ON THE APPLICATION OF THE CALCULUS OF QUATERNIONS TO THE THEORY OF THE MOON

By

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Sir William Rowan Hamilton made a communication respecting the application of the Calculus of Quaternions to the Theory of the Moon.

I. At two Meetings of the Royal Irish Academy, in the month of July, 1845, Sir William Rowan Hamilton had exhibited and illustrated the following general equation of motion of a system of bodies, with masses $m, m', \ldots$, and with vectors $\alpha, \alpha', \ldots$, and attracting each other according to Newton's law:

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \Sigma \frac{m'}{(\alpha - \alpha')\sqrt{\{-(\alpha - \alpha')^2\}}}. \tag{1}$$

He had, at that time, deduced from this equation the known laws of the centre of gravity, of areas, and of living force, for any such multiple system; and had shown that the corresponding, but less general, equation of relative motion of a binary system, which (by changing $\alpha - \alpha'$ to α, and $m + m'$ to $\textsc{m}$) becomes

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{\textsc{m}}{\alpha\sqrt{(-\alpha^2)}}, \tag{2}$$

can be rigorously integrated by the processes of his new calculus of quaternions, so as to conduct, with facility, when the principles and plan have been caught, to the known laws of elliptic, parabolic, or hyperbolic motion of one of the two attracting bodies about the other. (See the Proceedings of July 14th and 21st, 1845, Appendix to Volume III., pp. xxxvii., &c.)

At a subsequent Meeting of the Academy, in December, 1845, Sir W. Hamilton has shown that the general differential equation (1) might be put under this other form:

$$0 = \tfrac{1}{2}\Sigma \,.\, m \left(\delta\alpha \,\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}\,\delta\alpha\right) + \delta\Sigma \frac{mm'}{\sqrt{\{-(\alpha - \alpha')^2\}}}; \tag{3}$$

and that it might, theoretically, be integrated by an adaptation of that "General Method in Dynamics" which he had previously published in the Philosophical Transactions of the Royal Society of London, for the years 1834 and 1835; and which depended on a peculiar combination of the principles of variations and partial differentials, already illustrated by him, in earlier years, for the case of mathematical optics, in the Transactions of this Academy. (See Proceedings of December 8th, 1845, Appendix already cited, pp. lii., &c.)

At the same meeting of December, 1845, Sir W. Hamilton assigned the two following rigorous differential equations for the internal motions of a system of *three* bodies, with masses m, m', m'', and with vectors α, $\beta+\alpha$, $\gamma+\alpha$,—that is, for the motions of the two latter of these three bodies (regarded as points) about the former,—as consequences of the general equation (1):

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} = \frac{m+m'}{\beta\sqrt{(-\beta^2)}} + m''\left\{\frac{(\beta-\gamma)^{-1}}{\sqrt{\{-(\beta-\gamma)^2\}}} + \frac{\gamma^{-1}}{\sqrt{(-\gamma^2)}}\right\}; \tag{4}$$

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} = \frac{m+m''}{\gamma\sqrt{(-\gamma^2)}} + m'\left\{\frac{(\gamma-\beta)^{-1}}{\sqrt{\{-(\gamma-\beta)^2\}}} + \frac{\beta^{-1}}{\sqrt{(-\beta^2)}}\right\}. \tag{5}$$

It was remarked, that by regarding m, m', m'', as representing respectively the masses of the earth, moon, and sun, β and γ become the geocentric vectors of the two latter bodies; and that thus the laws of the disturbed motion of our satellite are contained in the two equations (4) and (5)—but especially in the first of those equations (the second serving chiefly to express the laws of the sun's relative motion).

