mu^{(t)},
\end{aligned}
$$

where

$$
\mu^{(t)} = \frac{(u)^{\alpha_2}}{[\alpha_2]^{\alpha_2}} \cdot \frac{(u')^{\alpha_3}}{[\alpha_3]^{\alpha_3}} \cdots \frac{(u^{(t-1)})^{\alpha_{t+1}}}{[\alpha_{t+1}]^{\alpha_{t+1}}} \times \frac{(w)^{\beta_1}}{[\beta_1]^{\beta_1}} \cdot \frac{(w')^{\beta_2}}{[\beta_2]^{\beta_2}} \cdots \frac{(w^{(t-1)})^{\beta_t}}{[\beta_t]^{\beta_t}},
$$

$$
\Sigma\,\alpha = \alpha_2 + \alpha_3 + \ldots \alpha_{t+1}, \quad \Sigma\,\beta = \beta_1 + \beta_2 + \ldots \beta_t,
$$

and $\alpha_2, \alpha_3 \ldots \beta_1, \beta_2 \ldots$ are any positive integers which satisfy the following relation:

$$t + 2 = 2\alpha_2 + 3\alpha_3 + \ldots (t+1)\alpha_{t+1} + \beta_1 + 2\beta_2 + \ldots t\beta_t. \tag{X''''}$$

It is easy to see, that if by these equations we calculate successively the coefficients $u^{(t)}$, $w^{(t)}$, as functions of u and w, and if we eliminate u by the assumed relation

$$u = z \,.\, w^2, \tag{Y''''}$$

z being a new variable; the resulting expressions will be of the form

$$\left.\begin{aligned} u' &= u_1 \,.\, w^3, \quad & u'' &= u_2 \,.\, w^4, \ldots \quad & u^{(t)} &= u_t \,.\, w^{t+2}, \\ w' &= w_1 \,.\, w^2, \quad & w'' &= w_2 \,.\, w^3, \ldots \quad & w^{(t)} &= w_t \,.\, w^{t+1}, \end{aligned}\right\} \tag{Z''''}$$

u_t, w_t, being rational and integer functions of z, not exceeding the t^{th} dimension; so that we may put

$$\left.\begin{aligned} u_t &= u_{t,0} + u_{t,1} \,.\, z + u_{t,2} \,.\, z^2 + \ldots u_{t,t'} \,.\, z^{t'} + \ldots u_{t,t} \,.\, z^t, \\ w_t &= w_{t,0} + w_{t,1} \,.\, z + w_{t,2} \,.\, z^2 + \ldots w_{t,t'} \,.\, z^{t'} + \ldots w_{t,t} \,.\, z^t, \end{aligned}\right\} \tag{A$^{(5)}$}$$

$u_{t,0}, u_{t,1}, \ldots w_{t,0}, w_{t,1}, \ldots$ being constant quantities, not containing the polar angle v, and depending only on the position of the given ray, and on the nature of the reflected system. In order therefore to complete our determination of the polar functions $u^{(t)}$, $w^{(t)}$, it is sufficient to calculate general expressions for the constants $u_{t,t'}$, $w_{t,t'}$, considered as functions of the indices t, t', and of the partial differentials $\dfrac{dx'}{da}$, $\dfrac{dx'}{db}$, $\dfrac{d^2x'}{da^2}, \ldots \dfrac{dy'}{da} \ldots$; since these differentials may, as we have before remarked, be deduced from the differentials of the characteristic function of the system.

To calculate these constants, the method which first presents itself, is to substitute in the equations $(1)^{(t)}$, $(2)^{(t)}$, in place of u', u'', w', w'', ... their values (Z''''), (A$^{(5)}$), and to compare the corresponding powers of z. Thus if we confine ourselves to the constants $u_{t,t}$, $w_{t,t}$, which multiply the highest powers of z, as the most important in our present investigations, because when w diminishes z increases without limit; we are to retain only those values of $\mu^{(t)}$ which give terms multiplied by z^t, and it is easy to see that these terms are distinguished by the relation

$$2 + \alpha_2 = 2\,\Sigma\,\alpha + \Sigma\,\beta; \tag{B$^{(5)}$}$$

putting them, then, under the form $\mu_{t,t} \,.\, w^{t+2} \,.\, z^t$, we have when $t = 1$,

$$\Sigma \,.\frac{d^{\Sigma\alpha+\Sigma\beta}x'}{da^{\Sigma\alpha}db^{\Sigma\beta}} \,.\, \mu_{1,1} = \frac{d^2x'}{da \,.\, db}; \quad \Sigma \,.\frac{d^{\Sigma\alpha+\Sigma\beta}y'}{da^{\Sigma\alpha}db^{\Sigma\beta}} \,.\, \mu_{1,1} = \frac{d^2y'}{da \,.\, db};$$

when $t = 2$,

$$\Sigma \,.\frac{d^{\Sigma\alpha+\Sigma\beta}x'}{da^{\Sigma\alpha}db^{\Sigma\beta}} \,.\, \mu_{2,2} = \tfrac{1}{2}\frac{d^2x'}{da^2} + \frac{d^2x'}{da \,.\, db} \,.\, w_{1,1} + \tfrac{1}{2} \,.\, \frac{d^2x'}{db^2} \,.\, w_{1,1}^2,$$
$$\Sigma \,.\frac{d^{\Sigma\alpha+\Sigma\beta}y'}{da^{\Sigma\alpha}db^{\Sigma\beta}} \,.\, \mu_{2,2} = \tfrac{1}{2}\frac{d^2y'}{da^2} + \frac{d^2y'}{da \,.\, db} \,.\, w_{1,1} + \tfrac{1}{2} \,.\, \frac{d^2y'}{db^2} \,.\, w_{1,1}^2;$$

and, when $t > 2$,

$$\Sigma\,.\frac{d^{\Sigma\alpha+\Sigma\beta}x'}{da^{\Sigma\alpha}db^{\Sigma\beta}}\,.\,\mu_{t,t} = \frac{d^2x'}{da\,.\,db}\,.\,w_{t-1,t-1} + \tfrac{1}{2}\,.\,\frac{d^2x'}{db^2}\,.\,\Sigma\,.w_{s,s}\,.\,w_{t-s,t-s},$$
$$\Sigma\,.\frac{d^{\Sigma\alpha+\Sigma\beta}y'}{da^{\Sigma\alpha}db^{\Sigma\beta}}\,.\,\mu_{t,t} = \frac{d^2y'}{da\,.\,db}\,.\,w_{t-1,t-1} + \tfrac{1}{2}\,.\,\frac{d^2y'}{db^2}\,.\,\Sigma\,.w_{s,s}\,.\,w_{t-s,t-s},$$

the sums in the second members being taken from $s = 1$, to $s = t-1$: and since the equations $(1)^{(t)}$, $(2)^{(t)}$, give, by comparison of the highest powers of z,

$$0 = \frac{dx'}{da}\,.\,u_{t,t} + \frac{d^2x'}{db^2}\,.\,w_{t,t} + \frac{d^{\Sigma\alpha+\Sigma\beta}x'}{da^{\Sigma\alpha}db^{\Sigma\beta}}\,.\,\mu_{t,t},$$
$$0 = \frac{d^2y'}{db^2}\,.\,w_{t,t} + \frac{d^{\Sigma\alpha+\Sigma\beta}y'}{da^{\Sigma\alpha}db^{\Sigma\beta}}\,.\,\mu_{t,t},$$

we have successively

$$(1)_{(1,1)}\cdots\quad 0 = \frac{dx'}{da}\,.\,u_{1,1} + \frac{d^2x'}{db^2}\,.\,w_{1,1} + \frac{d^2x'}{da\,.\,db},$$
$$(2)_{(1,1)}\cdots\quad 0 = \frac{d^2y'}{db^2}\,.\,w_{1,1} + \frac{d^2y'}{da\,.\,db},$$
$$(1)_{(2,2)}\cdots\quad 0 = \frac{dx'}{da}\,.\,u_{2,2} + \frac{d^2x'}{db^2}\,.\,w_{2,2} + \tfrac{1}{2}\frac{d^2x'}{da^2} + \frac{d^2x'}{da\,.\,db}\,.\,w_{1,1} + \tfrac{1}{2}\frac{d^2x'}{db^2}\,.\,w^2_{1,1},$$
$$(2)_{(2,2)}\cdots\quad 0 = \frac{d^2y'}{db^2}\,.\,w_{2,2} + \tfrac{1}{2}\frac{d^2y'}{da^2} + \frac{d^2y'}{da\,.\,db}\,.\,w_{1,1} + \tfrac{1}{2}\frac{d^2y'}{db^2}\,.\,w^2_{1,1},$$
$$(1)_{(3,3)}\cdots\quad 0 = \frac{dx'}{da}\,.\,u_{3,3} + \frac{d^2x'}{db^2}\,.\,w_{3,3} + \frac{d^2x'}{da\,.\,db}\,.\,w_{2,2} + \frac{d^2x'}{db^2}\,.\,w_{1,1}\,.\,w_{2,2},$$
$$(2)_{(3,3)}\cdots\quad 0 = \frac{d^2y'}{db^2}\,.\,w_{3,3} + \frac{d^2y'}{da\,.\,db}\,.\,w_{2,2} + \frac{d^2y'}{db^2}\,.\,w_{1,1}\,.\,w_{2,2},$$

the two last of which equations reduce themselves, by means of the two first, to the following form:

$$u_{3,3} = u_{1,1}\,.\,w_{2,2}, \qquad w_{3,3} = 0;$$

a similar reduction gives in general, when $t > 3$,

$$(1)_{(t,t)}\cdots\quad u_{t,t} = u_{1,1}\,.\,w_{t-1,t-1},$$
$$(2)_{(t,t)}\cdots\quad 0 = w_{t,t} + \tfrac{1}{2}\,.\,\Sigma\,.w_{s,s}\,.\,w_{t-s,t-s},$$

the sum being taken from $s = 2$, to $s = t-2$; so that the four first constants $u_{1,1}$, $w_{1,1}$, $u_{2,2}$, $w_{2,2}$, being determined by the four equations $(1)_{1,1}$, $(2)_{1,1}$, $(1)_{2,2}$, $(2)_{2,2}$, all the succeeding constants of the same kind, $u_{3,3}$, $u_{4,4}, \ldots$ $w_{3,3}$, $w_{4,4}, \ldots$ are given by the following general expressions, which may be deduced from the formulæ $(1)_{t,t}$, $(2)_{t,t}$, either by successive elimination, or by the calculus of finite differences;

$$\left.\begin{aligned} w_{2\tau,2\tau} &= 2^\tau\,.\,[\tfrac{1}{2}]^\tau\,.\,[0]^{-\tau}\,.\,(w_{2,2})^\tau; & w_{2\tau+1,2\tau+1} &= 0;\\ u_{2\tau+1,2\tau+1} &= u_{1,1}\,.\,w_{2\tau,2\tau}; & u_{2\tau+2,2\tau+2} &= 0;\end{aligned}\right\} \qquad (\mathrm{C}^{(5)})$$

τ being any integer number > 0, and $[\frac{1}{2}]^{\tau}$, $[0]^{-\tau}$, being known factorial symbols. In a similar manner we might calculate general expressions for the other constants of the form $u_{t,t'}$, $w_{t,t'}$; but it seems preferable to employ the following method, founded on the properties of partial differentials, and on the development of functions into series.

