ON “GAUCHE” POLYGONS IN CENTRAL SURFACES OF THE SECOND ORDER

By

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Sir William Rowan Hamilton gave an account of some geometrical reasonings, tending to explain and confirm certain results to which he had been previously conducted by the method of quaternions, respecting the inscription of gauche polygons in central surfaces of the second order.

1. It is a very well known property of the conic sections, that if three of the four sides of a plane quadrilateral inscribed in a given plane conic be cut by a rectilinear transversal in three given points, the fourth side of the same variable quadrilateral is cut by the same fixed right line in a fourth point likewise fixed. And whether we refer to the relation of involution discovered by Desargues, or employ other principles, it is easy to extend this property to *surfaces* of the second order, so far as the inscription in them of *plane* quadrilaterals is concerned. If then we merely wish to pass from one point P to another point R of such a surface, under the condition that some other point Q of the same surface shall exist, such that the two successive and rectilinear chords, PQ and QR, shall pass respectively through some two given *guide-points*, A and B, internal or external to the surface; we are allowed to *substitute*, for this pair of guide-points, *another pair*, such as B′ and A′, situated *on the same straight line* AB; and may choose *one* of these two new points *anywhere* upon that line, provided that the *other* be then suitably chosen. In fact, if C and C′ be the two (real or imaginary) points in which the surface is crossed by the given transversal AB, we have only to take care that the three pairs of points AA′, BB′, CC′, shall be in involution. And it is important to observe, that in order to determine one of the new guide-points, B′ and A′, when the other is given, it is by no means necessary to employ the points C, C′, of intersection of the transversal with the surface, which may be as often imaginary as real. We have only to *assume* at pleasure a point P upon the given surface; to draw from it the chords PAQ, QBR; and then if A′ be given, and B′ sought, to draw the two new chords RA′S, SB′P; or else if A′ is to be found from B′, to draw the chords PB′S, SA′R. For example, if we choose to throw off the new guide-point B′ to infinity, or to make it a *guide-star*, in the direction of the given line AB, we have only to draw, from the assumed initial and superficial point P, a rectilinear chord PS of the surface, which shall be parallel to AB, and then to join SR, and examine in what point A′ this joining line crosses the given line AB. The point A′ *thus* found will be entirely independent of the assumed initial point P, and will satisfy the condition required: in such a matter that if, from any *other* assumed superficial point P′, we draw the chords P′AQ′, Q′BR′, and the parallel P′S′ to AB, the chord R′S′ shall pass through the *same* point A′. All this follows easily from principles perfectly well known.

2. Since then for *two* given guide-points we may thus substitute the system of a guide-star and a guide-point, it follows that for *three* given guide-points we may substitute a guide-star and two guide-points; and, therefore, by a repetition of the same process, may substitute anew a system of two stars and one point. And so proceeding, for a system of n given guide-points, through which n successive and rectilinear chords of the surface are to pass, we may substitute a system of $n-1$ guide-stars, and of a single guide-point. The problem of inscribing, in a given surface of the second order, a gauche polygon of n sides, which are required to pass successively through n given points, is, therefore, in general, reducible, by operations with straight lines alone, to the problem of inscribing in the same surface another gauche polygon, of which the *last* side shall pass through a new fixed point, while all its *other* $(n-1)$ sides shall be parallel to so many fixed straight lines. And if the *first* n sides of an inscribed polygon of $n+1$ sides, $\mathrm{PP}_1\mathrm{P}_2 \ldots \mathrm{P}_n$, be obliged to pass, in order, through n given points, $\mathrm{A}_1\mathrm{A}_2 \ldots \mathrm{A}_n$, namely, the side or chord PP_1 through A_1, &c., it will then be possible, in general, to inscribe also *another* polygon $\mathrm{PQ}_1\mathrm{Q}_2 \ldots \mathrm{P}_n$, having the same first and nth points, P and P_n, and therefore the *same final* or closing side $\mathrm{P}_n\mathrm{P}$, but having the other n sides *different*, and such that the $n-1$ first of these sides, $\mathrm{PQ}_1, \mathrm{Q}_1\mathrm{Q}_2, \ldots \mathrm{Q}_{n-2}\mathrm{Q}_{n-1}$, shall be respectively parallel to $n-1$ given right lines, while the nth side $\mathrm{Q}_{n-1}\mathrm{P}_n$ shall pass through a fixed point B_n. The analogous reductions for polygons in conic sections have long been familiar to geometers.