The part of this equation (4), which is independent of the sun's mass m'', is of the form (2), and contains the laws of the undisturbed elliptic motion of the moon; the remainder is the disturbing part of the equation, and contains the laws of the chief lunar perturbations. A commencement was made of the development of this disturbing part, according to ascending powers of the vector of the moon, and descending powers of the vector of the sun; and an approximate expression was thereby obtained, which may be written thus:

$$m''\left\{\frac{(\beta-\gamma)^{-1}}{\sqrt{\{-(\beta-\gamma)^2\}}} + \frac{\gamma^{-1}}{\sqrt{(-\gamma^2)}}\right\} = m''\frac{(\beta+3\gamma^{-1}\beta\gamma)}{2(-\gamma^2)^{\frac{3}{2}}}. \tag{6}$$

There was also given a geometrical interpretation of this result, corresponding to a certain decomposition of the sun's disturbing force into two others, of which the greater is triple of the less, while the angle between them is bisected by the geocentric vector of the sun; and the lesser of these two component forces is in the direction of the moon's geocentric vector prolonged, so that it is an ablatitious force, which was shown to be one of nearly constant amount.

Although the foregoing formulæ may be found in the Appendix already cited, to the Proceedings of the above-mentioned dates, yet it is hoped that, in consideration of the importance and difficulty of the subject, and the novelty of the processes employed, the Academy will not be displeased at having had this brief recapitulation laid before them, as preparatory to a sketch of some additional developments and applications of the same general view, which have since been made by the author. It may, for the same reason, be not improper here to state again, what was stated on former occasions, that all expressions involving *vectors* α, α', &c., such as are considered in this new sort of algebraical geometry, and enter into the foregoing equations, admit of being translated into others, which shall involve, instead of those vectors, three times as many rectangular *co-ordinates*, x, y, z, x', y', z', &c., by means of relations of the forms

$$\alpha = ix + jy + kz, \quad \alpha' = ix' + jy' + kz', \quad \text{\&c.}; \tag{7}$$

where $i\ j\ k$ are the three original and coordinate vector units of Sir William Hamilton's theory of quaternions, and satisfy the fundamental equations

$$\left.\begin{aligned} &i^2 = j^2 = k^2 = -1; \\ &ij = k, \quad jk = i, \quad ki = j; \\ &ji = -k; \quad kj = -i; \quad ik = -j; \end{aligned}\right\} \tag{8}$$

which were communicated to the Royal Irish Academy at the Meeting of the 13th November, 1843. (See the Proceedings of that date, and the author's First Series of Researches respecting Quaternions, which Series has lately been printed in the Transactions of the Academy, Vol. XXI. Part 2.)

II. It is evident, from inspection of the equations above recapitulated, that every transformation of the *vector function*,

$$\phi(\alpha) = \alpha^{-1}(-\alpha^2)^{-\frac{1}{2}} \tag{9}$$

which represents, in direction and amount, the attraction exerted by one mass-unit, situated at the beginning of the vector α, on another mass-unit situated at the end of that vector, must be important in the theory of the Moon; and generally in the investigation, by quaternions, of the mathematical consequences of the Newtonian Law of Attraction. The integration of the equation of motion (2) of a binary system was deduced, in the communication of July, 1845, from a transformation of that vector function, which may now be written thus:

$$\alpha^{-1}(-\alpha^2)^{-\frac{1}{2}} = \frac{2\mathrm{d}\,.\,\alpha(-\alpha^2)^{-\frac{1}{2}}}{\alpha\,\mathrm{d}\alpha - \mathrm{d}\alpha\,\alpha}; \tag{10}$$

where d is, as in former equations, the characteristic of differentiation. And the hodographic theory of the motion of a system of bodies, attracting each other according to the same Newtonian law, so far as it was symbolically stated to the Academy, at the meeting of the 14th of December, 1846, depends essentially on the same transformation. In fact, if we make

$$d\alpha = \tau\,\mathrm{d}t, \quad \alpha = \int \tau\,\mathrm{d}t; \tag{11}$$

and if, by the use of notations explained in former communications, we employ the letters U and V as the characteristics of the operations of taking the versor and the vector of a quaternion, writing, therefore,

$$\mathrm{U}(\alpha) = \alpha(-\alpha^2)^{-\frac{1}{2}}; \quad \mathrm{V}\,.\,\alpha\tau = -\mathrm{V}\,.\,\tau\alpha = \tfrac{1}{2}(\alpha\tau - \tau\alpha); \tag{12}$$

