THE HODOGRAPH, OR A NEW METHOD OF EXPRESSING IN SYMBOLICAL LANGUAGE THE NEWTONIAN LAW OF ATTRACTION

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Sir William R. Hamilton made a communication respecting a new mode of geometrically conceiving, and of expressing in symbolical language, the Newtonian law of attraction, and the mathematical problem of determining the orbits and perturbations of bodies which are governed in their motions by that law.

Whatever may be the complication of the accelerating forces which act on any moving body, regarded as a moving point, and, therefore, however complex may be its *orbit*, we may always imagine a succession of straight lines, or vectors, to be drawn from some one point, as from a common origin, in such a manner as to represent, by their directions and lengths, the varying directions and degrees (or quantities) of the velocity of the moving point: and the curve which is the locus of the ends of the straight lines so drawn may be called the *hodograph* of the body, or of its motion, by a combination of the two Greek words, ὁδός, a *way*, and γράφω, *to write* or *describe*; because the vector of this hodograph, which may also be said to be the *vector of velocity* of the body, and which is always parallel to the tangent at the corresponding point of the orbit, marks out or indicates at once the direction of the momentary path or way in which the body is moving, and the rapidity with which the body, at that moment, is moving in that path or way. This hodographic curve is even more immediately connected than the orbit with the *forces* which act upon the body, or with the varying resultant of those forces, for the tangent to the hodograph is always parallel to the direction of this resultant; and if the element of the hodograph be divided by the element of the time, the quotient of this division represents (to the usual units) the intensity of the same resultant force; so that the whole accelerating force which acts on the body at any one instant is represented, both in direction and magnitude, by the element of the hodograph, divided by the element of the time. We have also the general proportion, that the *force is to the velocity*, in any varied motion of a point, *as the element of the hodograph is to the corresponding element of the orbit*.

These general remarks respecting varied motion, under the influence of *any* accelerating forces whatever, having been premised, let it be now supposed that the force is constantly directed towards some one *fixed point* or *centre*, which it will then be natural to choose for the origin of the vectors of the hodograph. The straight lines drawn to the moving body from the centre of force being called, in like manner, the vectors of the orbit, or the *vectors of position* of the body, we see that each such vector of position is now parallel to the tangent of the hodograph drawn at the extremity of the vector of velocity, as the latter vector was seen to be parallel to the tangent of the orbit, drawn at the extremity of the vector of position; so that the two vectors, and the two tangents drawn at their extremities, enclose at each moment a *parallelogram*, of which it is easily seen that the *plane and area are constant*, although its position and its shape are generally variable from one moment to another, in the motion thus performed under the influence of a central force. If, therefore, this constant area be given, and if either the hodograph or the orbit be known, the other of these two curves can be deduced, by a simple and uniform process, of which account the two curves themselves may be called *reciprocal hodographs*.

The opposite angles of a parallelogram being equal, it is evident, that if the central force be attractive, any one vector of position is inclined to the next following element of the orbit as the same angle as that at which the corresponding vector of velocity is inclined to the next preceding element of the hodograph. Also, if from either extremity of any small element of the curve, a chord of the circle which osculates to that curve along that element be drawn and bisected, the element subtends, at the middle point of this chord, an angle equal to the angle between the two tangents drawn at the two extremities of the element; that is, here, if the curve be the hodograph, to the angle between the two near vectors of position, which are parallel to the two extreme tangents of its element. We have, therefore, two small and similar triangles, from which results the following proportion, that *the half chord of curvature of the hodograph* (passing through, or tending towards the fixed centre of force) *is to the radius vector of the orbit* as the element of the hodograph is to the element of the orbit, that is, by what was lately seen, *as the force is to the velocity.**

But also, the radius of curvature of the hodograph is to the half chord of curvature of the same curve, as the radius vector of the orbit is to the perpendicular let fall from the fixed centre on the tangent to the same orbit; therefore, by compounding equal ratios, we obtain this other proportion: *the radius of curvature of the hodograph is to the radius vector of the orbit, as the rectangle under the same radius vector and the force is to the rectangle under the velocity and the perpendicular*, or to the constant parallelogram under the vectors of position and velocity. If, therefore, the law of the inverse square hold good, so that the second and third terms of this last proportion vary inversely as each other, while the fourth term remains unchanged, the first term must be also constant; that is, *with Newton's law of force* (supposed here to act towards a fixed centre) *the curvature of the hodograph is constant*: and, consequently, this curve, having been already seen to be *plane*, is now perceived to be a *circle*, of which the radius is equal to the attracting mass divided by the constant double areal velocity of the orbit. Reciprocally, we see that *no other law* of force would conduct to the same result: so that the Newtonian law may be *characterized* as being the *Law of the Circular Hodograph.*