3. Let us now consider the inscribed gauche quadrilateral $\mathrm{PQ}_1\mathrm{Q}_2\mathrm{Q}_3$, of which the four corners coincide with the four points of the last-mentioned polygon. In the plane $\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$ of the second and third sides of this gauche quadrilateral, draw a new chord $\mathrm{Q}_1\mathrm{R}_2$, which shall have its direction conjugate to the direction of PQ_1, with respect to the given surface. This new direction will itself be fixed, as being parallel to a fixed plane, and conjugate to a fixed direction, not generally conjugate to that plane; and hence in the plane inscribed quadrilateral $\mathrm{R}_2\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3$, the three first sides having fixed directions, the fourth side $\mathrm{Q}_3\mathrm{R}_2$ will also have its direction fixed: which may be proved, either as a limiting form of the theorem referred to in (1), respecting four points in one line, or from principles still more elementary. And there is no difficulty in seeing that because PQ_1 and $\mathrm{Q}_1\mathrm{R}_2$ have fixed and conjugate directions, the chord PR_2 is bisected by a fixed diameter of the surface, whose direction is conjugate to both of their's; or in other words, that if O be the centre of the surface, and if we draw the *variable* diameter PON, the variable *chord* NR_2 will then be parallel to the *fixed* diameter just mentioned. So far, then, as we only concern ourselves to construct the fourth or closing side $\mathrm{Q}_3\mathrm{P}$ of the gauche quadrilateral $\mathrm{PQ}_1\mathrm{Q}_2\mathrm{Q}_3$, whose three first sides have given or fixed directions, we may substitute for it another gauche quadrilateral $\mathrm{PNR}_2\mathrm{Q}_3$, inscribed in the same surface, and such that while its first side PN passes through the centre O, its second and third sides, NR_2 and $\mathrm{R}_2\mathrm{Q}_3$, are parallel to two fixed right lines. In other words, we may substitute, for a system of *three guide-stars*, a system of the *centre and two stars*, as guides for the three first sides; or, if we choose, instead of drawing successively three chords, PQ_1, $\mathrm{Q}_1\mathrm{Q}_2$, $\mathrm{Q}_2\mathrm{Q}_3$, parallel to three given lines, we may draw a first chord PR_2, so as to be bisected by a given diameter, and then a second chord $\mathrm{R}_2\mathrm{Q}_3$, parallel to a given right line.

4. Since, for a system of *three* stars, we may substitute a system of the centre and *two* stars, it follows that for a system of *four* stars we may substitute a system of the centre and

three stars; or, by a repetition of the same process, may substitute a system of the centre, the same centre *again*, and two stars; that is, ultimately, a system of *two* stars may be substituted for a system of *four* stars, the two employments of the centre as a guide having simply neutralized each other, as amounting merely to a *return* from N to P, after having *gone* from P to the diametrically opposite point N. For five stars we may therefore substitute three; and for six stars we may substitute four, or two. And so proceeding we perceive that, for *any* proposed system of guide-stars, we may substitute *two* stars, if the proposed number be even; or *three*, if that number be odd. And by combining this result with what was found in (2), we see that for any given system of n guide-points we may substitute a system of *two stars and a point*, if n be *odd*; or if n be *even*, then in that case we may substitute a system of *three* stars and a point: which may again be changed, by (3), to a system of the *centre, two stars, and one point.*