To make use of these properties, I observe that if we put

$$r \,.\, w^2 = \theta^2, \quad a = \zeta \,.\, \theta^2, \quad b = \eta \,.\, \theta, \tag{D$^{(5)}$}$$

and substitute for $u^{(t)}$, $w^{(t)}$ their expressions (Z''''), (A$^{(5)}$), the series (T''') will take the form

$$\left.\begin{aligned} \zeta &= z + (u_{1,0} + u_{1,1} \,.\, z) \,.\, \theta + (u_{2,0} + u_{2,1} \,.\, z + u_{2,2} \,.\, z^2) \,.\, \theta^2 + \ldots + u_{t,t'} \,.\, z^{t'} \,.\, \theta^t + \&\text{c.}, \\ \eta &= 1 + (w_{1,0} + w_{1,1} \,.\, z) \,.\, \theta + (w_{2,0} + w_{2,1} \,.\, z + w_{2,2} \,.\, z^2) \,.\, \theta^2 + \ldots + w_{t,t'} \,.\, z^{t'} \,.\, \theta^t + \&\text{c.}, \end{aligned}\right\} \tag{E$^{(5)}$}$$

equations which give by differentiation

$$\begin{aligned} \frac{d^{t+t'}\zeta}{d\theta^t \,.\, dz^{t'}} &= [t]^t \,.\, [t']^{t'} \,.\, u_{t,t'} + [t+1]^t \,.\, [t']^{t'} \,.\, u_{t+1,t'} \,.\, \theta + [t]^t \,.\, [t'+1]^{t'} \,.\, u_{t,t'+1} \,.\, z + \&\text{c.}, \\ \frac{d^{t+t'}\eta}{d\theta^t \,.\, dz^{t'}} &= [t]^t \,.\, [t']^{t'} \,.\, w_{t,t'} + [t+1]^t \,.\, [t']^{t'} \,.\, w_{t+1,t'} \,.\, \theta + [t]^t \,.\, [t'+1]^{t'} \,.\, w_{t,t'+1} \,.\, z + \&\text{c.}, \end{aligned}$$

and therefore, when $\theta = 0$, $z = 0$,

$$u_{t,t'} = [0]^{-t} \,.\, [0]^{-t'} \,.\, \frac{d^{t+t'}\zeta}{d\theta^t \,.\, dz^{t'}}; \quad w_{t,t'} = [0]^{-t} \,.\, [0]^{-t'} \,.\, \frac{d^{t+t'}\eta}{d\theta^t \,.\, dz^{t'}}; \tag{F$^{(5)}$}$$

in order therefore to obtain general expressions for the constants $u_{t,t'}$, $w_{t,t'}$ it is sufficient to calculate expressions for these partial differentials of ζ, η. Now, if from the two equations (S'''), [68.], we subtract the two others

$$x' = \frac{dx'}{da} \,.\, ur + \tfrac{1}{2} \,.\, \frac{d^2x'}{db^2} \,.\, w^2r, \quad y' = \tfrac{1}{2} \,.\, \frac{d^2y'}{db^2} \,.\, w^2r,$$

which result from the formulæ (U''') of the same paragraph; if we then eliminate $b^2 - w^2r$, and put for abridgment,

$$\tfrac{1}{2} \,.\, \frac{d^2y'}{da^2} \,.\, a^2 + \frac{d^2y'}{da \,.\, db} \,.\, ab + \tfrac{1}{6} \,.\, \frac{d^3y'}{da^3} \,.\, a^3 + \&\text{c.} = -\frac{d^2y'}{db^2} \,.\, \Phi(a,b),$$

$$\frac{d^2x'}{db^2} \,.\, \Phi(a,b) + \tfrac{1}{2} \,.\, \frac{d^2x'}{da^2} \,.\, a^2 + \frac{d^2x'}{da \,.\, db} \,.\, ab + \tfrac{1}{6} \,.\, \frac{d^3x'}{da^3} \,.\, a^3 + \&\text{c.} = -\frac{dx'}{da} \,.\, F(a,b);$$

we shall have the following two equations,

$$a = ur + F, \quad b^2 = w^2r + 2\Phi; \tag{G$^{(5)}$}$$

which, when we put

$$u = zw^2, \quad w^2r = \theta^2, \quad a = \zeta\theta^2, \quad b = \eta\theta,$$

become

$$\zeta = z + f(\zeta, \eta, \theta), \quad \eta^2 = 1 + 2\phi(\zeta, \eta, \theta); \tag{H$^{(5)}$}$$

f, ϕ, being functions such that

$$F = \theta^2 \,.\, f, \quad \Phi = \theta^2 \,.\, \phi, \tag{I$^{(5)}$}$$

and therefore

$$\left.\begin{aligned} 0 &= \frac{d^2 y'}{db^2} \,.\, \phi + \Sigma \,.[0]^{-m} \,.\, [0]^{-m'} \,.\, \frac{d^{m+m'} y'}{da^m \,.\, db^{m'}} \,.\, \zeta^m \,.\, \eta^{m'} \,.\, \theta^{2m+m'-2}, \\ -\frac{dx'}{da} \,.\, f &= \frac{d^2 x'}{db^2} \,.\, \phi + \Sigma \,.[0]^{-m} \,.\, [0]^{-m'} \,.\, \frac{d^{m+m'} x'}{da^m \,.\, db^{m'}} \,.\, \zeta^m \,.\, \eta^{m'} \,.\, \theta^{2m+m'-2}, \end{aligned}\right\} \tag{K$^{(5)}$}$$

m, m', being any positive integers which satisfy the following relation

$$2m + m' > 2. \tag{L$^{(5)}$}$$

If then we eliminate ζ, η, between the equations (H$^{(5)}$), so as to find expressions for those variables as functions of z and θ; it will remain to differentiate those expressions, (t) times for θ, and (t') times for z, and to put after the differentiations $\theta = 0$, $z = 0$; since the partial differentials thus obtained, multiplied by the factorial quantity $[0]^{-t} \,.\, [0]^{-t'}$, will give, by (F$^{(5)}$), the general expressions that we are in search of, for the constants $u_{t,t'}$, $w_{t,t'}$.

To perform this elimination, we may employ the theorems which Laplace has given, in the second book of the *Mécanique Céleste*, for the development of functions into series. Laplace has there shewn that if we have any number (r) of equations of the form

$$x = \phi(t + \alpha z), \quad x' = \psi(t' + \alpha' z'), \quad x'' = \Pi(t'' + \alpha'' z''), \text{ \&c.},$$

in which z, z' ,z'', &c. are functions of x, x', x'', &c. and if we develope any other function u of the same variables, according to the powers and products of α, α', α'', &c. in a series of which the general term is represented by $q_{n,n',n'',\ldots} \alpha^n \,.\, \alpha'^{n'} \,.\, \alpha''^{n''} \,.$ &c.; we shall have, to determine the coefficient $q_{n,n',n'',\ldots}$, a formula which may thus be written,

$$q_{n,n',n'',\ldots} = [0]^{-n} \,.\, [0]^{-n'} \,.\, [0]^{-n''} \ldots \frac{d^{n+n'+n''+\ldots-r}}{dt^{n-1} \,.\, dt'^{n'-1} \,.\, dt''^{n''-1} \ldots} \left(\frac{d^r u_{\prime}}{d\alpha \,.\, d\alpha' \,.\, d\alpha'' \ldots} \right),$$

$u_{\prime}$ being a function formed by changing in u the original variables x, x', x'', … into other variables determined by the following equations

$$x = \phi(t + \alpha z^n), \quad x' = \psi(t' + \alpha' z'^{n'}), \quad x'' = \Pi(t'' + \alpha'' z''^{n''}), \text{ \&c.},$$

the functions ϕ, ψ, Π, z, z', z'', &c. retaining the same forms as before, and α, α', α'', … being supposed to vanish after the differentiations. Laplace has also shewn, that when there are but two variables x, x', the partial differential $\left(\dfrac{d^r u_{\prime}}{d\alpha \,.\, d\alpha' \,.\, d\alpha'' \ldots}\right)$, determined in this manner, reduces itself to

$$\left(\frac{ddu_{\prime}}{d\alpha \,.\, d\alpha'}\right) = Z^n \,.\, Z'^{n'} \,.\, \left(\frac{dd\mathrm{u}}{dt \,.\, dt'}\right) + Z'^{n'} \,.\, \left(\frac{d \,.\, Z^n}{dt'}\right) \,.\, \left(\frac{d\mathrm{u}}{dt}\right) + Z^n \,.\, \left(\frac{d \,.\, Z'^{n'}}{dt}\right) \,.\, \left(\frac{d\mathrm{u}}{dt'}\right),$$

in which Z, Z', u represent the values that z, z', u, take, when we suppose $\alpha = 0$, $\alpha' = 0$. If then the original equations are of the form

$$x = t + \alpha z, \quad x' = \sqrt{(t' + \alpha' z')},$$

and if we change the function u to x and x' successively, we find the following developments for those two variables, according to the powers and products of α, α';

$$x = \Sigma [0]^{-n} \,.\, [0]^{-n'} \,.\, \alpha^n \alpha'^{n'} \,.\, \frac{d^{n+n'-2}}{dt^{n-1} \,.\, dt'^{n'-1}} \left(\frac{Z'^{n'}}{2\sqrt{t'}} \,.\, \frac{d \,.\, Z^n}{d\sqrt{t'}}\right),$$

$$x' = \Sigma [0]^{-n} \,.\, [0]^{-n'} \,.\, \alpha^n \alpha'^{n'} \,.\, \frac{d^{n+n'-2}}{dt^{n-1} \,.\, dt'^{n'-1}} \left(\frac{Z^n}{2\sqrt{t'}} \,.\, \frac{d \,.\, Z^n}{dt}\right),$$

in which Z, Z' are formed from z, z', by changing x to t, and x' to $\sqrt{t'}$. Applying these results to the equations ($\mathrm{H}^{(5)}$), which are of the form

$$\zeta = z + \alpha f, \quad \eta = \sqrt{(z' + \alpha' \phi)};$$

we find the following expressions for ζ, η, as functions of z and θ,

$$\left.\begin{aligned} \zeta &= \Sigma \,.[0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, \frac{d^{n+n'-2} \,.\, \zeta^{(n,n')}}{dz^{n-1} \,.\, dz'^{n'-1}}, \\ \eta &= \Sigma \,.[0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, \frac{d^{n+n'-2} \,.\, \eta^{(n,n')}}{dz^{n-1} \,.\, dz'^{n'-1}}, \end{aligned}\right\} \qquad (\mathrm{M}^{(5)})$$

in which

$$\zeta^{(n,n')} = \frac{\phi^{n'}}{2\sqrt{z'}} \,.\, \frac{d \,.\, f^n}{d\sqrt{z'}}, \quad \eta^{(n,n')} = \frac{f^n}{2\sqrt{z'}} \,.\, \frac{d \,.\, \phi^{n'}}{dz}, \qquad (\mathrm{N}^{(5)})$$

f, ϕ, being deduced from the formulæ ($\mathrm{K}^{(5)}$) by changing ζ to z, and η to $\sqrt{z'}$, and in which we may make after the differentiations $z' = 1$. And differentiating these developments ($\mathrm{M}^{(5)}$) in the manner already prescribed, we find, finally, the following general expressions for the constants $u_{t,t'}$, $w_{t,t'}$;

$$\left.\begin{aligned} u_{t,t'} &= [0]^{-t} \,.\, [0]^{-t'} \,.\, \Sigma \,.[0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, \frac{d^{t+t'+n+n'-2} \,.\, \zeta^{(n,n')}}{d\theta^t \,.\, dz^{t'+n-1} \,.\, dz'^{n'-1}}, \\ w_{t,t'} &= [0]^{-t} \,.\, [0]^{-t'} \,.\, \Sigma \,.[0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, \frac{d^{t+t'+n+n'-2} \,.\, \eta^{(n,n')}}{d\theta^t \,.\, dz^{t'+n-1} \,.\, dz'^{n'-1}}, \end{aligned}\right\} \qquad (\mathrm{O}^{(5)})$$

n, n', being any positive integers, and (θ, z, z') being changed after differentiations to $(0, 0, 1)$. It may be useful to observe, that by the formulae $(\mathrm{N}^{(5)})$, and by the nature of the developments, we are to make

$$\left.\begin{aligned}
&\frac{d^{-1}\,.\,\zeta^{(0,n')}}{dz^{-1}} = 0; \qquad \frac{d^{-1}\,.\,\eta^{(n,0)}}{dz'^{-1}} = 0;\\
&\frac{d^{-1}\,.\,\zeta^{(n,0)}}{dz'^{-1}} = f^n; \qquad \frac{d^{-1}\,.\,\eta^{(0,n')}}{dz^{-1}} = \frac{\phi^{n'}}{2\sqrt{z'}};\\
&\frac{d^{-2}\,.\,\zeta^{(0,0)}}{dz^{-1}\,.\,dz'^{-1}} = z; \qquad \frac{d^{-2}\,.\,\eta^{(0,0)}}{dz^{-1}\,.\,dz'^{-1}} = \sqrt{z'}.
\end{aligned}\right\} \tag{P$^{(5)}$}$$

These expressions $(\mathrm{O}^{(5)})$, may be put under other forms, some of which are more convenient for calculation. If, for abridgment, we write them thus

$$u_{t,t'} = \Sigma\,.u_{t,t'}^{(n,n')}, \quad w_{t,t'} = \Sigma\,.w_{t,t'}^{(n,n')}, \tag{Q$^{(5)}$}$$

