EXERCISES IN QUATERNIONS

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1. Although the following paper, or series, will be founded on the same *principles* as the communications on Symbolical Geometry in the present *Journal*, and on Quaternions in the *Philosophical Magazine*, yet its *plan* will be in many respects different. And the writer hopes that without either, on the one hand, too much repeating from those papers, or, on the other hand, interfering with their continuation, he may be able, by remarks, rules, formulæ, and examples which will be submitted in the present Exercises, to give some acceptable assistance to those mathematicians, or to those mathematical students, who may do him the honour of desiring to familiarise themselves with the Calculus of Quaternions; or who may even be disposed to give that Calculus a trial, as a branch of symbolical science, and as an instrument of geometrical and physical research.

2. The general *conception* of *directed lines* has occurred to several authors; and many have also perceived the existence of an important, and indeed fundamental *analogy* between the geometrical composition and decomposition of *motions*, and the algebraical addition and subtraction of positive and negative *numbers*: which analogy has been felt to be so close and strong, as to invite and justify the application of the *names* and *marks* of addition and subtraction to the operations of constructing the intermediate and transverse diagonals of a parallelogram, when two coinitial sides of that parallelogram are given, as two directed lines, of which (what thus come to be called) the *sum* and *difference* are to be taken. With these, as *preliminary conceptions*, to the introduction of which into science the present writer is aware that he cannot in any manner pretend, he wishes to be allowed to regard his readers as being *already* familiar, *before* entering on the study of the quaternions: although that study will certainly tend to impress them still more on the mind. They are, indeed, *common* to his own and to several other systems, to which systems, in *other* respects, the theory of the quaternions offers rather a *contrast* than a resemblance.

3. Yet, at this stage, he desires to invite attention to an unusual mode of *notation*, which appears to him to embody under a convenient and simple form, and under one which adapts itself as readily to *lines in space* as to lines in a single plane, those preliminary and fundamental conceptions. He denotes, by the symbol

$$\text{B} - \text{A} \qquad (1),$$

that finite straight line which is drawn *from* the point A *to* the point B; and which line, when *thus* denoted, is understood to have a determined length, direction, and situation in space, as soon as the two points A and B themselves are supposed to receive determined poitions. And then, by merely writing the two formulæ,

$$(\mathrm{C} - \mathrm{B}) + (\mathrm{B} - \mathrm{A}) = \mathrm{C} - \mathrm{A} \qquad (2),$$

$$(\mathrm{C} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}) = \mathrm{C} - \mathrm{B} \qquad (3),$$

he *expresses*, under the form of two *identities*, those conceptions of the geometrical addition and subtraction of directed lines, as analogous to the composition and decomposition of motions, and as performed according to the same rules, in which conceptions themselves he has already carefully disclaimed all private or personal property. For, with the use above explained of the notation (1), the *geometrical identity* (2) expresses that if the directed line to C from B be *geometrically added* to the directed line to B from A, according to the rules of the composition of motions, the *geometrical sum*, or the *resultant line*, will be that *third* directed line which is drawn to C to A; whatever may be the positions in space of the three points A B C. And the *other* geometrical identity (3) expresses in like manner this converse proposition, that if the directed line from A to B, which is, by the notation (1), denoted by the symbol $\mathrm{B} - \mathrm{A}$, be *geometrically subtracted* from the directed line $\mathrm{C} - \mathrm{A}$, which is drawn from A to C, according to the laws of decomposition of motion, the *geometrical difference* obtained by this subtraction will be that directed line $\mathrm{C} - \mathrm{B}$, which is drawn from the final point B of the subtrahend line $\mathrm{B} - \mathrm{A}$ to the final point C of that other given and coinitial directed line $\mathrm{C} - \mathrm{A}$, from which the subtraction is to be performed.

4. Instead of compounding only *two* successive motions, or *rectilinear steps* in space, we may compound *any number* of such steps; or, in other words, instead of considering a triangle A B C, we consider a *polygon* A B C D E . . .: and the known results for such more complex cases may still be expressed with great simplicity, and under the form of *geometrical identities*, by adhering to the same method of notation. Thus, for a rectilinear quadrilateral, A B C D, whether this be or be not in one plane, we shall always have the formula

$$(\mathrm{D} - \mathrm{C}) + (\mathrm{C} - \mathrm{B}) + (\mathrm{B} - \mathrm{A}) = \mathrm{D} - \mathrm{A} \qquad (4);$$