5. Let us now consider more closely the system of two guide-stars, and one guide-point; and for this purpose let us conceive that the two first sides PQ_1 and Q_1Q_2 of an inscribed gauche quadrilateral $PQ_1Q_2P_3$ are parallel to two given right lines, while the third side Q_2P_3 is obliged to pass through a fixed point B_3; the first point P, and therefore also the quadrilateral itself, being in other respects variable. In the plane PQ_1Q_2 of the two first sides, which is evidently parallel to a fixed plane, inscribe a chord Q_2S, whose direction shall be conjugate to that of the fixed line OB_3, and therefore shall itself also be fixed, O being still the centre of the surface; and draw the chord PS. Then, in the plane inscribed quadrilateral PQ_1Q_2S, the three first sides have fixed directions, and therefore, by (3), the direction of the fourth side SP is also fixed. In the plane SQ_2P_3, which contains the given point B_3, draw through that point an indefinite right line B_3C_3, parallel to SQ_2; the line so drawn will have a given position, and will be intersected, at some finite or infinite distance from B_3, by the chord SP_3, which is situated in the same plane with it, namely, in the plane SQ_2P_3. But if we consider the section of the surface, which is made by this last plane, and observe that the two first sides of the triangle SQ_2P_3 pass, by the construction, through a star or point at infinity conjugate to B_3, and through the point B_3 itself, we shall see that, in virtue of a well-known and elementary principle respecting triangles in conics, the third side P_3S must pass through the point D_3, if D_3 be the pole of the right line B_3C_3, which contains upon it the two conjugate points; this *pole* being taken with respect to the plane *section* lately mentioned. If then we denote by D_3E_3 the indefinite right line which is, with respect to the *surface*, the *polar* of the fixed line B_3C_3, we see that the chord SP_3 must intersect this reciprocal polar also, besides intersecting the line B_3C_3 itself. Conversely this condition, of intersecting these two fixed polars, is sufficient to enable us to draw the chord SP_3 when the point S has been determined, by drawing from the assumed point P the chord PS parallel to a fixed right line. We may then *substitute*, for a system of two guide-stars and one guide-point, the system of *one guide-star* and *two guide-lines*; these *lines* being (as has been seen) a pair of *reciprocal polars*, with respect to the given surface.

6. If, then, it be required to inscribe a polygon $PP_1P_2 \ldots P_{2n}$ with any odd number $2n + 1$ of sides, which shall pass successively through the same number of given points, $A_1A_2 \ldots A_{2n+1}$, we may begin by *assuming* a point P upon the given surface, and drawing through the given points $2n + 1$ successive chords, which will in general conduct to a final

point P_{2n+1}, *distinct* from the assumed initial point P. And then, by processes of which the nature has been already explained, we can find a point S such that the chord PS shall be parallel to a fixed right line, or shall have a direction independent of the assumed and variable position of P; and that the chord SP_{2n+1} shall at the same time cross two other fixed right lines, which are reciprocal polars of each other. In order then to find a *new* point P, which shall satisfy the conditions of the proposed problem, or shall be such as to *coincide* with the point P_{2n+1}, deduced from it as above, we see that it is necessary and sufficient to oblige this sought point P to be situated at one or other extremity of a certain chord PS, which shall at once be parallel to a fixed line, and shall also cross two fixed polars. It is clear then that we need only draw two planes, containing respectively these two polars, and parallel to the fixed direction; for the right line of intersection of these two planes will be the *chord of solution* required; or in other words, it will cut the surface in the two (real or imaginary) points, P and S, which are adapted, and are alone adapted, to be positions of the first corner of the polygon to be inscribed.

7. But if it be demanded to inscribe in the same surface a polygon $PP_1P_2 \ldots P_{2n-1}$, with an *even* number $2n$ of sides, passing successively through the same *even* number of given points, $A_1\,A_2 \ldots A_{2n}$, *the problem then acquires a character totally distinct.* For if, after assuming an initial point P upon the surface, we pass, by $2n$ successive chords, drawn through the given points A_1, &c., to a final point P_{2n} upon the surface, which will thus be in general distinct from P; it will indeed be possible to assign generally two fixed polars, across which, as two given guide-lines, a certain variable chord SP_{2n} is to be drawn, like the chord SP_{2n+1} of (6); but the chord PS will *not*, in *this* question, be *parallel to a given line*, or directed to a given star; it will, on the contrary, by (3) (4) (5), be *bisected by a given diameter*, which we may call AB; or, we prefer to state the result so, it will be now the *supplementary chord* NS of the same diametral section of the surface (N being still the point of that surface *opposite* to P), which will have a given direction, and *not* the chord PS itself. In fact, at the end of (4), we reduced the system of $2n$ guide-points to a system of the centre, two stars, and one point; and in (5) we reduced the system of two stars and a point to the system of a star and two polars. In order then to find a point P which shall *coincide* with the point P_{2n} deduced from it as above, or which shall be adapted to be the first corner of an inscribed polygon of $2n$ sides passing respectively through the $2n$ given points, $A_1 \ldots A_{2n}$, we must endeavour to find a chord PS which shall be at once bisected by the fixed diameter AB, and shall *also* intersect the two fixed polars above mentioned. And conversely, if we can find any such chord PS, it will necessarily be at least *one chord of solution* of the problem; understanding hereby, that if we set out with *either* extremity, P or S, and draw from it $2n$ successive chords PP_1, &c., or SS_1, &c., through the $2n$ given points A_1, &c., we shall be brought *back* hereby (as the question requires) to the point with which we started. For, in a process which we have proved to admit of being *substituted* for the process of drawing the $2n$ chords, we shall be brought first from P to S, and then back from S to P; or else first from S to P, and then back from P to S: provided that the chord of solution PS has been selected so as to satisfy the conditions above assigned.