$u_{t,t'}^{(n,n')}$, $w_{t,t'}^{(n,n')}$, denoting the terms of $u_{t,t'}$, $w_{t,t'}$, which correspond to any given values of the integers n, n'; we have, by $(\mathrm{O}^{(5)})$,

$$\left.\begin{aligned}
u_{t,t'}^{(n,n')} &= [0]^{-t}\,.\,[0]^{-t'}\,.\,[0]^{-n}\,.\,[0]^{-n'}\,.\,2^{n'}\frac{d^{t+t'+n+n'-2}\,.\,\zeta^{(n,n')}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}};\\
w_{t,t'}^{(n,n')} &= [0]^{-t}\,.\,[0]^{-t'}\,.\,[0]^{-n}\,.\,[0]^{-n'}\,.\,2^{n'}\frac{d^{t+t'+n+n'-2}\,.\,\eta^{(n,n')}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}};
\end{aligned}\right\} \tag{R$^{(5)}$}$$

and if in $(\mathrm{N}^{(5)})$ we change f, ϕ, to their values $(\mathrm{I}^{(5)})$,

$$\begin{aligned}
f &= \theta^{-2}\,.\,F(a,b) = \theta^{-2}\,.\,F(\theta^2 z, \theta\sqrt{z'}),\\
\phi &= \theta^{-2}\,.\,\Phi(a,b) = \theta^{-2}\,.\,\Phi(\theta^2 z, \theta\sqrt{z'}),
\end{aligned}$$

we find the following developments,

$$\zeta^{(n,n')} = \Sigma\,.\zeta_{m,m'}^{(n,n')}; \quad \eta^{(n,n')} = \Sigma\,.\eta_{m,m'}^{(n,n')}; \tag{S$^{(5)}$}$$

in which

$$\left.\begin{aligned}
\zeta_{m,m'}^{(n,n')} &= \tfrac{1}{2}\,.\,[0]^{-m}\,.\,[0]^{-m'}\,.\,\theta^{2m+m'+1-2(n+n')}\,.\,z^m\,.\,z'^{\frac{m'-1}{2}}\frac{d^{m+m'}\left(\Phi^{n'}\,.\,\dfrac{d\,.\,F^n}{db}\right)}{da^m\,.\,db^{m'}},\\
\eta_{m,m'}^{(n,n')} &= \tfrac{1}{2}\,.\,[0]^{-m}\,.\,[0]^{-m'}\,.\,\theta^{2m+m'+1-2(n+n')}\,.\,z^m\,.\,z'^{\frac{m'-1}{2}}\frac{d^{m+m'}\left(F^{n}\,.\,\dfrac{d\,.\,\Phi^{n'}}{da}\right)}{da^m\,.\,db^{m'}},
\end{aligned}\right\} \tag{T$^{(5)}$}$$

a, b, being supposed to vanish after the differentiations, and m, m', being any integer numbers: but the only values of these integers which do not make the partial differentials

$$\frac{d^{t+t'+n+n'-2}\zeta^{(n,n')}_{m,m'}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}}, \qquad \frac{d^{t+t'+n+n'-2}\eta^{(n,n')}_{m,m'}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}},$$

vanish when $\theta = 0$, $z = 0$, are those for which, in $\zeta^{(n,n')}_{m,m'}$,

$$t = 2m + m' + 1 - 2(n + n'), \quad t' + n - 1 = m,$$

and, in $\eta^{(n,n')}_{m,m'}$,

$$t = 2m + m' + 2 - 2(n + n'), \quad t' + n - 1 = m;$$

we have, therefore, when $\theta = 0$, $z = 0$,

$$\left.\begin{aligned} \frac{d^{t+t'+n+n'-2}\zeta^{(n,n')}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}} &= \frac{d^{t+t'+n+n'-2}\zeta^{(n,n')}_{t'+n-1,t+1+2n'-2t'}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}}, \\ \frac{d^{t+t'+n+n'-2}\eta^{(n,n')}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}} &= \frac{d^{t+t'+n+n'-2}\eta^{(n,n')}_{t'+n-1,t+2n'-2t'}}{d\theta^t\,dz^{t'+n-1}\,dz'^{n'-1}}, \end{aligned}\right\} \qquad (\mathrm{U}^{(5)})$$

in which the second members may be calculated by $(\mathrm{T}^{(5)})$. In this manner, the expressions $(\mathrm{R}^{(5)})$ become, after reductions,

$$\left.\begin{aligned} u^{(n,n')}_{t,t'} &= [0]^{-t'} \,.\, [0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, [0]^{-(t+2+2n'-2t')} \,.\, \left[\frac{t+2}{2} + n' - t'\right]^{n'} \\ &\quad\times \frac{d^{t-t'+n+2n'}\left(\Phi^{n'} \,.\, \dfrac{d\,.\,F^n}{db}\right)}{d^{t'+n-1}\,db^{t+1+2n'-2t'}}, \\ w^{(n,n')}_{t,t'} &= [0]^{-t'} \,.\, [0]^{-n} \,.\, [0]^{-n'} \,.\, 2^{n'} \,.\, [0]^{-(t+1+2n'-2t')} \,.\, \left[\frac{t+1}{2} + n' - t'\right]^{n'} \\ &\quad\times \frac{d^{t-t'+n+2n'-1}\left(F^n \,.\, \dfrac{d\,.\,\Phi^{n'}}{da}\right)}{d^{t'+n-1}\,db^{t+2n'-2t'}}; \end{aligned}\right\} \qquad (\mathrm{V}^{(5)})$$

and substituting these expressions in the developments $(\mathrm{Q}^{(5)})$, we get new developments for the constants $u_{t,t'}$, $w_{t,t'}$, in which we have only to differentiate for the two variables (a, b) instead of the three (θ, z, z').

Again, if we observe that by the nature of the functions F, Φ, we have when $a = 0$, $b = 0$,

$$\left.\begin{aligned} F = 0 \quad \frac{dF}{da} = 0, \quad \frac{dF}{db} = 0, \quad \frac{d^2F}{db^2} = 0, \\ \Phi = 0 \quad \frac{d\Phi}{da} = 0, \quad \frac{d\Phi}{db} = 0, \quad \frac{d^2\Phi}{db^2} = 0; \end{aligned}\right\} \qquad (\mathrm{W}^{(5)})$$

we shall easily deduce relations between the integer numbers t, t', n, n', which reduce the summation of these developments to the addition of a finite number of terms. For we may prove, either by these equations ($\mathrm{W}^{(5)}$), or by the condition ($\mathrm{L}^{(5)}$) which contains them, that the partial differentials

$$\frac{d^{m+m'}}{da^m\,db^{m'}}\left(\Phi^{n'}\,.\,\frac{d\,.\,F^n}{db}\right),\qquad \frac{d^{m+m'}}{da^m\,db^{m'}}\left(F^{n}\,.\,\frac{d\,.\,\Phi^{n'}}{da}\right),$$

which enter into the formulæ ($\mathrm{T}^{(5)}$), vanish, unless in the first

$$m+m'+2>2(n+n');\quad 2m+m'+2>3(n+n');$$

and, in the second,

$$m+m'+2>2(n+n');\quad 2m+m'+3>3(n+n').$$

Hence, by ($\mathrm{V}^{(5)}$), the partial differentials which enter into the expression for $u_{t,t'}^{(n,n')}$, $w_{t,t'}^{(n,n')}$, vanish unless in the first

$$n<t-t'+2;\quad n+n'<t+1;\tag{$\mathrm{X}^{(5)}$}$$

and, in the second

$$n<t-t'+1;\quad n+n'<t+1.\tag{$\mathrm{Y}^{(5)}$}$$

Thus, in calculating the constants $u_{t,t'}$, $w_{t,t'}$, by the formulæ ($\mathrm{Q}^{(5)}$),

$$u_{t,t'}=\Sigma\, u_{t,t'}^{(n,n')},\quad w_{t,t'}=\Sigma\, w_{t,t'}^{(n,n')},$$

we may reject all values of n, n', which are too great to satisfy these relations ($\mathrm{X}^{(5)}$), ($\mathrm{Y}^{(5)}$); we may also, by ($\mathrm{V}^{(5)}$), reject not only all negative values of the same integers n, n', but all for which the factorial index $t+2+2n'-2t'$ is negative in $u_{t,t'}^{(n,n')}$, or $t+1+2n'-2t'$ in $w_{t,t'}^{(n,n')}$; and by ($\mathrm{R}^{(5)}$), ($\mathrm{P}^{(5)}$) we may reject the value $n=0$ in the former, and $n'=0$ in the latter. Finally, we may remark, that since a factorial vanishes, when its base is less than its index, if both be positive integers, the expression ($\mathrm{V}^{(5)}$) for $u_{t,t'}^{(n,n')}$ vanishes if t be even, and $t'>\dfrac{t+2}{2}$; and similarly the expression for $w_{t,t'}^{(n,n')}$ vanishes if t be odd, and $t'>\dfrac{t+1}{2}$: from which it follows, that if the developments (T''') of the preceding paragraph, be put by (Y'''') ($\mathrm{D}^{(5)}$), ($\mathrm{E}^{(5)}$), under the form

$$\left.\begin{aligned} a&=\Sigma\,.u_{t,t'}\,.\,z^{t'}\theta^{t+2}=\Sigma\,.u_{t,t'}\,.\,u^{t'}\,.\,w^{t+2-2t'}\sqrt{r}^{\,t+2},\\ b&=\Sigma\,.w_{t,t'}\,.\,z^{t'}\theta^{t+1}=\Sigma\,.w_{t,t'}\,.\,u^{t'}\,.\,w^{t+1-2t'}\sqrt{r}^{\,t+1},\end{aligned}\right\}\tag{$\mathrm{Z}^{(5)}$}$$

the negative even powers of (θ) or of (w) will all disappear.

Let us verify these general results, respecting the constants $u_{t,t'}$, $w_{t,t'}$, by applying them to the particular case $t'=t$, which as we have before remarked, is the most important in our present investigations, and which we have already resolved by an entirely different method. In this case, when $t = 1$, we find by our present method,

$$u_{1,1} = u_{1,1}^{(1,0)} = \frac{d^2 F}{da \,.\, db}; \quad w_{1,1} = w_{1,1}^{(0,1)} = \frac{d^2 \Phi}{da \,.\, db};$$

when $t = 2$,

$$\begin{aligned} u_{2,2} = u_{2,2}^{(1,0)} + u_{2,2}^{(1,1)} &= \tfrac{1}{2}\frac{d^{2-1}\left(\Phi^0 \dfrac{dF}{db}\right)}{da^2\, db^{-1}} + \tfrac{1}{2}\frac{d^{2+1}\left(\Phi^1 \dfrac{dF}{db}\right)}{da^2\, db^1} = \tfrac{1}{2}\frac{d^2 F}{da^2} + \frac{d^2 F}{da \,.\, db}\frac{d^2 \Phi}{da \,.\, db}, \\ w_{2,2} = w_{2,2}^{(0,1)} + w_{2,2}^{(0,2)} &= \tfrac{1}{2}\frac{d^2 \Phi}{da^2} + \tfrac{1}{8}\frac{d^4 \,.\, \Phi^2}{da^2\, db^2} = \tfrac{1}{2}\frac{d^2 \Phi}{da^2} + \tfrac{1}{2}\left(\frac{d^2 \,.\, \Phi}{da \,.\, db}\right)^2; \end{aligned}$$

and when $t > 2$, if we put

$$u_{t,t} = u_{2\tau+1,2\tau+1}, \quad w_{t,t} = w_{2\tau,2\tau},$$

(since $u_{t,t}$ vanishes if t be an even number > 2, and $w_{t,t}$ if t be an odd number > 2), we have the following formulæ,

$$u_{2\tau+1,2\tau+1} = \Sigma \,.\, u_{2\tau+1,2\tau+1}^{(1,n')}; \quad w_{2\tau,2\tau} = \Sigma \,.\, w_{2\tau,2\tau}^{(0,n')}; \tag{A$^{(6)}$}$$

the sum being taken in each from $n' = \tau$ to $n' = 2\tau$. We have also, by (V$^{(5)}$),