for a pentagon, whether plane or gauche,

$$(\mathrm{E} - \mathrm{D}) + (\mathrm{D} - \mathrm{C}) + (\mathrm{C} - \mathrm{B}) + (\mathrm{B} - \mathrm{A}) = \mathrm{E} - \mathrm{A} \qquad (5);$$

and similarly for other polygons, in space or in one plane: the line which is drawn from the initial to the final point of any *unclosed* polygon being regarded (in this as in many other systems) as the *geometrical sum of all the successive sides* of the figure which it thus serves to *close*; exactly as in the formula (2) the *directed base* $\mathrm{C} - \mathrm{A}$ of the triangle ABC was the geometrical *sum* of the two successive *sides* $\mathrm{B} - \mathrm{A}$ and $\mathrm{C} - \mathrm{B}$, obtained by *adding* the second of those two sides to the first.

5. If the closing line be drawn in the order of succession of the sides, or in the order of the motion along the polygon which has been above supposed to be performed; or if the polygon be given as closed; then the *sum of all* the successive lines, including the closing line,

will be a *null line*, because the motion thus conceived would simply bring a moving point *back* to its original or initial position. Accordingly, in the notation above proposed, we shall have the following formulæ of identity, which *express* this conception of *return*:

$$\text{A} - \text{A} = (\text{A} - \text{B}) + (\text{B} - \text{A}) = (\text{A} - \text{C}) + (\text{C} - \text{B}) + (\text{B} - \text{A}) = \&\text{c}. \qquad (6).$$

6. We may agree to *suppress the symbol of a null line*, when it occurs as written to the *left-hand* of any complex symbol denoting the result of any geometrical addition or subtraction; and then, by changing C to B in the identity (2), and to A in the identity (3), we shall obtain the formulæ,

$$+(\text{B} - \text{A}) = (\text{B} - \text{B}) + (\text{B} - \text{A}) = \text{B} - \text{A} \qquad (7);$$

$$-(\text{B} - \text{A}) = (\text{A} - \text{A}) - (\text{B} - \text{A}) = \text{A} - \text{B} \qquad (8);$$

which allow of our interpreting the two isolated but *affected* symbols of lines,

$$+(\text{B} - \text{A}) \quad \text{and} \quad -(\text{B} - \text{A}) \qquad (9),$$

as denoting respectively the directed line B − A *itself*, and the *opposite* of that line, namely the directed line A − B: two lines being said to be mutually opposites, when the beginning and end of the first line coincide respectively with the end and the beginning of the second. A null line is *its own* opposite,

$$+(\text{A} - \text{A}) = -(\text{A} - \text{A}) \qquad (10);$$

but any *actual line*, that is, a line B − A with any finite length, is distinguished form the opposite line A − B by the contrast between their *directions.* A line B − A may be *subtracted* from another line C − A, by *adding* the second line C − A to the *opposite* A − B of the first; for we have, by the identities (2) and (3),

$$(\text{C} - \text{A}) - (\text{B} - \text{A}) = \text{C} - \text{B} = (\text{C} - \text{A}) + (\text{A} - \text{B}) \qquad (11).$$

7. Two directed lines being regarded as *equal* to each other, when, and only when, their *directions* as well as their *lengths* are identical, although their *situations* will generally be different, the *equation*

$$\text{D} - \text{C} = \text{B} - \text{A} \qquad (12)$$

will express, generally, that the four points A B D C are the four successive corners of a *parallelogram*; the corner D being opposite to A, and C to B: and we may still retain this mode of speaking, even when the altitude or the area or the parallelogram vanishes, by the four points A B D C coming to range themselves on one right line. From this signification of the equation (12) it is evident that this equation admits of *inversion*, and of *alternation*, so that it may be written thus (inversely),

$$\text{C} - \text{D} = \text{A} - \text{B} \qquad (13),$$

because the opposites of equal lines are equal; or thus (alternately),

$$\text{D} - \text{B} = \text{C} - \text{A} \qquad (14),$$

by Euclid I. 33, or because the two *paths of transport*, from A and B to C and D respectively, must be *themselves* equal directed lines, in order to allow of the first given directed line $\text{B} - \text{A}$ being carried, *without rotation*, by the simultaneous motion of its two extreme points along those two paths of transport, so as to come to *coincide* with the second given directed line $\text{D} - \text{C}$; which second line would not be (in the foregoing sense) *equal* to the first, unless this perfect coincidence could be effected by such transport without rotation. (The writer may remark, in passing, that he agrees with those who hold that all such considerations as these, of *motions abstracted from causes of motion*, do not vitiate, in any degree however small, the *purity* of geometrical science: to think otherwise would be indeed, as he conceives, to condemn, so far, those ancient geometers, including Euclid, who generated surfaces, for example the sphere, by motion.) Directed lines which are *equal* to the same directed line are also equal to each other; and the sums and differences of equal directed lines, similarly taken, are equal directed lines. Lines which are opposites of equal directed lines may be said, by an extension of the former definition of opposites, to be themselves also opposite lines.