8. *To inscribe then*, for example, *a gauche chiliagon in an ellipsoid*, $PP_1 \ldots P_{999}$, or $SS_1 \ldots S_{999}$, under the condition that *its thousand successive sides shall pass successively*

through a thousand given points $A_1 A_2 \ldots A_{1000}$, we are conducted to seek to inscribe, in the same given ellipsoid, a *chord* PS, which shall be at once *bisected by a given diameter* AB, and also *crossed by a given chord* CD *and by the polar of that given chord.* Now in general when any two proposed right lines intersect each other, their respective polars also intersect, namely, in the pole of the plane of the two lines proposed. Since then the sought chord PS intersects the polar of the given chord CD, it follows that the polar of the same sought chord PS must intersect the given chord CD itself. We may therefore reduce the problem to this form: to find a chord PS of the ellipsoid which shall be bisected by a given diameter AB, and shall also be such that while it intersects a given chord CD in some point E, its polar intersects the prolongation of that given chord, in some other point F.

9. The two sought points E, F, as being situated upon two polars, are of course *conjugate* relatively to the *surface*; they are therefore also conjugate relatively to the *chord* CD, or, in other words, they cut that given chord *harmonically.* The four diametral planes ABC, ABE, ABD, ABF, compose therefore an harmonic pencil; the second being, *in this pencil*, harmonically conjugate to the fourth; and being at the same time, on account of the polars, conjugate to it also with respect to the *surface*, as one diametral plane to another. When the ellipsoid becomes a *sphere*, the conjugate planes ABE, ABF become *rectangular*; and consequently the sought plane ABE *bisects the angle* between the two given planes ABC and ABD. *This solves at once the problem for the sphere*; for if, conversely, we thus bisect the given dihedral angle CABD by a plane ABE, cutting the chord CD in E, and if we take the harmonic conjugate F on the same given chord prolonged, and draw from E to F lines meeting ordinately the given diameter AB, these two right lines will be situated in two rectangular or conjugate diametral planes, and will satisfy all the other conditions requisite for their being polars of each other; but each intersects the given chord CD, or that chord prolonged, and therefore each intersects also, by (8), the polar of that chord; each therefore satisfies all the transformed conditions of the problem, and gives a chord of solution, real or imaginary. More fully, the ordinate EE′ to the diameter AB, drawn from the *internal* point of harmonic section E of the chord CD, gives, when prolonged both ways to meet the surface, the *chord of real solution*, PS; and the other ordinate FF′ to the same diameter AB, which is drawn from the *external* point of section F of the same chord CD, and which is itself wholly external to the surface, is the *chord of imaginary solution.* But because when we return from the sphere to the *ellipsoid*, or other surface of the second order, the condition of *bisection* of the given dihedral angle CABD is no longer fulfilled by the sought plane ABE, a slight generalization of the foregoing process becomes necessary, and can easily be accomplished as follows.

10. Conceive, as before, that on the diameter AB the ordinate EE′ is let fall from the internal point of section E, and likewise the ordinates CC′ and DD′ from C and D; and draw also, parallel to that diameter, the right lines CC″, DD″, EE″, from the same three points C, D, E, so as to terminate on the diametral plane through O which is conjugate to the same diameter; in such a manner that OC″, OD″, OE″ shall be parallel and equal to the ordinates C′C, D′D, E′E; and that the segments CE, ED of the chord CD shall be proportional to the segments C″E″, E″D″ of the base C″D″ of the triangle C″OD″, which is situated in the diametral plane, and has the centre O for its vertex. For the case of the *sphere*, the vertical angle C″OD″ of this triangle is, by (9), bisected by the line OE″; wherefore the sides OC″, OD″,