$$\left.\begin{aligned} u_{2\tau+1,2\tau+1}^{(1,n')} &= [0]^{-(2\tau+1)} \,.\, [0]^{-n'} \,.\, [0]^{-(2n'-2\tau+1)} \,.\, 2^{n'} \,.\, [n' - \tau + \tfrac{1}{2}]^{n'} \,.\, \frac{d^{2n'+1} \,.\, \Phi^{n'} \dfrac{dF}{db}}{da^{2\tau+1} db^{2n'-2\tau}}, \\ w_{2\tau,2\tau}^{(0,n')} &= [0]^{-2\tau} \,.\, [0]^{-n'} \,.\, [0]^{-(2n'-2\tau+1)} \,.\, 2^{n'} \,.\, [n' - \tau + \tfrac{1}{2}]^{n'} \,.\, \frac{d^{2n'} \,.\, \Phi^{n'}}{da^{2\tau} db^{2n'-2\tau}}, \end{aligned}\right\} \tag{B$^{(6)}$}$$

in which, by (W$^{(5)}$),

$$\left.\begin{aligned} \frac{d^{2n'+1} \,.\, \Phi^{n'} \dfrac{dF}{db}}{da^{2\tau+1} \,.\, db^{2n'-2\tau}} &= (2\tau + 1) \,.\, \frac{d^2 F}{da \,.\, db} \cdot \frac{d^{2n'} \,.\, \Phi^{n'}}{da^{2\tau} \,.\, db^{2n'-2\tau}}, \\ \frac{d^{2n'} \,.\, \Phi^{n'}}{da^{2\tau} \,.\, db^{2n'-2\tau}} &= [2\tau]^{2\tau} [n']^{2n'-2\tau} \,.\, 2^{n'-2\tau} \left(\frac{d^2 \Phi}{da^2}\right)^{2\tau-n'} \left(\frac{d^2 \Phi}{da \,.\, db}\right)^{2n'-2\tau}, \end{aligned}\right\} \tag{C$^{(6)}$}$$

and, by the properties of factorials,

$$\left.\begin{aligned} [0]^{-(2n'-2\tau+1)} [n' - \tau + \tfrac{1}{2}]^{n'} \,.\, 2^{2n'-2\tau} &= [\tfrac{1}{2}]^{\tau} \,.\, [0]^{-(n'-\tau)}, \\ [0]^{-n'} \,.\, [n']^{2n'-2\tau} &= [0]^{-\tau} \,.\, [\tau]^{n'-\tau}; \end{aligned}\right\} \tag{D$^{(6)}$}$$

thus the formulæ ($A^{(6)}$) reduce themselves to

$$\left.\begin{aligned} u_{2\tau+1,2\tau+1} &= \frac{d^2F}{da\,.\,db}\,.\,[0]^{-\tau}\,.\,[\tfrac{1}{2}]^{\tau}\,.\,\Sigma\,.[0]^{-(n'-\tau)}[\tau]^{n'-\tau}\left(\frac{d^2\Phi}{da^2}\right)^{\tau-(n'-\tau)}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{2(n'-\tau)}, \\ w_{2\tau,2\tau} &= [0]^{-\tau}\,.\,[\tfrac{1}{2}]^{\tau}\,.\,\Sigma\,.[0]^{-(n'-\tau)}[\tau]^{n'-\tau}\left(\frac{d^2\Phi}{da^2}\right)^{\tau-(n'-\tau)}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{2(n'-\tau)}, \end{aligned}\right\} \quad (E^{(6)})$$

or, finally, effecting the summations,

$$\left.\begin{aligned} u_{2\tau+1,2\tau+1} &= \frac{d^2F}{da\,.\,db}\,.\,[0]^{-\tau}\,.\,[\tfrac{1}{2}]^{\tau}\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau}, \\ w_{2\tau,2\tau} &= [0]^{-\tau}\,.\,[\tfrac{1}{2}]^{\tau}\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau}: \end{aligned}\right\} \quad (F^{(6)})$$

and if we eliminate $\dfrac{d^2\Phi}{da^2}$, $\dfrac{d^2\Phi}{da\,.\,db}$, $\dfrac{d^2F}{da^2}$, $\dfrac{d^2F}{da\,.\,db}$, by the following relations

$$\left.\begin{aligned} 0 &= \frac{d^2y'}{db^2}\,.\,\frac{d^{m+m'}\Phi}{da^m\,.\,db^{m'}}+\frac{d^{m+m'}y'}{da^m\,.\,db^{m'}}, \\ -\frac{dx'}{da}\,.\,\frac{d^{m+m'}F}{da^m\,.\,db^{m'}} &= \frac{d^2x'}{db^2}\,.\,\frac{d^{m+m'}\Phi}{da^m\,.\,db^{m'}}+\frac{d^{m+m'}x'}{da^m\,.\,db^{m'}}, \end{aligned}\right\} \quad (G^{(6)})$$

in which m, m', are any positive integers satisfying the conditions ($L^{(5)}$), we arrive again at the same expressions for the constants of the form $u_{t,t}$, $w_{t,t}$, as those given by the equations $(1)_{1,1}$, $(2)_{1,1}$, $(1)_{2,2}$, $(2)_{2,2}$, ($C^{(5)}$), which we obtained before by reasonings of so different a nature. It results from these equations, that if in the developments ($Z^{(5)}$), for the components of aberration at the mirror, we confine ourselves to the terms of the form

$$u_{2\tau+1,2\tau+1}\,.\,u^{2\tau+1}\,.\,w^{1-2\tau}\,.\,\sqrt{r}^{2\tau+3}, \quad w_{2\tau,2\tau}\,.\,u^{2\tau}\,.\,w^{1-2\tau}\,.\,\sqrt{r}^{2\tau+1},$$

which correspond to the greatest negative powers of w, or of the sine of the polar angle v; the sums of these terms, taken from $\tau=0$, to $\tau=\infty$, may be calculated by the binomial theorem, and are thus expressed:

$$\left.\begin{aligned} \Sigma_0^\infty\,.w_{2\tau,2\tau}u^{2\tau}w^{1-2\tau}\sqrt{r}^{2\tau+1} &= w\sqrt{r}\,.\,\Sigma_0^\infty\,.[0]^{-\tau}\,.\,[\tfrac{1}{2}]^{\tau}(2w_{2,2}u^2w^{-2}r)^{\tau} \\ &= \sqrt{(w^2r+2w_{2,2}u^2r^2)}; \\ \Sigma_0^\infty\,.u_{2\tau+1,2\tau+1}u^{2\tau+1}w^{1-2\tau}\sqrt{r}^{2\tau+3} &= u_{1,1}\,.\,ur\,.\,\sqrt{(w^2r+2w_{2,2}u^2r^2)}. \end{aligned}\right\} \quad (H^{(6)})$$

We shall return to this remarkable result, and examine its optical meaning.

As another application of our general formulæ for the constants $u_{t,t'}$, $w_{t,t'}$, let us take the terms of the form

$$w_{2\tau+2,2\tau+1}\,.\,u^{2\tau+1}\,.\,w^{1-2\tau}\,.\,\sqrt{r}^{2\tau+3},$$

which correspond to the next greatest negative powers of w, or of the sine of the polar angle v, in the development ($\mathrm{Z}^{(5)}$) for b, that is, for the aberration at the mirror measured in a direction perpendicular to the tangent plane of the caustic surface, and considered as depending on the polar coordinates r and v, which determine the magnitude and direction of the aberration on the perpendicular plane at the focus. We have, by what precedes,

$$\begin{aligned} w_{2\tau+2,2\tau+1} &= \Sigma\,.w^{(0,n')}_{2\tau+2,2\tau+1} + \Sigma\,.w^{(1,n')}_{2\tau+2,2\tau+1} \\ &= [0]^{-(2\tau+1)}.\,\Sigma\,.[0]^{-n'}2^{n'}[0]^{-(2n'-2\tau+1)}[n'-\tau+\tfrac{1}{2}]^{n'}\frac{d^{2n'+1}\left(\Phi^{n'}+F\,.\,\dfrac{d\,.\,\Phi^{n'}}{da}\right)}{da^{2\tau+1}\,.\,db^{2n'-2\tau}}; \end{aligned} \tag{I$^{(6)}$}$$

and the summation here indicated, with reference to the variable integer n', may be performed by partial differential and factorial transformations, similar to those which we have already employed in finding the sums of the expressions ($\mathrm{B}^{(6)}$). Thus, we may eliminate the variable n' from the partial differential index, by putting, in virtue of ($\mathrm{W}^{(5)}$),

$$\begin{aligned} &[0]^{-(2\tau+1)}[0]^{-n'}\frac{d^{2n'+1}\left(\Phi^{n'}+F\,.\,\dfrac{d\,.\,\Phi^{n'}}{da}\right)}{da^{2\tau+1}\,.\,db^{2n'-2\tau}} = [2n'-2\tau]^{2n'-2\tau} \\ &\times\Sigma\,.[0]^{-s}.\,[0]^{-(3-s)}[0]^{-s'}.\,[0]^{-(n'-1-s')}\left(\tfrac{1}{2}\,.\,\frac{d^2\Phi}{da^2}\right)^{n'-1-s'}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{s'}\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,.\,db^s}, \end{aligned} \tag{K$^{(6)}$}$$

the integers s, s', being new variables connected by the relation

$$s+s'=2n'-2\tau, \tag{L$^{(6)}$}$$

which gives, by the properties of factorials,

$$[2n'-2\tau]^{2n'-2\tau}[0]^{-s'} = [2n'-2\tau]^s = 2^s[n'-\tau]^s + 2^{s-2}[s]^2[n'-\tau]^{s-1}, \tag{M$^{(6)}$}$$

observing that by ($\mathrm{K}^{(6)}$), s is included between the limits 0 and 3; and, by the same properties,

$$\left.\begin{aligned} [0]^{-(n'-1-s')} &= [0]^{-(\tau-\tau')}[\tau-\tau']^{n'-\tau-s+1-\tau'}, \\ [0]^{-(2n'-2\tau+1)}2^{n'}[n'-\tau+\tfrac{1}{2}]^{n'} &= [\tfrac{1}{2}]^{\tau'}[\tfrac{1}{2}-\tau']^{\tau-\tau'}2^{2\tau-n'}[0]^{(n'-\tau)}, \end{aligned}\right\} \tag{N$^{(6)}$}$$

τ' being an arbitrary integer; so that if we put

$$\left.\begin{aligned} W &= [0]^{-(n'-\tau-s)}[\tau-\tau']^{n'-\tau-s+1-\tau'}\left(\frac{d^2\Phi}{da^2}\right)^{2\tau-n'+s-1}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{2n'-2\tau-s}, \\ W' &= [0]^{-(n'-\tau-s+1)}[\tau-\tau']^{n'-\tau-s+1-\tau'}\left(\frac{d^2\Phi}{da^2}\right)^{2\tau-n'+s-1}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{2n'-2\tau-s}, \end{aligned}\right\} \tag{O$^{(6)}$}$$

the expression ($\mathrm{I}^{(6)}$) resolves itself into the two following parts:

$$
\begin{aligned}
w_{2\tau+2,2\tau+1} &= 2[\tfrac{1}{2}]^{\tau'}[\tfrac{1}{2}-\tau']^{\tau-\tau'}[0]^{-(\tau-\tau')}\,\Sigma^{(s)}[0]^{-s}[0]^{-(3-s)}\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,db^s}\,.\,\Sigma^{(n')}\,W \\
&+ 2^{-1}[\tfrac{1}{2}]^{\tau'}[\tfrac{1}{2}-\tau']^{\tau-\tau'}[0]^{-(\tau-\tau')}\,\Sigma^{(s)}[0]^{-(s-2)}[0]^{-(3-s)}\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,db^s}\,.\,\Sigma^{(n')}\,W',
\end{aligned}
\tag{$\mathrm{P}^{(6)}$}
$$

in which $\Sigma^{(n')}$, $\Sigma^{(s)}$, denote summations with reference to the two independent variables n' and s, and which can be calculated separately, by making in the first $\tau'=1$, and in the second $\tau'=0$: for this gives, by the binomial theorem,

$$
\left.
\begin{aligned}
\Sigma^{(n')}\,W &= \left(\frac{d^2\Phi}{da\,.\,db}\right)^s\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau-1}, \\
\Sigma^{(n')}\,W' &= \left(\frac{d^2\Phi}{da\,.\,db}\right)^{s-2}\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau},
\end{aligned}
\right\}
\tag{$\mathrm{Q}^{(6)}$}
$$