the equation (2) of the internal motion of a binary system becomes

$$\mathrm{d}\tau = \frac{-\mathrm{M}\,\mathrm{dU}\left(\int \tau\,\mathrm{d}t\right)}{\mathrm{V}\left(\tau \int \tau\,\mathrm{d}t\right)}; \tag{13}$$

where the denominator in the second member is constant, by the law of the equable description of areas. Hence, this second member, like the first, is an exact differential; and an immediate integration, introducing an arbitrary, but constant vector ϵ, coplanar with α and τ, gives the *law of the circular hodograph*, under the symbolical form

$$\tau = \frac{\mathrm{M}\left(\epsilon - \mathrm{U}\int \tau\,\mathrm{d}t\right)}{\mathrm{V}\,.\,\tau\int \tau\,\mathrm{d}t}: \tag{14}$$

the constant part of this expression (14) for the vector of the velocity, τ, being the vector of the centre of the hodograph, drawn from that one of the two bodies which is regarded as the centre of force; while the variable part of the same expression for τ represents the variable radius of the same hodographic circle, or the vector of a point on its circumference, drawn from its own centre of figure as the origin.

Multiplying this integral equation (14) by $\int \tau\,\mathrm{d}t$, taking the vector part of the product, dividing by M, and multiplying both members of the result into the constant denominator of the second member of (13) or of (14), we find, by the rules of the present calculus,

$$\frac{-\left(\mathrm{V}\,.\,\tau\int \tau\,\mathrm{d}t\right)^2}{\mathrm{M}} = \mathrm{S}\,.\,\epsilon\int \tau\,\mathrm{d}t + \mathrm{T}\,.\int \tau\,\mathrm{d}t; \tag{15}$$

where S and T are the characteristics of the operations of taking respectively the scalar and tensor of a quaternion, so that, as applied to the present question, they give the results,

$$\mathrm{T}\,.\int \tau\,\mathrm{d}t = \mathrm{T}\alpha = \sqrt{(-\alpha^2)} = r; \tag{16}$$

and

$$\mathrm{S}\,.\,\epsilon\int \tau\,\mathrm{d}t = \tfrac{1}{2}(\epsilon\alpha + \alpha\epsilon) = er\cos v; \tag{17}$$

where

$$e = \mathrm{T}\epsilon = \sqrt{(-\epsilon^2)} = \text{const.}; \tag{18}$$

while v is the *angle* (of true anomaly) which the variable *vector* α of the orbit makes with the fixed vector $-\epsilon$ in the plane of that orbit; and r denotes the *length* of α, or what is usually called (and may still in this theory be named) the *radius* vector of the relative orbit. The first member of the equation (15) is a positive and constant number, representing the quotient which is obtained when the square of the double areal velocity in the relative orbit is divided by the sum of the two masses; if then we denote, as usual, this constant quotient (or semiparameter) by p, and observe that the constant e is also numerical (expressing, as usual, the eccentricity of the orbit), we shall obtain again, by this process, as by that of July, 1845, the polar equation of the orbit, under the well-known form,

$$r = \frac{p}{1 + e\cos v}. \tag{19}$$

This sketch of a process for employing the general transformation (10) in the theory of a binary system, may make it easier, than it would otherwise be, to understand how the following equation for the motion of a multiple system,

$$d\tau = \Sigma \frac{(m+\Delta m)\,\mathrm{d}\mathrm{U}\left(\int \Delta\tau\,\mathrm{d}t\right)}{\mathrm{V}\left(\Delta\tau\,.\int \Delta\tau\,\mathrm{d}t\right)}, \tag{20}$$

(where $m+\Delta m$, $\tau+\Delta\tau$, are the mass and the vector of velocity of an attracting body, as m, τ are those of an attracted one, which is analogous to, and includes, the equation (13) for the motion of a binary one, and which agrees with a formula communicated to the Academy in December, 1846), was obtained by the present author; and how it may hereafter be applied.