or their equals, the ordinates $\mathrm{C'C}$, $\mathrm{D'D}$, are, in this case, proportional to the segments $\mathrm{C''E''}$, $\mathrm{E''D''}$ of the base, or to the segments CE, ED of the chord: while the squares of the ordinates are, for the same case of the sphere, equal to the rectangles $\mathrm{AC'B}$, $\mathrm{AD'B}$, under the segments of the diameter AB. Hence, *for the sphere, the squares of the segments of the given chord are proportional to the rectangles under the segments of the given diameter*, these latter segments being found by letting fall ordinates from the ends of the chord; or, in symbols, we have the proportion,

$$\mathrm{CF}^2 : \mathrm{DF}^2 :: \mathrm{CE}^2 : \mathrm{ED}^2 :: \mathrm{AC'B} : \mathrm{AD'B}.$$

But, by the general principles of *geometrical deformation, the property, thus stated, cannot be peculiar to the sphere. It must extend*, without any further modification, *to the ellipsoid*; and it gives at once, for that surface, the two points of harmonic section, E and F, of the given chord CD, through which points the two sought *chords of real and imaginary solution* are to pass; *these chords of solution are therefore completely determined*, since they are to be also ordinates, as before, to the given diameter AB. *The problem of inscription for the ellipsoid is therefore fully resolved; not only* when, as in (6), the number of sides of the polygon is *odd*, but *also* in the more difficult case (7), when the number of sides is *even*.

11. If the given surface be a hyperboloid of *two sheets, one* of the two fixed polars will still intersect that surface, and the fixed chord CD may still be considered as *real*. If the given diameter AB be also real, the proportion in (10) still holds good, without any modification from imaginaries, and determines still a real point E, with its harmonic conjugate F, through one of other of which two points still passes a *chord of real solution*, while through the other point of section still is drawn a *chord of imaginary solution*, reciprocally polar to the former. But if the *diameter* AB be *imaginary*, or in other words, if it fail to meet the proposed hyperboloid at all, we are led to consider, instead of it, an *ideal diameter* $\mathrm{A'B'}$, having the same *real direction*, but terminating, in a well-known way, on a certain *supplementary surface*; in such a manner that while A and B are now *imaginary points*, the points $\mathrm{A'}$ and $\mathrm{B'}$ are *real*, although *not really situated on the given surface*; and that

$$\mathrm{OA}^2 = \mathrm{OB}^2 = -\mathrm{OA}'^2 = -\mathrm{OB}'^2.$$

The points $\mathrm{C'}$ and $\mathrm{D'}$ are still real, and so are the rectangles $\mathrm{AC'B}$ and $\mathrm{AD'B}$, although A and B are imaginary; for we may write,

$$\mathrm{AC'B} = \mathrm{OA}^2 - \mathrm{OC}'^2, \quad \mathrm{AD'B} = \mathrm{OA}^2 - \mathrm{OD}'^2,$$

and the proportion in (10) becomes now,

$$\mathrm{CF}^2 : \mathrm{DF}^2 :: \mathrm{CE}^2 : \mathrm{ED}^2 :: \mathrm{OC}'^2 + \mathrm{OA}'^2 : \mathrm{OD}'^2 + \mathrm{OA}'^2.$$

It gives therefore still a *real point of section* E, and a *real conjugate point* F; and through these two points of section of CD we can still draw *two real right lines*, which shall still ordinately cross the real direction of AB, and shall still be two reciprocal polars, satisfying all the transformed conditions of the question, and coinciding with two chords of real and imaginary solution. *For the double-sheeted hyperboloid*, therefore, *as well* as for the ellipsoid, the problem of inscribing a *gauche chiliagon*, or other *even-sided polygon*, whose sides shall pass successively, and in order, through the same given number of points, is solved by a system of *two polar chords*, which we have assigned geometrical processes to determine; and the solutions are *still*, in general, *four* in number; *two* of them being still *real*, and *two imaginary*.