and reduces the expression ($\mathrm{P}^{(6)}$) to the following form,

$$
\begin{aligned}
w_{2\tau+2,2\tau+1} = [-\tfrac{1}{2}]^{\tau-1}[0]^{-(\tau-1)}&\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau-1} \\
&\times\Sigma^{(s)}[0]^{-s}[0]^{-(3-s)}\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,.\,db^s}\left(\frac{d^2\Phi}{da\,.\,db}\right)^s \\
&+[\tfrac{1}{2}]^{\tau}[0]^{-\tau}\left(\frac{d^2\Phi}{da^2}+\left(\frac{d^2\Phi}{da\,.\,db}\right)^2\right)^{\tau} \\
&\times\Sigma^{(s)}\,\frac{[0]^{-(s-2)}[0]^{-(3-s)}}{2}\,\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,.\,db^s}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{s-2},
\end{aligned}
\tag{$\mathrm{R}^{(6)}$}
$$

in which the first sum contains only four terms, and the second only two, however great may be the value of (τ). And if we multiply this expression ($\mathrm{R}^{(6)}$) by $u^{2\tau+1}\,.\,w^{1-2\tau}\sqrt{r^{2\tau+3}}$, and sum with reference to (τ), from $\tau=0$ to $\tau=\infty$, we find

$$
\begin{aligned}
&\Sigma_0^\infty\,.w_{2\tau+2,2\tau+1}u^{2\tau+1}w^{1-2\tau}\sqrt{r^{2\tau+3}} \\
&\quad= u^3r^3\{w^2r+2w_{2,2}\,.\,u^2r^2\}^{-\frac{1}{2}}\,\Sigma^{(s)}[0]^{-s}[0]^{-(3-s)}\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,.\,db^s}\left(\frac{d^2\Phi}{da\,.\,db}\right)^s \\
&\qquad+ ur\{w^2r+2w_{2,2}\,.\,u^2r^2\}^{\frac{1}{2}}\,\Sigma^{(s)}\,\frac{[0]^{-(s-2)}[0]^{-(3-s)}}{2}\,\frac{d^3.\left(\Phi+F\,.\,\dfrac{d\,.\,\Phi}{da}\right)}{da^{3-s}\,.\,db^s}\left(\frac{d^2\Phi}{da\,.\,db}\right)^{s-2},
\end{aligned}
\tag{$\mathrm{S}^{(6)}$}
$$

if we observe that $w_{2,2} = \frac{1}{2}\left(\dfrac{d^2\Phi}{da^2} + \left(\dfrac{d^2\Phi}{da\,.\,db}\right)^2\right)$. We might easily extend the principles of these summations, but it is better to make use of the results to which we have already arrived, for the solution of our second problem.

(II.) We proposed, first, to find general expressions for the polar functions $u^{(t)}$, $w^{(t)}$, which enter as coefficients into the developments (T‴), and to examine what negative powers they contain of the sine of the polar angle v; and secondly, to eliminate these negative powers, and so to transform the series (T‴) into others which shall contain none but positive powers of any variable quantity. The I$^{\text{st.}}$ of these problems has been completely resolved by the discussions in which we have just been engaged. We have seen that the functions $u^{(t)}$, $w^{(t)}$, are of the form

$$u^{(t)} = \Sigma_{(t')}\,{}_0^t\,.\,u_{t,t'}\,.\,u^{t'}\,.\,w^{t+2-2t'}; \quad w^{(t)} = \Sigma_{(t')}\,{}_0^t\,.\,w_{t,t'}\,.\,u^{t'}\,.\,w^{t+1-2t'}; \tag{T$^{(6)}$}$$

$\Sigma_{(t')}\,{}_0^t$ denoting a summation with reference to t' from $t' = 0$ to $t' = t$; $u_{t,t'}$, $w_{t,t'}$ constants, which we have given general formulæ to determine; and u, w, functions, which in the notation of (L⁗) [68.], have for expressions

$$u = \frac{\epsilon\,.\,\sin.(v - v')}{P\,.\,\cos.\,v'}, \quad w = \pm\sqrt{(\epsilon\,.\,\sin.\,v)}, \tag{U$^{(6)}$}$$

v being the polar angle, and ϵ, P, v' constants which enter into the equations of the curves (M⁗). Substituting these values in the series (T‴) which may be thus written,

$$\left.\begin{aligned} a &= \Sigma_{(\tau)}\,{}_0^\infty\,.\,u^{(2\tau)}r^{\tau+1} \pm \sqrt{r}\,.\,\Sigma_{(\tau)}\,{}_0^\infty\,.\,u^{(2\tau+1)}\,.\,r^{\tau+1}, \\ b &= \Sigma_{(\tau)}\,{}_0^\infty\,.\,w^{(2\tau+1)}r^{\tau+1} \pm \sqrt{r}\,.\,\Sigma_{(\tau)}\,{}_0^\infty\,.\,w^{(2\tau)}\,.\,r^{\tau}, \end{aligned}\right\} \tag{V$^{(6)}$}$$

and observing that as the negative powers of w are all odd, those of $(\sin.\,v)$ are all fractional, we find the following transformed developments:

$$\left.\begin{aligned} a = {}& \Sigma_{(\tau)}\,{}_0^\infty(\epsilon r\sin.\,v)^{\tau+1}\,.\,\Sigma_{(t')}\,{}_0^{\tau+1}\,.\,u_{2\tau,t'}\,.\,P^{-t'}\left(\frac{\sin.(v-v')}{\cos.\,v'\,.\,\sin.\,v}\right)^{t'} \\ & \pm \Sigma_{(\tau)}\,{}_0^\infty(\epsilon r\sin.\,v)^{\tau+\frac{3}{2}}\,.\,\Sigma_{(t')}\,{}_0^{2\tau+1}\,.\,u_{2\tau+1,t'}\,.\,P^{-t'}\left(\frac{\sin.(v-v')}{\cos.\,v'\,.\,\sin.\,v}\right)^{t'}, \\ b = {}& \Sigma_{(\tau)}\,{}_0^\infty(\epsilon r\sin.\,v)^{\tau+1}\,.\,\Sigma_{(t')}\,{}_0^{\tau+1}\,.\,w_{2\tau+1,t'}\,.\,P^{-t'}\left(\frac{\sin.(v-v')}{\cos.\,v'\,.\,\sin.\,v}\right)^{t'} \\ & \pm \Sigma_{(\tau)}\,{}_0^\infty(\epsilon r\sin.\,v)^{\tau+\frac{1}{2}}\,.\,\Sigma_{(t')}\,{}_0^{2\tau}\,.\,w_{2\tau,t'}\,.\,P^{-t'}\left(\frac{\sin.(v-v')}{\cos.\,v'\,.\,\sin.\,v}\right)^{t'}, \end{aligned}\right\} \tag{W$^{(6)}$}$$

which have the advantage of exhibiting to the eye, the manner wherein the rectangular components a, b, of aberration at the mirror, depend on the polar components r, v, of aberration at the caustic surface. To eliminate from these developments (W$^{(6)}$) the negative powers of (sin. v), without introducing those of any other variable, or the positive powers of any quantity which (like the z of the preceding problem) becomes infinite when the polar radius r assumes a particular direction; let us resume the summations, expressed by the equations ($\mathrm{H}^{(6)}$). It results from those equations, or from the formulæ ($\mathrm{C}^{(5)}$) ($\mathrm{F}^{(6)}$), on which they were founded, that if, in order to begin with the greatest negative powers of (sin. v), we reject at first all but the greatest values of t' in the developments ($\mathrm{W}^{(6)}$), namely $t' = 2\tau + 1$ in a, and $t' = 2\tau$ in b, and denote by a_1, b_1, the sums of the terms that remain, we shall have

$$a_1 = \frac{d^2F}{da \,.\, db} \,.\, \frac{r \,.\, \sin.(v - v')}{P \,.\, \cos.\, v'} \,.\, \epsilon^{\frac{3}{2}} \mathcal{C}^{\frac{1}{2}}, \quad b_1 = \epsilon^{\frac{1}{2}} \mathcal{C}^{\frac{1}{2}}; \tag{$\mathrm{X}^{(6)}$}$$

in which

$$\mathcal{C} = r \,.\, \sin.\, v + \left(\frac{d^2\Phi}{da^2} + \left(\frac{d^2\Phi}{da \,.\, db} \right)^2 \right) \left(\frac{\sqrt{\epsilon} \,.\, r \,.\, \sin.(v - v')}{P \,.\, \cos.\, v'} \right)^2, \tag{$\mathrm{Y}^{(6)}$}$$

F, Φ having the same meanings as in the foregoing problem. To find the optical meaning of the binomial function ($\mathcal{C}$), let us consider the points upon the plane of aberration for which that function vanishes. It is evident that at these points (sin. v) is small; if then we change $r \,.\, \sin.(v - v')$ to $-r \,.\, \sin.\, v'$, the condition $\mathcal{C} = 0$ becomes by ($\mathrm{Y}^{(6)}$)

$$0 = \sin.\, v + \left(\frac{d^2\Phi}{da^2} + \left(\frac{d^2\Phi}{da \,.\, db} \right)^2 \right) \,.\, (\sqrt{\epsilon} \,.\, P^{-1} \,.\, \tan.\, v')^2 \,.\, r,$$

that is, in the notation of paragraph [61.],

$$2i^2C \,.\, \sin.\, v = (AC - B^2)r, \tag{$\mathrm{Z}^{(6)}$}$$

which, by the same paragraph, is the equation of the osculating circle to the section of the caustic surface; from which it follows, that in this approximation, the function ($\mathcal{C}$) is, for any other point upon the plane of aberration, the distance of that point from the osculating circle just mentioned, measured in a direction parallel to the normal of the caustic surface. More accurately, if we put

$$r \,.\, \sin.\, v = y'' \,.\, \sin.\, v', \quad r \,.\, \sin.(v - v') = x'' \,.\, \sin.\, v', \tag{$\mathrm{A}^{(7)}$}$$

x'', y'', will be the oblique coordinates of the point r, v, referred to two axes in the plane of aberration, of which one touches the caustic surface at the focus of the given ray, while the other is inclined to this tangent at an angle $= v'$; and the equation ($\mathrm{Y}^{(6)}$) will become

$$\mathcal{C} = y'' \,.\, \sin.\, v' - \left(\frac{AC - B^2}{2i^2C} \right) \,.\, x''^2, \tag{$\mathrm{B}^{(7)}$}$$

which shews that ($\mathcal{C}$) vanishes for the points of a parabola, which has its diameter parallel to the axis of y'', and has contact of the second order with the section of the caustic surface; and that for any other point upon the plane of aberration, ($\mathcal{C}$) is equal to the distance from this parabola measured in a direction parallel to its own diameter, and then projected upon the normal. If, therefore, in the developments ($\mathrm{W}^{(6)}$), we change $r \,.\, \sin.\, v$, $r \,.\, \sin.(v - v')$, to their values $y'' \,.\, \sin.\, v'$, $x'' \,.\, \sin.\, v'$ ($\mathrm{A}^{(7)}$); if we then eliminate $y'' \,.\, \sin.\, v'$, by changing it, in virtue of ($\mathrm{B}^{(7)}$), to the binomial

$$\mathcal{C} + \left(\frac{AC - B^2}{2i^2 C}\right) . \, x''^2,$$

and develope every fractional power of this binomial according to the ascending powers of x'', and the descending powers of $\mathcal{C}$, we see that the new developments will contain no negative powers of this latter variable, except those which arise from the terms that we rejected in effecting the summations ($\mathrm{X}^{(6)}$): and I am going to shew, that if in place of the parabola $\mathcal{C} = 0$, which has contact of the second order with the section of the caustic surface, we take that section itself, whose equation referred to the coordinates (x'', y'') is of the form $\mathcal{C}' = 0$, in which

$$\mathcal{C}' = \mathcal{C} - \Sigma_{(\nu)}{}_0^\infty \, . \, [0]^{-(\nu+3)} \left(\frac{d^{\nu+3}\mathcal{C}}{dx''^{\nu+3}}\right) . \, x''^{\nu+3} = (y'' - y''_0) \sin.\, v', \tag{$\mathrm{C}^{(7)}$}$$

y''_0 being the ordinate of the section, and $\mathcal{C}'$ the distance from that curve, measured in a direction parallel to the axis of y'', and then projected on the normal; it is sufficient to change the fractional powers of $y'' \,.\, \sin.\, v'$ to those of $\mathcal{C}' + y''_0 \,.\, \sin.\, v'$, in order to obtain development for a and b, which shall satisfy the condition of the question, containing no negative powers of any variable quantity, but only positive and integer powers and products of x'' and of $\sqrt{\mathcal{C}'}$.