Another mode of arriving at the same simple but important result, is to observe, that because the radius of curvature of the hodograph is equal to the element of that curve, divided by the angle between the tangents at its extremities, or (in the case of a central force) by the angle between two corresponding vectors of the orbit, which angle is equal to the double of the elementary area divided by the square of the distance (of the body from the centre of

* By an exactly similar reasoning, the following known proportion may be proved anew, namely, that the force is to the velocity as that velocity is to the half chord of curvature of the orbit, whatever the law of central force may be.

force), while the element of the hodograph has been seen to be equal to the force multiplied by the element of time, or multiplied by the same double element of orbital area, and divided by the constant of double areal velocity, therefore this radius of curvature of the hodograph must, for any central force, be equal to the force multiplied by the square of the distance, and divided by the double areal velocity.

The point on the hodograph which is the termination of any one vector of velocity may be called the *hodographic representative* of the moving body, and the foregoing principles show, that with a central force varying as the inverse square of the distance, this representative point describes, in any proposed interval of time, a *circular arc*, which contains the same number of degrees, minutes and seconds, as the angle contemporaneously described round the centre of force by the body itself in its orbit, or by the revolving vector of position; because, whatever that angle may be, an equal angle is described in the same time by the revolving tangent to the hodograph. Thus, with the law of Newton, *the angular motion of a body in its orbit is exactly represented, with all its variations, by the circular motion on the hodograph*; and this remarkable result may be accepted, perhaps, as an additional motive for the use of the new term which it is here proposed to introduce.

Whatever the law of central force may be, if the square of the velocity in the orbit be subtracted from the double rectangle under the force and distance, which has been seen to be equal to the rectangle under the same velocity and the chord of curvature of the hodograph, the remainder is the rectangle under the segments into which that chord is cut by the centre of force, being positive when this section takes place internally, but negative when the section is external, that is,when the centre of force is outside the osculating circle of the hodograph. In the case of the law of the inverse square, this latter curve is its own osculating circle, and the rectangle under the segments of the chord is, therefore, constant, by an elementary theorem of geometry contained in the third book of Euclid; if, then, the square of the velocity be subtracted from the double of the attracting mass, divided by the distance of the body from the centre of force, at which that mass is conceived to be placed, the remainder is a constant quantity, which is positive if the orbit be a closed curve, that is, if the centre of force be situated in the *interior* of the circular hodograph.

In the case of a closed orbit, the positive constant, which is thus equal to the product of the segments of a hodographic chord, or *the constant product of any two opposite velocities of the body* is easily seen, by the foregoing principles, to be equal to *the attracting mass divided by the semisum of the two corresponding distances of the body*, which semisum is therefore seen to be *constant* and may be called (as in fact it is) the *mean distance.* The *law of living force*, involving this mean distance, may therefore be deduced as an elementary consequence of this mode of hodographic representation, for the case of a closed orbit; together with the corresponding forms of this law, involving a null or a negative constant, instead of the reciprocal of the mean distance, for the two cases of an orbit which is not closed, namely, when the centre of force is on, or is outside the circumference of the hodographic circle.

Whichever of these situations the centre of force may have, we may call the straight *line* drawn from it to the centre of the hodograph, the hodographic *vector of eccentricity*; and the *number* which expresses the ratio of the length of this vector to the radius of the hodograph will represent, if the orbit be closed, the ratio of the semidifference to the semisum of the two extreme distances of the body from the centre of force, and may be called generally the *numerical eccentricity* of the hodograph, or of the orbit (without violating the received meaning of the term).

Whatever the value of this numerical eccentricity may be, the constant area of the parallelogram under the vectors of position and velocity may always be treated as the sum or difference of two other parallelograms, of which one is equal to the rectangle under the constant radius of the hodographic circle and the varying radius vector of the orbit, while the other is equal to the parallelogram under the vectors of position and eccentricity; and hence it is not difficult to infer that the length of the vector of position, or of the radius vector of the orbit, varies in a constant ratio, expressed by the numerical eccentricity, to the perpendicular let fall from its extremity, that is, from the position of the body, on a constant straight line or *directrix*, which is situated in the plane of the orbit, and is parallel to the hodographic vector of eccentricity. The *orbit*, therefore, whether it be closed or not, is always (with the law of the inverse square) a *conic section*, having the centre of force for a *focus*—a theorem which has indeed been known since the time of Newton, but has not perhaps been proved before from principles so very elementary.*