12. If the given surface be a hyperboloid of *one sheet*, then not only may the diameter AB be real or imaginary, but also the chord CD may or may not cease to be real; for the two fixed polars will *now* either *both meet* the surface, or else *both fail* to meet it in any two real points. When AB and CD are both real, the proportion in (10), being put under the form

$$\mathrm{CF}^2 : \mathrm{DF}^2 :: \mathrm{CE}^2 : \mathrm{ED}^2 :: \mathrm{OA}^2 - \mathrm{OC}'^2 : \mathrm{OA}^2 - \mathrm{OD}'^2,$$

shews that the point of section E and its conjugate F will be real, if the points C′ and D′ fall *both* on the diameter AB *itself*, or *both* on that diameter *prolonged*; that is, if the extremities Cand D lie *both within* or *both without* the interval between the two parallel tangent planes to the surface which are drawn at the points A and B: under these conditions therefore there will still be *two real right lines*, which may still be called the *two chords of solution*; but because these lines will still be two reciprocal polars, they will now (like the two fixed polars above mentioned) either *both meet* the hyperboloid, or else *both fail* to meet it; and consequently there will now be either *four real*, or else *four imaginary* solutions. If AB and CD be still both real, but if the chord CD have *one* extremity *within* and the *other* extremity *without* the interval between the two parallel tangent planes, the proportion above written will assign *a negative ratio* for the squares of the segments of CD; the points of section E and F, and the *two polar chords* of solution, become therefore, in *this* case, *themselves imaginary*; and of course, by still stronger reason, the four solutions of the problem become then imaginary likewise. If CD be real, but AB imaginary, the proportion in (11) conducts to two real points of section, and consequently to two real chords, which may, however, correspond, as above, either to four real or to four imaginary solutions of the problem. And, finally, it will be found that the same conclusion holds good also in the remaining case, namely, when the chord CD becomes imaginary, whether the diameter AB be real or not; that is, when the two fixed polars do not meet, in any real points, the single-sheeted hyperboloid.

13. Although the case last mentioned may still be treated by a modification of the proportion assigned in (10), which was deduced from considerations relative to the sphere, yet in order to put the subject in a clearer (or at least in another) point of view, we may now resume the problem for the ellipsoid as follows, without making any use of the spherical deformation. It was required to find two lines, reciprocally polar to each other, and ordinately crossing a given diameter AB of the ellipsoid, which should also cut a given chord CD of the same surface, internally in some point E, and externally in some other point F. Bisect CD in G, and conceive EF to be bisected in H; and besides the four old ordinates to the diameter AB, namely CC′, DD′, EE′, and FF′, let there be now supposed to be drawn, as two new ordinates to the same diameter, the lines GG′ and HH′. Then G′ will bisect C′D′, and H′ will bisect E′F′; while the centre O of the ellipsoid will still bisect AB. And because the points E′ and F′ are

harmonic conjugates, not only with respect to the points A and B, but also with respect to the points C′ and D′, we shall have the following equalities:

$$\begin{aligned} \mathrm{H'F'}^2 &= \mathrm{H'E'}^2 = \mathrm{H'A'} \,.\, \mathrm{H'B'} = \mathrm{H'C'} \,.\, \mathrm{H'D'} \\ &= \mathrm{H'O}^2 - \mathrm{OA}^2 = \mathrm{H'G'}^2 - \mathrm{G'C'}^2. \end{aligned}$$

Hence,

$$\mathrm{OH'}^2 - \mathrm{G'H'}^2 = \mathrm{OA}^2 - \mathrm{C'G'}^2,$$

that is,

$$\mathrm{OH'} = \frac{\mathrm{OA}^2 + \mathrm{OG'}^2 - \mathrm{C'G'}^2}{2\mathrm{OG'}} = \frac{\mathrm{OA}^2 + \mathrm{OC'} \,.\, \mathrm{OD'}}{\mathrm{OC'} + \mathrm{OD'}}.$$

Now each of these two last expressions for OH′ remains real, and assigns a real and determinate position for the point H′, even when the points C′, D′, or the points A, B, or when both these pairs of points at once become imaginary; for the points O and G′ are still in all cases real, and so are the squares of OA and C′G′, the rectangle under OC′ and OD′, and the sum OC′ + OD′. Thus H′ can always be found, as a real point, and hence we have a real value for the square of H′E′, or H′F′, which will enable us to assign the points E′ and F′ themselves, or else to pronounce that they are imaginary.