To demonstrate this theorem, let us resume the equations ($\mathrm{G}^{(5)}$), putting them by ($\mathrm{U}^{(6)}$), ($\mathrm{A}^{(7)}$) under the form

$$\epsilon P^{-1} \tan.\, v' \,.\, x'' = a - F(a, b); \quad \epsilon \,.\, \sin.\, v' \,.\, y'' = b^2 - 2\Phi(a, b). \tag{$\mathrm{D}^{(7)}$}$$

Conceive a parallel to the axis of y'', drawn through the point x'', y'', upon the plane of aberration; this parallel will meet the section of the caustic surface in a point having for coordinates x'', y''_0, and the ray which has that point for focus will cross the perpendicular plane at the mirror in another point whose coordinates may be called a_0, b_0; to determine these coefficients we have by ($\mathrm{D}^{(7)}$),

$$\epsilon P^{-1} \tan.\, v' \,.\, x'' = a_0 - F_0; \quad \epsilon \,.\, \sin.\, v' \,.\, y''_0 = b_0^2 - 2\Phi_0, \tag{$\mathrm{E}^{(7)}$}$$

F_0, Φ_0, representing for abridgment the functions $F(a_0, b_0)$, $\Phi(a_0, b_0)$; we have also, by the nature of y''_0,

$$\frac{dx''}{da_0} \cdot \frac{dy''_0}{db_0} = \frac{dx''}{db_0} \cdot \frac{dy''_0}{da_0}, \quad \text{that is,} \quad \left(1 - \frac{dF_0}{da_0}\right)\left(b_0 - \frac{d\Phi_0}{db_0}\right) = \frac{dF_0}{db_0} \cdot \frac{d\Phi_0}{da_0}, \tag{$\mathrm{F}^{(7)}$}$$

from which it follows that the locus of the point a_0, b_0, on the perpendicular plane at the mirror, has for tangent the right line

$$b_0 = a_0 \,.\, \frac{d^2\Phi}{da \,.\, db} = a_0 \tan.(v' + \tfrac{1}{2}\pi), \tag{$\mathrm{G}^{(7)}$}$$

and that we can develope a_0, b_0 in series of the form

$$\left.\begin{aligned} a_0 &= \epsilon P^{-1} \tan. v'' \,.\, x'' + \frac{d^2 a_0}{dx''^2} \cdot \frac{x''^2}{2} + \frac{d^3 a_0}{dx''^3} \cdot \frac{x''^3}{2\,.\,3} + \&\text{c.}, \\ b_0 &= -\epsilon \,.\, P^{-1} \,.\, x'' + \frac{d^2 b_0}{dx''^2} \cdot \frac{x''^2}{2} + \frac{d^3 b_0}{dx''^3} \cdot \frac{x''^3}{2\,.\,3} + \&\text{c.} \end{aligned}\right), \tag{H$^{(7)}$}$$

This being laid down, let us subtract (E$^{(7)}$) from (D$^{(7)}$); we find

$$\left.\begin{aligned} a - a_0 &= F - F_0 \\ &= \frac{dF_0}{da_0}(a - a_0) + \frac{dF_0}{db_0}(b - b_0) + \Sigma \,.\, \frac{d^{m+m'} F_0}{da_0^m \,.\, db_0^{m'}} \cdot \frac{(a - a_0)^m}{[m]^m} \cdot \frac{(b - b_0)^{m'}}{[m']^{m'}}, \\ \tfrac{1}{2}(b^2 - b_0^2 - \epsilon\mathcal{C}') &= \Phi - \Phi_0 \\ &= \frac{d\Phi_0}{da_0}(a - a_0) + \frac{d\Phi_0}{db_0}(b - b_0) + \Sigma \,.\, \frac{d^{m+m'} \Phi_0}{da_0^m \,.\, db_0^{m'}} \cdot \frac{(a - a_0)^m}{[m]^m} \cdot \frac{(b - b_0)^{m'}}{[m']^{m'}}, \end{aligned}\right\} \tag{I$^{(7)}$}$$

and therefore, by (F$^{(7)}$),

$$\begin{aligned} (b - b_0)^2 &= \epsilon\mathcal{C}' + 2 \,.\, \Sigma \,.\, \frac{d^{m+m'} \Phi_0}{da_0^m \,.\, db_0^{m'}} \cdot \frac{(a - a_0)^m}{[m]^m} \cdot \frac{(b - b_0)^{m'}}{[m']^{m'}} \\ &\quad + \frac{2 \,.\, \dfrac{d\Phi_0}{da_0}}{1 - \dfrac{dF_0}{da_0}} \,.\, \Sigma \,.\, \frac{d^{m+m'} F_0}{da_0^m \,.\, db_0^{m'}} \cdot \frac{(a - a_0)^m}{[m]^m} \cdot \frac{(b - b_0)^{m'}}{[m']^{m'}}, \end{aligned} \tag{K$^{(7)}$}$$

in which $m + m' > 1$,

$$\left.\begin{aligned} \frac{d^{m+m'} F_0}{da_0^m \,.\, db_0^{m'}} &= \Sigma \,.\, \frac{d^{m+m'+m''+m'''} F}{da^{m+m''} \,.\, db^{m'+m'''}} \cdot \frac{a_0^{m''}}{[m'']^{m''}} \cdot \frac{b_0^{m'''}}{[m''']^{m'''}}, \\ \frac{d^{m+m'} \Phi_0}{da_0^m \,.\, db_0^{m'}} &= \Sigma \,.\, \frac{d^{m+m'+m''+m'''} \Phi}{da^{m+m''} \,.\, db^{m'+m'''}} \cdot \frac{a_0^{m''}}{[m'']^{m''}} \cdot \frac{b_0^{m'''}}{[m''']^{m'''}}, \end{aligned}\right\} \tag{L$^{(7)}$}$$

a, b, being supposed to vanish after the differentiations in these second members: and it is easy to see that by means of these equations we can develope $a - a_0$, $b - b_0$, and therefore also a, b, according to the positive integer powers and products of x'', $\sqrt{\mathcal{C}'}$. With respect to the coefficients of these developments, they may be calculated by differentiating the equations that we have just established; they may also be deduced from the coefficients of the series (T$'''$), by relations which will be elsewhere indicated.

In the mean time let us remark, that instead of measuring the distance from the caustic section in a direction parallel to the axis of y'', we may measure it parallel to any other line upon the plane of aberration. If for instance, to simplify our remaining calculations, we resume the rectangular coordinates x', y', of which the former is a tangent, and the latter a normal to the section; if from the point (x', y') we draw a line parallel to this normal, and denote by x'_0, y'_0, the coordinates of the point where this line meets the section, and by δ the intercepted portion, so that

$$x' - x'_0 = 0, \quad y' - y'_0 = \delta; \tag{M$^{(7)}$}$$

if also we call a_0, b_0, the coordinates of the point in which the ray that passes through (x'_0, y'_0) is crossed by the perpendicular plane at the mirror; we shall have the equations

$$\left.\begin{aligned} x' &= \frac{dx'}{da} \,.\, a + \Sigma \,.\, \frac{d^{m+m'}x'}{da^m \,.\, db^{m'}} \,.\, \frac{a^m}{[m]^m} \,.\, \frac{b^{m'}}{[m']^{m'}}, \\ y' &= \qquad\qquad \Sigma \,.\, \frac{d^{m+m'}y'}{da^m \,.\, db^{m'}} \,.\, \frac{a^m}{[m]^m} \,.\, \frac{b^{m'}}{[m']^{m'}}, \end{aligned}\right\} \qquad (\mathrm{S}''') \quad [68.]$$

$$\left.\begin{aligned} x'_0 &= \frac{dx'}{da} \,.\, a_0 + \Sigma \,.\, \frac{d^{m+m'}x'}{da^m \,.\, db^{m'}} \,.\, \frac{a_0^m}{[m]^m} \,.\, \frac{b_0^{m'}}{[m']^{m'}}, \\ y'_0 &= \qquad\qquad \Sigma \,.\, \frac{d^{m+m'}y'}{da^m \,.\, db^{m'}} \,.\, \frac{a_0^m}{[m]^m} \,.\, \frac{b_0^{m'}}{[m']^{m'}}, \end{aligned}\right\} \tag{N$^{(7)}$}$$

$$\left.\begin{aligned} 0 &= \frac{dx'_0}{da_0} \,.\, (a - a_0) + \frac{dx'_0}{db_0} \,.\, (b - b_0) + \Sigma \,.\, \frac{d^{m+m'}x'_0}{da_0^m \,.\, db_0^{m'}} \,.\, \frac{(a-a_0)^m}{[m]^m} \,.\, \frac{(b-b_0)^{m'}}{[m']^{m'}}, \\ \delta &= \frac{dy'_0}{da_0} \,.\, (a - a_0) + \frac{dy'_0}{db_0} \,.\, (b - b_0) + \Sigma \,.\, \frac{d^{m+m'}y'_0}{da_0^m \,.\, db_0^{m'}} \,.\, \frac{(a-a_0)^m}{[m]^m} \,.\, \frac{(b-b_0)^{m'}}{[m']^{m'}}, \end{aligned}\right\} \tag{O$^{(7)}$}$$

$$\frac{dx'_0}{da_0} \,.\, \frac{dy'_0}{db_0} = \frac{dx'_0}{db_0} \,.\, \frac{dy'_0}{da_0} \tag{P$^{(7)}$}$$

in which $m + m' > 1$; and by these equations we can change the developments (T$'''$) [68.] into series of the form

$$\left.\begin{aligned} a &= \frac{da}{dx'} \,.\, x' \pm \frac{da}{d\sqrt{\delta}} \,.\, \sqrt{\delta} + \frac{d^2a}{dx'^2} \,.\, \frac{x'^2}{2} \pm \frac{d^2a}{dx' \,.\, d\sqrt{\delta}} \,.\, x'\sqrt{\delta} + \&\text{c.}, \\ b &= \frac{db}{dx'} \,.\, x' \pm \frac{db}{d\sqrt{\delta}} \,.\, \sqrt{\delta} + \frac{d^2b}{dx'^2} \,.\, \frac{x'^2}{2} \pm \frac{d^2b}{dx' \,.\, d\sqrt{\delta}} \,.\, x'\sqrt{\delta} + \&\text{c.}, \end{aligned}\right\} \tag{Q$^{(7)}$}$$

which contain no negative powers of any variable quantity, and which we are going to apply to the solution of the succeeding problems.

(III.) We must be more brief in the discussion of these remaining problems, namely to transform the integral expressions of the preceding paragraph, and to effect the double integrations within the limits of the present. Applying to the series (Q$^{(7)}$) the geometrical and optical reasonings of [68.], we find for the quantities $\Delta^{(\alpha)}$, $S^{(\mu)}$, $Q^{(s)}$, which were there represented by the developments (D$''''$), (E$''''$), (F$''''$), the following transformed expressions:

$$\Delta^{(\alpha)} = \frac{D}{\sqrt{\delta}}; \quad S^{(\mu)} = \iint \frac{S \,.\, dx' \,.\, dy'}{\sqrt{\delta}}; \quad Q^{(s)} = \iint \frac{Q \,.\, dx' \,.\, dy'}{\sqrt{\delta}}; \tag{R$^{(7)}$}$$

in which δ is, as in ($\mathrm{M}^{(7)}$), the distance of any assigned point x', y' upon the plane of aberration from the section of the caustic surface, measured in a direction parallel to the normal of that curve; and D, S, Q, are rational and integer functions of x' and δ, or of x' and y', which when those variables vanish, that is, at the focus of the given ray, reduce themselves to the following values:

$$D = \frac{\Delta^{(\mu)} \,.\, \rho_1 \,.\, \rho_2}{i\sqrt{\frac{1}{2}C}}; \quad S = \frac{\rho_1 \,.\, \rho_2}{i\sqrt{\frac{1}{2}C}}; \quad Q = \frac{\Delta^{(\mu)} \,.\, \rho_1 \,.\, \rho_2}{i\sqrt{\frac{1}{2}C}}; \tag{S$^{(7)}$}$$

$\Delta^{(\mu)}$, ρ_1, ρ_2, i, c, having the same meanings as in [68.]. If then we integrate the two last of these expressions ($\mathrm{R}^{(7)}$) within the double limits afforded by the equations

$$\delta = 0, \quad x'^2 + y'^2 = r^2, \tag{T$^{(7)}$}$$

of which the former represents the caustic section, and the latter the circular circumference, we shall have the required expressions for the quantities that we denoted by $S^{(r)}$, $Q^{(r)}$, at the beginning of the present paragraph.