III. The vector function $\phi(\alpha)$ in (9) may be called the TRACTOR corresponding to the vector of position α, or simply the tractor of α; and another general transformation of this tractor, which is more intimately connected than the foregoing with the problem of *perturbation*, may be obtained by supposing the vector α to receive any small but finite increment β, representing a new but shorter vector, which begins, or is conceived to be drawn, in any arbitrary direction, from the point of space where the vector α ends; and, by then *developing*, in conformity with the rules of quaternions, the *new tractor* $\phi(\beta+\alpha)$, (answering to the new vector $\beta+\alpha$, which is drawn from the beginning of α to the end of β), according to the ascending powers of this added vector β. In this manner we find

$$\begin{aligned}\phi(\beta+\alpha) &= \{-(\beta+\alpha)^2\}^{-\frac{1}{2}}(\beta+\alpha)^{-1}\\ &= \{-\alpha^2(1+\alpha^{-1}\beta)(1+\beta\alpha^{-1})\}^{-\frac{1}{2}}\{\alpha(1+\alpha^{-1}\beta)\}^{-1}\\ &= (1+\beta\alpha^{-1})^{-\frac{1}{2}}(1+\alpha^{-1}\beta)^{-\frac{3}{2}}\alpha^{-1}(-\alpha^2)^{-\frac{1}{2}};\end{aligned} \tag{21}$$

that is,

$$\phi(\beta+\alpha) = \Sigma_{n,n'}\phi_{n,n'} \tag{22}$$

if we make, for abridgment,

$$\phi_{n,n'} = m_{n,n'}(\beta\alpha)^n(\alpha\beta)^{n'}\alpha^{-1}(-\alpha^2)^{-\frac{1}{2}-n-n'} \tag{23}$$

where

$$m_{n,n'} = \frac{1\,.\,3\,\ldots\,(2n-1)}{2\,.\,4\,\ldots\,(2n)} \times \frac{3\,.\,5\,\ldots\,(2n'+1)}{2\,.\,4\,\ldots\,(2n')}. \tag{24}$$

The attraction $\phi(\beta+\alpha)$ which a mass-unit, situated at the beginning of the vector $\beta+\alpha$, exerts on another mass-unit situated situated at the end of that vector, is thus decomposed into an infinite but convergent series of other forces, $\phi_{n,n'}$, of which the *intensities* are determined by the *tensors*, and of which the *directions* are determined by the *versors*, of the expressions included in the formula (23); or by the following expressions, which are derived from it by the rules of the calculus of quaternions:

$$\mathrm{T}\phi_{n,n'} = m_{n,n'}\left(\mathrm{T}\frac{\beta}{\alpha}\right)^{n+n'}(\mathrm{T}\alpha)^{-2}; \tag{25}$$

$$\mathrm{U}\phi_{n,n'} = (\mathrm{U} \,.\, \beta\alpha)^{n-n'}(\mathrm{U}\alpha)^{-1} = \left(\mathrm{U}\frac{\beta}{-\alpha}\right)^{n-n'} \mathrm{U}(-\alpha). \tag{26}$$

Let a, b, be the lengths (or tensors) of the vectors α, β, and let C be the angle between them, which angle we may conceive to express the amount of the positive rotation, in their common plane, from the direction of $-\alpha$ to the direction of $+\beta$; then the positive or negative rotation in the same plane, from the same direction of $-\alpha$, to the direction of the component force $\phi_{n,n'}$, is measured as follows:

$$\textit{angle}\text{, from } -\alpha \text{ to force } \phi_{n,n'} = (n - n')\mathrm{C}; \tag{27}$$

and

$$\textit{intensity} \text{ of same component force} = m_{n,n'} \left(\frac{b}{a}\right)^{n+n'} a^{-2}. \tag{28}$$

The case $n = 0$, $n' = 0$, answers to the old tractor $\phi(\alpha)$, or to a force of which the intensity is represented by a^{-2}, while its direction is the same as that of $-\alpha$.

IV. Thus, if the vector α be conceived to being at a point B, and to end at a point C, while the vector β shall be conceived to begin at C and to end at A; and if we conceive a unit-mass at B to attract two other masses, regarded as collected into points, and as situated respectively at C and at A; this attraction of B will *disturb* the relative motion of A about C, if A be supposed to be nearer than B is to C, by producing a *series of groups of smaller and smaller forces*, of which groups it may be sufficient here to consider the two following.