14. We see at the same time, from the values $\mathrm{H'O}^2 - \mathrm{OA}^2$ and $\mathrm{H'G'}^2 - \mathrm{C'G'}^2$ above assigned for $\mathrm{H'E'}^2$ or $\mathrm{H'F'}^2$, that these two sought points E′ and F′ must both be real, unless the two fixed points A and C′ are themselves both real, since O, G′, H′, are, all three, real points. But for the ellipsoid, and for the double sheeted hyperboloid, we can in general *oblige* the points C, D, and their projections C′, D′, to become imaginary, by selecting *that one* of the two fixed polars which does *not* actually meet the surface; for *these* two sorts of surfaces, the two polar chords of solution of the problem of inscription of a gauche polygon with an even number of sides passing through the same number of given points, and therefore found anew to be two *real lines*, although only one of them will actually intersect the surface, and only two of the four polygons will (as before) be real. And even for the single-sheeted hyperboloid, in order to render the two chords of solution *imaginary lines*, it is necessary that the two given polars should actually meet the surface; for otherwise the polar lines deduced will still be real. It is necessary also, for the imaginariness of the two lines deduced, that the given diameter AB should be itself a real diameter, or in other words that it should actually intersect the hyperboloid. But even when the given chord CD and the given diameter AB are thus *both* real, and when the surface is a single-*sheeted* hyperboloid, it does not *follow* that the two chords of solution *may not* be real lines. We shall only have *failed to prove* their reality by the expressions recently referred to. We must *resume*, for this case, the reasonings of (12), or some others equivalent to them, and we find, as in that section of this Abstract, for the imaginariness of the two sought polar lines, the condition that *one* of the two extremities of the given and real chord CD shall fall *within*, and that the *other* extremity of that chord shall fall *without* the interval between the two real and parallel tangent planes to the single-sheeted hyperboloid, which are drawn at the extremities of the real diameter AB. Sir W. R. Hamilton confesses that the case where all these particular conditions are combined, so as to render *imaginary* the two polar lines of solution, had not occurred to him when he made to the Royal Irish Academy his communication of June, 1849.

15. It seems to him worth while to notice here that instead of the foregoing *metric* processes for finding (when they exist) the two lines of solution of the problem, the following *graphic* process of construction of those lines may always, at least in theory, be substituted, although in practice it will sometimes require modification for imaginaries. In the diametral plane ABC draw a chord KD′L, which shall be bisected at the known point D′ by the given diameter AB; and join CK, CL. These joining lines will cut that diameter in the two sought points E′, F′; which being in this manner found, the two sought lines of solution EE′, FF′, are constructed without any difficulty. For the sphere, the ellipsoid, and the hyperboloid of two sheets, although not always for the single-sheeted hyperboloid, this simple and graphic process can actually be applied, without any such modification from imaginaries as was above alluded to. The consideration of non-central surfaces does not enter into the object of the present communication; nor has it been thought necessary to consider in it any limiting or exceptional cases, such as those where certain positions or directions become indeterminate, by some *peculiar* combinations of the data, while yet they are *in general* definitely assignable, by the processes already explained.

16. Sir William Rowan Hamilton is unwilling to add to the length of this communication by any historical references; in regard to which, indeed, he does not consider himself prepared to furnish anything important, as supplementary to what seems to be pretty generally known, by those who feel an interest in such matters. He has however taken some pains to inquire, from a few geometrical friends, whether it is *likely* that he has been anticipated in his results respecting the inscription of *gauche* polygons in *surfaces* of the second order; and he has not hitherto been able to learn that any such anticipation is thought to exist. Of course he knows that he must, consciously and unconsciously, be in many ways indebted to his scientific contemporaries, for their instructions and suggestions on these and on other subjects; and also to his acquaintance, imperfect as it may be, with what has been done in earlier times. But he conceives that he only does justice to the yet infant Method of Quaternions (communicated to the Royal Irish Academy for the first time in 1843), when he states that he considers himself to owe, to that new method of geometrical research, not merely the *results* stated to the Academy in the summer of 1849, respecting these inscriptions of gauche polygons, and several other connected although hitherto unpublished results, which to him appear remarkable, but also the *suggestion* of the mode of *geometrical* investigation which has been employed in the present Abstract. No doubt the principles used in it have all been very elementary, and perhaps their combination would have cost no serious trouble to any experienced geometer who had chosen to attack the problem. But to his *own* mind the whole foregoing investigation presents itself as being (what in fact in his his case it *was*) a mere *translation of the quaternion analysis into ordinary geometrical language*, on this particular subject. And he will not complicate the present Abstract by giving, on *this* occasion, any account of those *other* theorems respecting polygons in surfaces, to which the Calculus of Quaternions has conducted him, but of which he has not seen how to *translate the proofs* (for it is easy to translate the *results*) into the usual language of *geometry.*