(IV.) To effect these double integrations, let us put the functions S, Q, under the form

$$S = \Sigma \,.S_{m,m'} \,.\, x'^m \,.\, \delta^{m'}, \quad Q = \Sigma \,.Q_{m,m'} \,.\, x'^m \,.\, \delta^{m'}, \tag{U$^{(7)}$}$$

and let us change the differential product $dx' \,.\, dy'$ to $dx' \,.\, d\delta$, which is permitted, because in forming this product y' varies independently of x'. In this manner the expressions ($\mathrm{R}^{(7)}$) become

$$\int 2\, d\sqrt{\delta} \int S\, dx', \quad \int 2\, d\sqrt{\delta} \int Q\, dx',$$

in which

$$\left.\begin{aligned} \int S\, dx' &= \Sigma \,.S_{m,m'} \,.\, \frac{x_2'^{m+1} - x_1'^{m+1}}{m+1} \,.\, \delta^{m'}, \\ \int Q\, dx' &= \Sigma \,.Q_{m,m'} \,.\, \frac{x_2'^{m+1} - x_1'^{m+1}}{m+1} \,.\, \delta^{m'}, \end{aligned}\right\} \tag{V$^{(7)}$}$$

x_1', x_2', being the extreme values of x', corresponding to any given value of δ, that is, the abscissæ of the points where the little circumference is crossed by any given parallel to the section of the caustic surface. To determine these values, we have the equation

$$x'^2 + (y_0' + \delta)^2 = r^2, \tag{W$^{(7)}$}$$

y_0' being the ordinate of the section, and r the radius of the circle: and putting this equation under the form

$$x'^2 + y_0'^2 = \delta'^2 - 2y_0'\delta, \quad \text{in which} \quad \delta' = \pm\sqrt{r^2 - \delta^2}, \tag{X$^{(7)}$}$$

we can, by Laplace's theorem, develope x'^{m+1} according to the positive integer powers of δ, δ', the term of least dimension being δ'^{m+1}; from which it follows that the integrals (V$^{(7)}$) may be put under the form

$$\left.\begin{aligned} \int S\,dx' &= 2\delta' S' = 2\,\Sigma\,.S'_{n,n'}\,.\,\delta^n \delta'^{2n'+1}, \\ \int Q\,dx' &= 2\delta' Q' = 2\,\Sigma\,.Q'_{n,n'}\,.\,\delta^n \delta'^{2n'+1}, \end{aligned}\right\} \tag{Y$^{(7)}$}$$

in which

$$S'_{0,0} = S_{0,0}, \quad Q'_{0,0} = Q_{0,0}, \tag{Z$^{(7)}$}$$

$S_{0,0}$, $Q_{0,0}$, being the values of S, Q, assigned by the formulæ (S$^{(7)}$). Multiplying (Y$^{(7)}$) by $2d\sqrt{\delta}$, integrating with reference to δ from $\delta = 0$ to $\delta = r$, and putting for abridgment

$$I_{n,n'} = \int_0^1 .z^n (1 - z^2)^{n'+\frac{1}{2}} d\sqrt{z}, \tag{A$^{(8)}$}$$

we find finally

$$\left.\begin{aligned} S^{(r)} &= 4r^{\frac{3}{2}}\,.\,\Sigma\,.S'_{n,n'}\,.\,I_{n,n'}\,.\,r^{n+2n'}, \\ Q^{(r)} &= 4r^{\frac{3}{2}}\,.\,\Sigma\,.Q'_{n,n'}\,.\,I_{n,n'}\,.\,r^{n+2n'}, \end{aligned}\right\} \tag{B$^{(8)}$}$$

and therefore when we suppose r infinitely small,

$$\left.\begin{aligned} S^{(dr)} &= 4S'_{0,0}\,.\,I_{0,0}\,.\,(dr)^{\frac{3}{2}} = \frac{4\rho_1\,.\,\rho_2\,.\,(dr)^{\frac{3}{2}}}{i\sqrt{\frac{1}{2}C}} \int_0^1 .\sqrt{1 - z^2}\,.\,d\sqrt{z}, \\ Q^{(dr)} &= 4Q'_{0,0}\,.\,I_{0,0}\,.\,(dr)^{\frac{3}{2}} = \frac{4\Delta^{(\mu)}\rho_1\,.\,\rho_2\,.\,(dr)^{\frac{3}{2}}}{i\sqrt{\frac{1}{2}C}} \int_0^1 .\sqrt{1 - z^2}\,.\,d\sqrt{z}, \end{aligned}\right\} \tag{C$^{(8)}$}$$

values which satisfy the relation (R$''''$) at the beginning of the present paragraph; and which shew, by the principles there laid down, that the density at the caustic surface is proportional to the following expression:

$$\Delta^{(\chi)} = \frac{\Delta^{(\mu)}\,.\,\rho_1\,.\,\rho_2}{i\sqrt{C}}; \tag{D$^{(8)}$}$$

that is, in passing from one point to another upon such a surface, or from a point upon one caustic surface to a point upon the other, the density of the reflected light varies directly as the density at the mirror multiplied by the product of the two focal distances, and inversely as the difference of these distances multiplied by the square root of the radius of curvature of the caustic curve. We see also that the definite integral (Q$''''$), which represents the area of the heartshaped curve that we considered in [68.], is equal to the first term of the development (B$^{(8)}$) for $S^{(r)}$,

$$\Pi = 4\epsilon^{\frac{3}{2}} r^{\frac{3}{2}} P^{-1}\,.\,\tan.\,v'\,.\int_0^1 .\sqrt{(1 - z^2)}\,.\,d\sqrt{z} = 4S'_{0,0}\,.\,I_{0,0}\,.\,r^{\frac{3}{2}}; \tag{E$^{(8)}$}$$

on which account we may call that curve a *pycnoid*, because if r be given, its area is proportional to the density at the caustic surface divided by the density at the mirror.

[70.] The expression that we have just found for the density at a caustic surface, becomes infinite in two cases, which require to be considered separately; namely, first when $i = 0$, that is, at the intersection of the two caustic surfaces, which, as I have shewn, reduces itself to a finite number of isolated points, the principal foci of the system; and secondly, when $C = 0$, that is, when the radius of curvature of the caustic curve vanishes. A point at which this latter circumstance takes place, is in general a cusp upon the caustic curve; and the locus of these points forms in general a curve consisting of two branches, each of which is a sharp edge on one of the two caustic surfaces. These cusps are also connected by remarkable relations, with the pencils to which the caustic curves belong; on which account we shall reserve the investigation of the density at such a cusp, until we come to treat more fully of the developable pencils of the system.

[71.] Let us then consider the points where the interval (i) vanishes, that is, let us investigate an expression for the density at a principal focus. In this case we have by the XIIth. section, the following approximate formulæ:

$$\left.\begin{aligned} x &= A\alpha^2 + 2B\alpha\beta + C\beta^2, \quad a = -\rho\alpha, \\ y &= B\alpha^2 + 2C\alpha\beta + D\beta^2, \quad b = -\rho\beta, \end{aligned}\right\} \qquad (\mathrm{F}^{(8)})$$

(x, y) being the coordinates of the point in which the near ray intersects the plane of aberration; (a, b) the coordinates of the point in which it intersects the perpendicular plane at the mirror; (ρ) the focal length or interval between these two planes; (α, β) the cosines of the angles which the near ray makes with the axes of (x) and (y), the given ray being the axis of (z); and (A, B, C, D) coefficients calculated in [62.], which have not the same meanings here, as in the four preceding paragraphs. These formulæ give, by elimination of α, the following biquadratic equation,

$$F'' \,.\, \beta^4 - 2\beta^2 \,.\, \{2(B^2 - AC)(By - Cx) + (AD - BC)(Ay - Bx)\} + (Ay - Bx)^2 = 0, \quad (\mathrm{G}^{(8)})$$

in which $F'' = (AD - BC)^2 - 4(B^2 - AC)(C^2 - BD)$; when F'' is negative, that is, when the principal focus is inside the little ellipses of aberration, [62.], this biquadratic $(\mathrm{G}^{(8)})$ has two of its roots real, and the other two imaginary; but when F'' is positive, that is, when the principal focus is outside those ellipses, then the roots are either all real, or all imaginary; so that in the first case, any given point (x, y), near the focus, will have two rays passing through it; whereas, in the second case, it will either have four such rays or none. As these two cases are thus essentially distinct, it will be convenient to consider them separately; let us therefore begin by investigating the density in the case where the principal focus is inside the little ellipses of aberration.

Ist. Case. $F'' < 0$.

[72.] In this case, if we consider any rectangle upon the plane of aberration, having for its four corners,

1st. $x,\ y$; 2d. $x + dx,\ y$; 3d. $x,\ y + dy$; 4th. $x + dx,\ y + dy$;

the rays that pass inside this little rectangle are diffused over two little parallelograms on the perpendicular plane at the mirror, the corners of the one being

$$1\text{st. } a,\ b; \quad 2\text{d. } a + \frac{da}{dx} \,.\, dx,\ b + \frac{db}{dx} \,.\, dx,$$

$$3\text{d. } a + \frac{da}{dy} \,.\, dy,\ b + \frac{db}{dy} \,.\, dy, \quad 4\text{th. } a + \frac{da}{dx} \,.\, dx + \frac{da}{dy} \,.\, dy,\ b + \frac{db}{dx} \,.\, dx + \frac{db}{dy} \,.\, dy,$$

and those of the other being composed in a similar manner of a', b'; a, b, a', b' being the two points in which the two rays that pass through the point (x, y) are crossed by the perpendicular plane at the mirror. The areas of these little parallelograms, have for expressions

$$\left(\frac{da}{dy} \cdot \frac{db}{dx} - \frac{da}{dx} \cdot \frac{db}{dy}\right) . \, dx \,.\, dy, \quad \left(\frac{da'}{dy} \cdot \frac{db'}{dx} - \frac{da'}{dx} \cdot \frac{db'}{dy}\right) . \, dx \,.\, dy,$$

and they are equal to one another, because $b' = -b$, $a' = -a$; also the area of the little retangle on the plane of aberration is $dx \,.\, dy$; if then we denote by $\Delta^{(\mu)}$ the density at the mirror, the density at the point (x, y) will be nearly

$$\Delta^{(\alpha)} = 2\Delta^{(\mu)} \left(\frac{da}{dy} \cdot \frac{db}{dx} - \frac{da}{dx} \cdot \frac{db}{dy}\right) \tag{H$^{(8)}$}$$

and it remains to calculate the coefficient in the second member. For this purpose, I observe that in general, when any four quantities a, b, x, y, are connected by two relations, so that a, b, are functions of x, y, and reciprocally, their partial differentials are connected by the following relation,

$$\left(\frac{da}{dy} \cdot \frac{db}{dx} - \frac{da}{dx} \cdot \frac{db}{dy}\right)\left(\frac{dy}{da} \cdot \frac{dx}{db} - \frac{dy}{db} \cdot \frac{dx}{da}\right) = 1; \tag{I$^{(8)}$}$$

it is sufficient therefore to calculate $\frac{dy}{da} \cdot \frac{dx}{db} - \frac{dy}{db} \cdot \frac{dx}{da}$. Now, the equations (F$^{(8)}$) give

$$\begin{aligned} \tfrac{1}{2}\rho^2 \,.\, dx &= (Aa + Bb) \,.\, da + (Ba + Cb) \,.\, db, \\ \tfrac{1}{2}\rho^2 \,.\, dy &= (Ba + Cb) \,.\, da + (Ca + Db) \,.\, db, \end{aligned}$$

$$\begin{aligned} \tfrac{1}{4} \,.\, \rho^4 \,.\, \left(\frac{dy}{da} \cdot \frac{dx}{db} - \frac{dy}{db} \cdot \frac{dx}{da}\right) &= (Ba + Cb)^2 - (Aa + Bb)(Ca + Db) \\ &= \rho^2 \,.\, \{(B\alpha + C\beta)^2 - (A\alpha + B\beta)(C\alpha + D\beta)\}; \end{aligned}$$

and if we put $(B\alpha + C\beta)^2 - (A\alpha + B\beta)(C\alpha + D\beta) = M$, we have by the same equations

$$\left.\begin{aligned} M \,.\, \alpha &= (B\alpha + C\beta) \,.\, y - (C\alpha + D\beta) \,.\, x, \\ M \,.\, \beta &= (B\alpha + C\beta) \,.\, x - (A\alpha + B\beta) \,.\, y; \end{aligned}\right\} \tag{K$^{(8)}$}$$

we have therefore

$$\Delta^{(\alpha)} = \frac{\Delta^{(\mu)} \,.\, \rho^2}{2M}, \quad M = \surd\{(By - Cx)^2 + (Ay - Bx)(Dx - Cy)\}. \qquad (\mathrm{L}^{(8)})$$

It results from this expression, that when the principal focus is inside the little ellipses of aberration considered in [62.], there exists another remarkable series of ellipses upon the plane of aberration, determined by the equation

$$M = \text{const.} \qquad (\mathrm{M}^{(8)})$$

and possessing this characteristic property that along every such ellipse the density of the reflected light is constant. The ellipses of this series ($\mathrm{M}^{(8)}$) are all concentric and similar, having their common centre at the principal focus, and having their axes situated on two remarkable lines, which are perpendicular to each other and to the given ray, and form with that ray *three natural axes* of coordinates passing through the principal focus.