The first and principal group consists of the two disturbing forces $\phi_{1,0}$ and $\phi_{0,1}$, and of these the first is purely *ablatitious*, or is directed along the prolongation of the side of the triangle ABC, which is drawn from C to A, and it has its intensity denoted by the expression $\frac{1}{2}ba^{-3}$, since we have for this force, and for its tensor and versor, the expressions

$$\phi_{1,0} = \tfrac{1}{2}\beta(-\alpha^2)^{-\frac{3}{2}}; \quad \mathrm{T}\phi_{1,0} = \tfrac{1}{2}ba^{-3}; \quad \mathrm{U}\phi_{1,0} = \mathrm{U}\beta. \tag{29}$$

The second disturbing force, of this last group, has for expression

$$\phi_{1,0} = \tfrac{3}{2}\alpha\beta\alpha^{-1}(-\alpha^2)^{-\frac{3}{2}} = \tfrac{3}{2}\alpha\beta\alpha^{-1}a^{-3}; \tag{30}$$

it intensity is *exactly triple* of that of the former force, being represented by $\frac{3}{2}ba^{-3}$; and its direction is the same as that of a straight line drawn from C to A$'$, if A$'$ be a point such that the line AA$'$ is perpendicularly bisected by the line BC (prolongued through C if necessary). These two principal disturbing forces evidently correspond to those which were considered for the case of our own satellite in a communication above alluded to; the second force being the one which was described in that former communication as being directed to what was there called the "fictitious moon," and was conceived to be as far from the sun in the heavens on one side, as the actual moon is on the other side, but in the same great circle.

If now we extend that mode of speaking so far as to conceive a similar *reflexion* of the sun with respect to the moon, and to call the point in the heavens so found the "fictitious sun," the moon being thus imagined to be seen midway among the stars between the actual and the fictitious sun: and if we farther imagine a "second fictitious sun," so placed that the actual sun shall appear to be midway between this and the first fictitious sun; we shall then be able to describe in words the directions of the three disturbing forces of the second group, and to say that they tend respectively, for the case of our own satellite, to these three (real or fictitious) suns. For these three *forces* will have, for their respective expressions, the three corresponding *terms* of the development of the *tractor* (22), namely, the following:

$$\phi_{2,0} = \tfrac{3}{8}\beta\alpha\beta(-\alpha^2)^{-\frac{5}{2}}; \quad \phi_{1,1} = \tfrac{3}{4}\beta^2\alpha(-\alpha^2)^{-\frac{5}{2}}; \quad \phi_{0,2} = \tfrac{15}{8}\alpha\beta\alpha\beta\alpha^{-1}(-\alpha^2)^{-\frac{5}{2}}; \tag{31}$$

of which the *intensities* are respectively

$$\tfrac{3}{8}b^2a^{-4}; \quad \tfrac{3}{4}b^2a^{-4}; \quad \tfrac{15}{8}b^2a^{-4}; \tag{32}$$

so that they are *exactly proportional to the three whole numbers*, 1, 2, 5; while they are *directed*, respectively, to the first fictitious sun, the actual sun, and the second fictitious sun. The *disturbing force of a superior planet*, exerted on an inferior one, may be developed or decomposed into a series of groups of lesser disturbing forces, the intensities of the several forces in each group being constantly proportional to whole numbers, in an exactly similar way; nor does the application of the principle and method of development thus employed terminate here. In the applications to the lunar theory, a and b, in the recent expressions, are to be regarded as denoting the variable distances of the sun and moon from the earth; and the expressions for the forces are to be multiplied by the mass of the sun. Nothing depends, so far, on any smallness of eccentricities or inclinations.

