ON THE UNDULATORY TIME OF PASSAGE OF LIGHT THROUGH A PRISM

By

William Rowan Hamilton

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On the undulatory Time of Passage of Light through a Prism. By WILLIAM R. HAMILTON, *Esq. Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland**.

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Since I communicated my little paper, On the Effect of Aberration in prismatic Interference, I have seen Professor Airy's remarks on Mr. Potter's experiment; in which it is suggested, that the observed central points, which tended towards the thickness of the prism, were not the points of simultaneous arrival of two homogenenous streams. From the well-known experience and skill of Professor Airy as an observer, I think it likely that he has assigned the true physical explanation of Mr. Potter's instructive experiment; though I wish that this experiment were repeated, with careful micrometrical measures. But I continue to think the mathematical correction just, which I proposed in my recent paper. In that paper, I took Mr. Potter's own account of his experiment; namely, that he had found, in the plane perpendicular to the edge, a tendency *towards* the thickness, and *from* a certain intermedial line, in the locus of points of simultaneous arrival to two near homogeneous streams: and I endeavoured to show, that according to the undulatory theory, this locus *ought*, during a considerable range, to tend in this direction and not in the opposite;—a mathematical result, which was contrary to Mr. Potter's conclusion. It is, I hope, unnecessary to repeat the expression of my sincere respect for the gentleman from whom I have found myself obliged to differ on this mathematical question. But as I only stated, in my former paper, a correction of Mr. Potter's formula for the difference of times of arrival of two homogeneous streams, arising from the prismatic aberration of figure, and showed the influence of this aberrational correction on the course of the sought locus, without showing how I obtained the correction itself,—it may be useful to give here an outline of the method which I employ, for the treatment of this, and of other similar questions; referring, for more full details, to the recent and forthcoming volumes of the Transactions of the Royal Irish Academy.

Let light be supposed to go, in a bent path $ABCD$, from an initial point A to a final point D, through any prism ordinary or extraordinary, undergoing a first refraction at the point of entrance B, and a second refraction at the point of emergence C, the prism being placed *in vacuo*, and its angle being small or large; and let the position of the final point D be marked by three rectangular coordinates x, y, z, of which the origin is taken on the edge of the prism; and let the position of the initial point A be marked by three other rectangular coordinates x', y', z', having the same origin, but not necessarily the same axes: let α, β, γ, be the cosines of the angles which the emergent or final direction CD makes with the rectangular axes of x, y, z; and let α', β', γ', be the cosines of the angles which the incident or initial direction AB makes with the rectangular axes of x', y', z'; finally, let V be the undulatory

* Communicated by the Author.

time of propagation from the initial to the final point, measured by the equivalent path *in vacuo*; and let it be considered as a function of the initial and final coordinates, which, by the position that we have assigned to the origin, is homogeneous of the first dimension. We shall then have the two following equations, deduced from my general methods,

$$V = \alpha x + \beta y + \gamma z - \alpha' x' - \beta' y' - \gamma' z', \tag{1}$$

$$0 = x\,\delta\alpha + y\,\delta\beta + z\,\delta\gamma - x'\,\delta\alpha' - y'\,\delta\beta' - z'\,\delta\gamma'; \tag{2}$$

that is, V is to be determined as a function of the extreme coordinates $x\ y\ z\ x'\ y'\ z'$, which I have called in my Theory of Systems of Rays the *Characteristic Function*, by the condition that it shall be a maximum or minimum, with respect to the quantities α, β, γ, α', β', γ', of the expression (1): attending to the two general relations,

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \quad \alpha'^2 + \beta'^2 + \gamma'^2 = 1, \tag{3}$$

and to two other relations between the final and initial cosines of direction $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, which result, in each particular case, from the prismatic connexion between the incident and emergent directions. And when the form of the *characteristic function* V is known, the six extreme cosines of direction may be deduced from it, by differentiation, as follows:

$$\left.\begin{array}{rrr} \alpha = \dfrac{\delta V}{\delta x}, & \beta = \dfrac{\delta V}{\delta y}, & \gamma = \dfrac{\delta V}{\delta z}, \\ -\alpha' = \dfrac{\delta V}{\delta x'}, & -\beta' = \dfrac{\delta V}{\delta y'}, & -\gamma' = \dfrac{\delta V}{\delta z'}, \end{array}\right\} \tag{4}$$