[73.] Suppose then that we have taken for our axes of coordinates, the three natural axes just mentioned, the ray from which the aberrations are to be measured being still the axis of z, we shall have the relation

$$AD - BC = 0, \qquad (\mathrm{N}^{(8)})$$

and the expression for the density at a point (r, v) upon the plane of aberration will become

$$\Delta^{(\alpha)} + \frac{\frac{1}{2}\Delta^{(\mu)} \,.\, \rho^2 \,.\, r^{-1}}{\surd\{(B^2 - AC) \,.\, \sin.^2 v + (C^2 - BD) \,.\, \cos.^2 v\}}, \qquad (\mathrm{O}^{(8)})$$

in which $B^2 - AC$, $C^2 - BD$, will both be positive, and proportional to the squares of the semiaxes of the ellipses of uniform density. Multiplying this expression by $r \,.\, dr \,.\, dv$, and integrating from $r = 0$ to $r = r$, and from $v = 0$ to $v = 2\pi$, we find for the whole number of the near reflected rays that pass within a small given distance (r) from the focus, the following approximate formula:

$$\iint \Delta^{(\alpha)} \,.\, r\, dr\, dv = 2\Delta^{(\mu)} \,.\, \rho^2 \,.\, r \int_0^{\frac{\pi}{2}} \frac{dv}{\surd\{(B^2 - AC) \,.\, \sin.^2 v + (C^2 - BD) \,.\, \cos.^2 v\}}, \qquad (\mathrm{P}^{(8)})$$

a transcendental of known form, which can be calculated either by elliptic arcs or by series. And if we denote this transcendental by T, and reason as in [69.], we find the following expression for the density at the focus itself, as compared with the density at another point of the same kind,

$$\Delta^{(\phi)} = \Delta^{(\mu)} \,.\, \rho^2 \,.\, T. \qquad (\mathrm{Q}^{(8)})$$

IId. Case. $F'' > 0$.

[74.] Let us now consider the case where $F'' > 0$, that is, where the principal focus is outside the little ellipses, [62.]. In this case the points in which the near reflected rays intersect the plane of aberration, are all comprised within the angle formed by the two limiting lines (Y″), [62.], namely, the common tangents to those ellipses of aberration; and if we take the bisector of this angle for the axis of x, the relation ($\mathrm{N}^{(8)}$) will reappear, and the equation of the limiting lines will become

$$\frac{y^2}{x^2} = \frac{C^2 - BD}{AC - B^2} = \frac{C}{A} = \frac{D}{B}. \tag{$\mathrm{R}^{(8)}$}$$

Moreover, the rays that pass inside any little rectangle $dx.dy$, within the angle formed by these limiting lines, are at the mirror diffused nearly perpendicularly over four little parallelograms, which are equal to one another, and have their sum $= \dfrac{\rho^2 \,.\, dx \,.\, dy}{M}$, M being the same function as in [72.], we have, therefore, for the density at the point x, y, the following approximate expression,

$$\Delta_{\prime}^{(\alpha)} = \frac{\Delta^{(\mu)} \,.\, \rho^2}{M}, \tag{$\mathrm{S}^{(8)}$}$$

and the *lines of uniform density* are still given by the equation

$$M = \text{const.},$$

which now represents a series of concentric and similar hyperbolas, having the principal focus for their common centre, and the limiting lines of aberration for their common asymptotes. And if we multiply this expression for the density $\Delta_{\prime}^{(\alpha)}$ by $r\,dr\,dv$, and integrate from $r = 0$ to $r = r$, and from $v = -v'$, to $v = +v'$, v' being the semiangle between the limiting lines, and consequently

$$\tan.\, v' = \sqrt{\frac{C}{A}} = \sqrt{\frac{D}{B}}; \tag{$\mathrm{T}^{(8)}$}$$

we find for the whole number of the near rays that pass within a given distance r from the focus

$$\iint \Delta_{\prime}^{(\alpha)} \,.\, r\,dr\,dv = 2\Delta^{(\mu)} \,.\, \rho^2 \,.\, rT_{\prime}, \tag{$\mathrm{U}^{(8)}$}$$

and, therefore, for the density at the focus itself as compared with that at another point of the same kind,

$$\Delta^{(\phi)} = \Delta^{(\mu)} \,.\, \rho^2 T_{\prime}, \tag{$\mathrm{V}^{(8)}$}$$

$T_{\prime}$ denoting the transcendental

$$T_{\prime} = \int_0^{v'} \frac{dv}{\sqrt{\{(C^2 - BD) \,.\, \cos.^2 v - (AC - B^2) \,.\, \sin.^2 v\}}}. \tag{$\mathrm{W}^{(8)}$}$$

[75.] The preceding expressions may be put under other forms, some of which are simpler. Thus, if we still suppose the axes of coordinates to coincide with the natural axes determined by the equation ($\mathrm{N}^{(8)}$), so that the axis of the reflected system may be the axis of z, and the common transverse axis of the lines of uniform density the axis of x; if also we denote by $\dfrac{\Delta'}{x}$

the density of a point upon this latter axis, and by $\dfrac{\Delta''}{y}$ or $\dfrac{\Delta''\sqrt{-1}}{y}$ the density of a point upon the axis of y; we shall have ($\mathrm{O}^{(8)}$) ($\mathrm{S}^{(8)}$) the following approximate expressions for the density at any other point upon the plane of aberration,

$$\left(\frac{x^2}{\Delta'^2} \pm \frac{y^2}{\Delta''^2}\right)^{-\frac{1}{2}} = \Delta' r^{-1}(1 - e^2 \,.\, \sin.^2 v)^{-\frac{1}{2}}, \tag{$\mathrm{X}^{(8)}$}$$

r, v being the polar coordinates of the point, and e the excentricity of the ellipses or hyperbolas at which the density is constant; and the formulæ ($\mathrm{Q}^{(8)}$) ($\mathrm{V}^{(8)}$) for the density at the principal focus become

$$\left.\begin{aligned} \Delta^{(\phi)} &= \Delta' \,.\, \int_0^{\frac{\pi}{2}} \frac{dv}{(1 - e^2 \,.\, \sin.^2 v)^{\frac{1}{2}}}, \\ \Delta_{\prime}^{(\phi)} &= \Delta' \,.\, \int_0^{\frac{\pi}{2}} \frac{dv}{(1 - e^2 \,.\, \sin.^2 v)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{$\mathrm{Y}^{(8)}$}$$

e being less than unity in the first and greater in the second. With respect to the value of this excentricity, it is equal to the cosecant of the imaginary or real angle v' determined by the formula ($\mathrm{T}^{(8)}$); it is also connected with the position of the ellipses of aberration, [62.], by this remarkable relation, that the segments into which the principal focus divides that diameter of such an ellipse upon which it is situated, are proportional to the squares of the semiaxes of the lines of uniform density; in such a manner, that when the principal focus is situated at the centre of the ellipses of aberration, the excentricity e vanishes, and the lines of uniform density become a series of concentric circles; and when on the contrary, the principal focus is on the circumference of the ellipses of aberration, then e becomes equal to unity, and the lines of uniform density become a set of rectilinear parallels to the axis of y, which axis in this case coincides with the common tangent to the little ellipses of aberration, drawn through the principal focus. When the latter circumstance happens, the two expressions ($\mathrm{Y}^{(8)}$) for the density at this principal focus, coincide with one another, and become

$$\Delta' \,.\, \int_0^{\frac{\pi}{2}} \frac{dv}{\cos.\, v} = \infty; \tag{$\mathrm{Z}^{(8)}$}$$

in this case therefore, we should be obliged to have recourse to new calculations, and to introduce the consideration of aberrations of the third order. We may remark that the quantity F'', the sign of which distinguishes between the two chief cases of aberration from a principal focus, becomes $= 0$, in the case which we have just been considering; and since, by Section XII., the sign of this quantity F'' determines also the nature of the roots of the cubic equation

$$\frac{d^3V}{dx^3} \,.\, dx^3 + 3 \,.\, \frac{d^3V}{dx^2 \,.\, dy} \,.\, dx^2 \,.\, dy + 3 \,.\, \frac{d^3V}{dx \,.\, dy^2} \,.\, dx \,.\, dy^2 + \frac{d^3V}{dy^3} \,.\, dy^3 = 0,$$

which by the same section assigns the directions of spheric inflexion upon the surfaces of constant action, and of focal inflexion on the osculating focal mirror; it follows that, in the present case this cubic equation has two of its roots equal, and therefore that two of the directions of focal or of spheric inflexion coincide. With respect to the value of these equal roots, we have from our present choice of the coordinate planes the equations $A = 0$, $B = 0$, and therefore by [62.], $\dfrac{d^3V}{dx^3} = 0$, $\dfrac{d^3V}{dx^2 \,.\, dy} = 0$: thus the cubic equation becomes

$$3 \,.\, \frac{d^3V}{dx \,.\, dy^2} \,.\, dx \,.\, dy^2 + \frac{d^3V}{dy^3} \,.\, dy^3 = 0,$$

from which it follows that the two directions of inflexion which coincide with one another are contained in the plane of xz, that is, in a plane passing through the axis of the reflected system, and cutting perpendicularly the lines of uniform density.

[76.] Many other remarks remain to be made, in order to illustrate and complete the theory of the present section; but as we shall have occasion, in treating of refracted systems, to resume this theory under a more general point of view, we shall only here add, that the function which we have called the *density*, may differ sensibly in many instances from the observed intensity of light; because in calculating this function, we have abstracted from all physical causes not included in that fundamental law of catoptrics, which is expressed by our original equation,

$$\cos.\,\rho l + \cos.\,\rho' l = 2 \cos.\,I \,.\, \cos.\,nl, \qquad \text{(A)} \quad [1.]:$$

or in the resulting formula

$$\alpha\, dx + \beta\, dy + \gamma\, dz = dV,$$

α, β, γ being the cosines of the angles which the ray passing through the point $x\ y\ z$ makes with the axes of coordinates, and V the characteristic function. To distinguish therefore that mathematical affection of the system to which the preceding calculations relate, from the physical intensity of which it is an element, we may give to it a separate name, and call it the *Geometrical Density.*

CONCLUSION TO THE FIRST PART.

The preceding pages contain the execution of the first part of our plan; being an attempt to establish general principles respecting the systems of rays produced by the ordinary reflexion of light, at any mirror or combination of mirrors, shaped and placed in any manner whatsoever; and to shew that the mathematical properties of such a system may all be deduced by analytic methods from the form of ONE CHARACTERISTIC FUNCTION: as, in the application of analysis to geometry, the properties of a plane curve, or of a curve surface, may all be deduced by uniform methods from the form of the function which characterises its equation. It remains to extend these principles to other optical systems; to shew that in every such system, whether the rays be straight or curved, whether ordinary or extraordinary, there exists a Characteristic Function analogous to that which we have already pointed out for the case of the systems produced by the ordinary reflexion of light; to simplify and generalise the methods that we have given, for calculating from the form of this function all the other properties of the system; to integrate various equations which present themselves in the determination of mirrors, lenses, and crystals satisfying assigned conditions; to establish some more general principles in the theory of SYSTEMS OF RAYS, and to terminate with a brief review of our own results, and of the discoveries of former writers. But we have trespassed too long at present on the indulgence of mathematicians, and of the Academy, and must defer to another occasion the completion of this extensive design.

W. R. HAMILTON.

Observatory,
April 1828.