V. The lunar theory is, very approximately, contained in the differential equation (4), provided that we regard γ as the elliptic vector of the sun, drawn from the common centre of gravity of the earth and moon; and the laws of the sun's relative elliptic motion, with respect to that centre of gravity, are then contained in the following differential equation, which takes the place of the equation (5):

$$\frac{d^2\gamma}{dt^2} = \frac{m + m' + m''}{\gamma\sqrt{(-\gamma^2)}}. \tag{33}$$

Indeed, when we come to consider the small disturbing forces which belong to the second group, and which depend on the inverse fourth power of the sun's distance, the corresponding terms of the development of the first member of the formula (6) are then, for greater accuracy, to be multiplied by the fraction $\dfrac{m - m'}{m + m'}$, which expresses the ratio of the difference to the sum of the masses of the earth and moon. But if we neglect, for the present, those small disturbing terms, we may regard that formula (6) as accurate, without as yet neglecting anything on account of smallness of eccentricities or of inclinations; and even without assuming any knowledge of the smallness of the moon's mass, as compared with the mass of the earth; γ still denoting, as just stated, the elliptic vector of the sun. And thus, if the moon's geocentric vector β be changed to the sum $\beta + \delta\beta$, where the term $\delta\beta$ is supposed to depend on the disturbing force, and to give a product which may be neglected when it is multiplied by or into the expression for that force, we shall have the following approximate differential equation, by developing the disturbed or *altered tractor* $\phi(\beta + \delta\beta)$, and confining ourselves to the first power of $\delta\beta$:

$$\frac{\mathrm{d}^2\delta\beta}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} - \frac{m+m'}{\beta\surd(-\beta^2)} = \frac{m+m'}{2(-\beta^2)^{\frac{3}{2}}}(\delta\beta + 3\beta^{-1}\,\delta\beta\,\beta) + \frac{m''}{2(-\gamma^2)^{\frac{3}{2}}}(\beta + 3\gamma^{-1}\beta\gamma). \tag{34}$$

The *disturbance* $\delta\beta$ of the moon's geocentric *vector* is thus exhibited as giving rise to an *alteration* $\delta\phi(\beta)$ in the corresponding *tractor* $\phi(\beta)$, which alteration is *analogous to a disturbing force*, and occasions the presence of the first of the two parts of the second member of the equation (34): which equation will be found to contain a considerable portion of the theory of the moon.

VI. The author will only mention here two very simple applications, which he has made of this equation (34), one to the Lunar Variation, and the other to the Regression of the Node. Treating *here* the sun's relative orbit as exactly circular, and the moon's as approximately such, neglecting the inclination, taking for units of their kinds the sum of the masses of the earth and moon, and the moon's mean distance and mean angular velocity, and employing, as usual, the letter m to denote (not now the earth's mass, but) the ratio of the sun's mean angular motion to the corresponding motion of the moon, the differential equation (34) becomes:

$$\frac{\mathrm{d}^2\delta\beta}{\mathrm{d}t^2} = \tfrac{1}{2}(\delta\beta + 3\beta^{-1}\,\delta\beta\,\beta) + \frac{m^2}{2}(\beta + 3\gamma^{-1}\beta\gamma); \tag{35}$$

in which the laws of the circular revolutions of the vectors β and γ give

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} = -\beta; \quad \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} = -m^2\gamma. \tag{36}$$

Assuming, from some general indications of this theory, an expression for the perturbation of the moon's vector, which shall be of the form

$$\delta\beta = m^2(A\beta + B\gamma^{-1}\beta\gamma + C\beta^{-1}\gamma^{-1}\beta\gamma\beta), \tag{37}$$

and neglecting all powers of m above the square, we find

$$\frac{\mathrm{d}^2\delta\beta}{\mathrm{d}t^2} = -m^2(A\beta + B\gamma^{-1}\beta\gamma + 3^2C\beta^{-1}\gamma^{-1}\beta\gamma\beta); \tag{38}$$

$$\beta^{-1}\,\delta\beta\,.\,\beta = m^2(A\beta + C\gamma^{-1}\beta\gamma + B\beta^{-1}\gamma^{-1}\beta\gamma\beta); \tag{39}$$

so that the three numerical coefficients, A, B, C, must satisfy the three following equations of condition:

$$-A = 2A + \tfrac{1}{2}; \quad \text{giving} \quad A = -\tfrac{1}{6}; \tag{40}$$