When the prism is ordinary, such as glass, or when being extraordinary its edge is an axis of elasticity; and when we take the edge for the axis of z and of z', and consider only rays in a plane perpendicular to this edge, we may make,

$$\left.\begin{array}{l} z = 0, \quad z' = 0, \quad \gamma = 0, \quad \gamma' = 0, \\ \alpha = \cos\theta, \quad \beta = \sin\theta, \quad \alpha' = \cos\theta', \quad \beta' = \sin\theta', \end{array}\right\} \tag{5}$$

θ being the emergent inclination to the axis of x, and θ' being the incident inclination to the axis of x'; and the undulatory time V, corresponding to any given coordinates $x\ y\ x'\ y'$, is the maximum or minimum, relatively to θ, of the expression

$$V = x\cos\theta + y\sin\theta - x'\cos\theta' - y'\sin\theta', \tag{6}$$

in which θ' is to be considered as a function of θ, depending on the prismatic connexion between the initial and final directions.

For an ordinary prism *in vacuo*, having its angle $= \varpi$, and its index $= \mu$, so that

$$\sin i = \mu \sin\frac{\varpi}{2}, \tag{7}$$

i being the angle of external incidence corresponding to the minimum of deviation, the relation between θ, θ', is

$$\mu^2 \sin \varpi^2 = \sin(i+\theta)^2 + \sin(i-\theta')^2 + 2\cos\varpi \,.\, \sin(i+\theta) \,.\, \sin(i-\theta'), \tag{8}$$

if the positive semiaxis of x be an emergent ray of minimum deviation, and the positive semiaxis of x' the corresponding incident ray prolonged, while the positive semiaxes of y, y', lie on the same side of the axes of x, x', as the prism. The relation (8) may be put under the approximate form,

$$\theta' = \theta - \frac{m}{4} \,.\, \theta^2, \tag{9}$$

when the angles θ, θ', are small, that is, when we consider rays having nearly the minimum of deviation, m being the same positive number as in my last paper, namely,

$$m = \frac{8 \sin\left(i + \dfrac{\varpi}{2}\right) \sin\left(i - \dfrac{\varpi}{2}\right)}{\sin 2i \,.\, \left(\cos \dfrac{\varpi}{2}\right)^2}; \tag{10}$$

and if, besides, we consider the ordinates y, y', as small, that is, if we suppose the light to pass near the edge of the prism, and neglect terms of the fourth dimension with respect to the small quantities y, y', θ, we shall have the undulatory time or characteristic function $V =$ the maximum or minimum, relatively to θ, of the expression,

$$V = x - x' + (y - y')\theta - \tfrac{1}{2}\left(x - x' - \frac{m}{2}y'\right)\theta^2 - \frac{m}{4}x'\theta^2. \tag{11}$$

In this manner we find, with the same order of approximation,

$$V = x - x' + \tfrac{1}{2} \,.\, \frac{(y-y')^2}{x-x'} + \frac{m}{4} \,.\, \frac{(xy' - x'y)(y-y')^2}{(x-x')^3}, \tag{12}$$

a result which may also be thus expressed:

$$V = \frac{m}{2}y' + \sqrt{\left(x - x' - \frac{m}{2}y'\right)^2 + (y-y')^2} - \frac{mx'}{4}\left(\frac{y-y'}{x-x'}\right)^3. \tag{13}$$

If we neglected the last term of this last expression, it would give, by (4) and (5), the following formula for the tangent of the inclination of the emergent ray,

$$\tan\theta = \frac{\beta}{\alpha} = \frac{y - y'}{x - x' - \dfrac{m}{2}y'}; \tag{14}$$

it would therefore imply that all the rays which diverged before incidence from the luminous point or primary image x', y', diverge after emergence from a prismatic focus or secondary image, having for coordinates,

$$x = x' + \frac{m}{2}y', \quad y = y' : \tag{15}$$

so that the last term, $-\dfrac{mx'}{4}\left(\dfrac{y-y'}{x-x'}\right)^3$, of the expression (13) for the undulatory time V, may be considered as an aberrational term, arising from and determining the aberration (of figure, not of colour) of the prism. Accordingly Mr. Potter, neglecting this aberration of figure, did not perceive this term in the expression of the undulatory time, and was led to the results respecting the locus of points of simultaneous arrival of two near homogeneous streams, which I attempted in my last paper to correct. In comparing my present notation with my former, we are to make $x' = -l$, and $y' = \pm a$; and we are also to observe that the present origin of x and y is on the edge of the prism. It seemed useful to give the present outline of a proof of the results stated in my former paper; because the methods which I have introduced for the solution of optical problems differ much from those usually received; and because it would perhaps be difficult, by those usual methods, to investigate the influence of the prismatic aberration of figure, on the undulatory time of propagation of homogeneous light.

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