Conceive a diameter of the hodograph to be drawn in a direction perpendicular to the vector of eccentricity; the extremities of this diameter correspond to the extremities of that chord of the orbit which is perpendicular to the shortest radius vector, and which is called the *parameter*; from which it follows that the *semiparameter of the orbit is equal to the constant area of the parallelogram under distance and velocity, divided by the radius of the hodograph*, and, consequently, that it is equal to the *square of the constant double areal velocity, divided by the attracting mass.*

It is evident that these results agree with and illustrate those by which Newton shewed that Kepler's laws were mathematical consequences of his own great law of attraction. In applying them to the undisturbed motion of any binary system of bodies, attracting each other according to that law, we have only to substitute the sum of the two masses for the single attracting mass already considered, and to treat one of the two bodies as if it were the fixed origin of the vector of a *relative hodograph*, which will still be circular as before. And even if we consider a ternary, or a *multiple system*, we may still regard *each* body as *tending* by its attraction, to cause every *other* to describe an orbit of which the hodographic representative would be a perfect circle.

When there is *one predominant mass*, as in the case of the solar system, we may in general regard each other body of the system as moving in an orbit about it, which is, on the same plan, *represented* by a *varying circular hodograph.* For if, at any one moment, we know *the two heliocentric vectors* of position and velocity of a planet, we know the plane and area of the parallelogram under those two vectors, and can conceive a parellelepiped constructed, of which this momentary parallelogram shall be the base, while the volume of the solid shall

* The *hodograph of the earth's annual motion* may be considered to be exhibited to *observation* in astronomy as the *curve of aberration of a star*; and it is known that this aberratic curve is a *circle*, notwithstanding the eccentricity of the earth's orbit; but the author is not aware that this circularity of the aberratic curve (for a star near the pole of the ecliptic) has ever been shown before to be a *consequence of the law of the inverse square*, except by the help of the properties of the *elliptic orbit*; whereas the spirit of the present communication is the *derive that orbit from the circle*, and to regard that circle itself as a sort of geometrical *picture of Newton's law*, instead of being only one of many corollaries from the laws of Kepler.

represent the sum of the masses of the sun and planet; and then the height of the same solid will be equal to the radius of the momentary hodograph; so that, in order to construct this hodograph, we shall only have to describe, in the plane and with the radius determined as above, a circle which shall touch the side parallel to the heliocentric vector of position, at the extremity of the vector of velocity, and shall have its concavity, at the point of contact, turned towards the sun. The moon, or any other satellite, may also be regarded as describing, about its primary, an orbit of which the hodographic representative shall still be a varying circle.

As formulæ which may assist in symbolically tracing out the consequences of this geometrical conception, Sir William Hamilton offers the following transformations of certain general equations for the motion of a system of bodies attracting each other according to Newton's law, which he communicated to the Royal Irish Academy in July, 1845. (See Proceedings, vol. III, part 2, Appendix III and V.)

The new forms of the equations are these:

$$\rho = \int \tau \, dt; \quad \sigma = \frac{m'}{\mathrm{V} \, . \, (\rho' - \rho)(\tau' - \tau)}; \quad \tau = \sum \int \sigma \, d\mathrm{U}(\rho' - \rho);$$

in which ρ and τ are the vectors of position and velocity of the mass m at the time t; ρ' and τ' are the two corresponding vectors of another mass m' at the same time; σ is another vector, perpendicular to the plane, and equal in length to the radius of the *momentary relative hodograph, representing the momentary relative orbit*, which the attraction of the mass m' tends to cause the body m to describe; d, $\int$, $\sum$ are marks of differentiation, integration and summation, and V, U are the characterics of operations of taking respectively the vector and versor of a quaternion. Or, eliminating ρ and σ, but retaining the *hodographic vector* τ, and using Δ as the mark of differencing, the conditions of the question may be included in the following formula, which the author hopes on a future occasion to develope:

$$\tau = \sum \int \frac{(m + \Delta m) \, d\mathrm{U} \left(\int \Delta\tau \, dt \right)}{\mathrm{V}(\Delta\tau \, . \int \Delta\tau \, dt)}.$$

Meanwhile it is conceived that any such attempt as the foregoing, to simplify or even to transform the important and difficult problem of investigating the mathematical consequences of the Newtonian law of attraction, is likely to be received at the present time with peculiar indulgence and interest, in consequence not only of the brilliant deductive discovery lately made of the new planet exterior to Uranus, but also of the extraordinary and exciting intelligence which has just arrived from Dorpat of the presumed discovery by Professor Mädler of a *central cluster* (the Pleiades), and of a *central sun* (Alcinoe, also called Eta Tauri): around which cluster, and which sun or star, it is believed by Mädler that our own sun and all the other stars of our sidereal system, including the milky way, but exclusive of the more distant nebulæ, are moving in enormous orbits, under the combined influences of their own mutual attractions, all regulated by the same great law.