and

$$-B = \tfrac{1}{2}(B + 3C) + \tfrac{3}{2}; \quad -9C = \tfrac{1}{2}(C + 3B); \tag{41}$$

giving

$$B = -\tfrac{19}{16}; \quad C = +\tfrac{3}{16}. \tag{42}$$

Thus, if we neglect eccentricities and inclination, and confine ourselves to the first power of the disturbing force, or to the second power of m, the perturbation of the moon's vector, produced by the sun's attraction, is composed of the three following terms:

$$\delta\beta = -\frac{m^2}{6}\beta - \frac{19m^2}{16}\gamma^{-1}\beta\gamma + \frac{3m^2}{16}\beta^{-1}\gamma^{-1}\beta\gamma\beta. \tag{43}$$

The first of these three terms expresses that the sun's ablatitious force, by partially counteracting the earth's attractive force on the moon, allows our satellite to revolve in a somewhat smaller orbit than would otherwise be consistent with the observed periodic time: the ratio of the diminished to the undiminished radius of the orbit being that of $1 - \dfrac{m^2}{6}$ to 1. The second term expresses a displacement of the moon, through perturbation, from its diminished circular orbit, of which displacement the constant magnitude or length bears to the radius of the undiminished orbit the ratio of $\dfrac{19m^2}{16}$ to unity; while the direction of this displacement is always *from* that fictitious moon, *to* which it has been seen that one of the two principal components of the sun's disturbing force is directed: an opposition of sign which may at first surprise, but which is exactly analogous to the *contraction* of the orbit produced by the *ablatitious* force (when the periodic time is given), and is to be explained upon similar principles. Finally, the third term of the formula (43) for $\delta\beta$, expresses that with the two foregoing displacements a third is to be combined, which is, like them, of constant amount, being equal to $\frac{3}{19}$ths of the second displacement, or bearing to the radius of the moon's orbit the ratio of $\dfrac{3m^2}{16}$ to unity; but being always directed *to* what, by an extension of a recently employed phaseology, might be called the second fictitious moon, being so placed that the actual moon is midway in the heavens between *this* fictitious moon and the one which was before considered. These two latter terms of (43) contain the chief laws of the *Lunar Variation*: and are easily shown to give the known terms in the expressions of the moon's parallax and longitude,

$$\delta\frac{1}{r} = m^2 \cos 2(☽ - \odot); \tag{44}$$

$$\delta\theta = \frac{11m^2}{8} \sin 2(☽ - \odot). \tag{45}$$

It may assist some readers to observe here, that when the inclination of the orbit is neglected, the longitudes of the first and second fictitious moons are, respectively,

$$2\odot - ☽, \quad \text{and} \quad 3☽ - 2\odot; \tag{46}$$

while those of the first and second fictitious suns, mentioned in a former section of this abstract, are, under the same condition,

$$2☽ - \odot, \quad \text{and} \quad 3\odot - 2☽. \tag{47}$$

VII. The law and quantity of the regression of the Moon's Node may also be calculated on principles of the kind above stated, but we must content ourselves with writing here the formula for the angular velocity of a planet's node generally, considered as depending on the variable *vector of position* α, the *vector of velocity* $\dfrac{\mathrm{d}\alpha}{\mathrm{d}t}$, and the *vector of acceleration* $\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}$, and also on a vector unit λ, supposed to be directed towards the north pole of a fixed ecliptic. The formula thus referred to is the following:

$$d\Omega = \frac{\mathrm{S} \,.\, \alpha\lambda \,.\, \mathrm{S} \,.\, \mathrm{d}^2\alpha \, \mathrm{d}\alpha \, \alpha}{(\mathrm{V} \,.\, \lambda \mathrm{V} \,.\, \alpha \, \mathrm{d}\alpha)^2}, \tag{48}$$

where S and V are, as before, the characteristics of the operations of taking the scalar and vector of a quaternion. The author proposes to give a fuller account of his investigations on this class of dynamical questions, when the Third Series of his Researches respecting Quaternions shall come to be printed in the Transactions of the Academy: the Second Series being devoted to subjects more purely geometrical; as the First Series (already printed) relates chiefly to others which are of a more algebraical character.