8. Since, under the condition expressed by the equation (12), the line $\text{D} - \text{A}$ is the *directed diagonal* of the parallelogram ABDC, intermediate between the two directed sides $\text{B} - \text{A}$ and $\text{C} - \text{A}$, and coinitial with them, it ought (by a known principle above mentioned) to be found to be their geometrical sum: and, accordingly,

$$\text{D} - \text{A} = (\text{D} - \text{C}) + (\text{C} - \text{A}) = (\text{B} - \text{A}) + (\text{C} - \text{A}) \qquad (15);$$

or, adding the sides in a different order, and employing the principle of *alternation*, whereby we pass from the equation (12) to (14),

$$\text{D} - \text{A} = (\text{D} - \text{B}) + (\text{B} - \text{A}) = (\text{C} - \text{A}) + (\text{B} - \text{A}) \qquad (16).$$

It is therefore allowed to *change the order* of the summands in the addition of any two directed lines; a conclusion which is easily extended to any number of such lines, in space or in one plane, so as to shew that geometrical *addition* is a *commutative operation*, or that the *sum* of any given system of directed lines is always equal to the same given directed line, in whatever *order* the summation is effected. Addition of directed lines is also an *associative* operation, in the sense that any number of successive summands may be collected of *associated* together (as is done in calculation by enclosing their symbols within brackets) into one partial group, and their sum then added as a single summand to the rest: for this comes merely (when its geometrical signification is examined) to drawing a *diagonal of a polygon*, plane or gauche. It is understood that in order to avail ourselves of the identity (2), for the purpose of adding an *arbitrary* but given line $\text{B}' - \text{A}'$ to another line $\text{B} - \text{A}$, when the beginning A' of the proposed *addend* line does not *already* coincide with the end B of the line to which the addition is to be performed, we are to *make* it coincide, by a transport without rotation; this process of construction being symbolically expressed by by the formula

$$(\text{B}' - \text{A}') + (\text{B} - \text{A}) = \text{C} - \text{A}, \quad \text{if } \text{B}' - \text{A}' = \text{C} - \text{B} \qquad (17).$$

9. When three points ABC are so related as to satisfy the equation

$$\text{C} - \text{B} = \text{B} - \text{A} \qquad (18),$$

which gives, by principles and notations already explained,

$$\mathrm{C} - \mathrm{A} = (\mathrm{B} - \mathrm{A}) + (\mathrm{B} - \mathrm{A}) \tag{19}$$

then, by a natural and obvious use of numerical coefficients, we may write also, as other expressions for the same relation of position between the three points (namely that the point B bisects the straight line connecting A and C), the two following equations, of which each includes the other:

$$\mathrm{C} - \mathrm{A} = 2(\mathrm{B} - \mathrm{A}); \quad \mathrm{B} - \mathrm{A} = \tfrac{1}{2}(\mathrm{C} - \mathrm{A}) \tag{20}$$

And generally, if a denote any positive or negative number, whether integral or fractional, and whether commensurable or incommensurable, the notation

$$\mathrm{C} - \mathrm{A} = a(\mathrm{B} - \mathrm{A}) \tag{21}$$

may conveniently be employed to express that the point C is situated on the same indefinite right line as the points A and B, being at the same side of A as B if the coefficient a be positive, but at the opposite side of A if a be negative, and at a distance from A which bears to the distance of B to A the ratio of $\pm a$ to 1. When the coefficient a becomes zero, then both members of (21) become null lines, and C coincides with A. With such an use of coefficients we shall have, as in ordinary algebra, the two identities

$$(a' \pm a)(\mathrm{B} - \mathrm{A}) = a'(\mathrm{B} - \mathrm{A}) \pm a(\mathrm{B} - \mathrm{A}) \tag{22}$$

$$a\{(\mathrm{B}' - \mathrm{A}') \pm (\mathrm{B} - \mathrm{A})\} = a(\mathrm{B}' - \mathrm{A}') \pm a(\mathrm{B} - \mathrm{A}) \tag{23}$$

which we may express in words by saying that the operation of *multiplication* of a directed line by a numerical coefficient is a *distributive operation*, whether relatively to the multiplying number, or relatively to the multiplied line; this *distributive* property of such *multiplication* of lines by numbers depending mainly on the *commutative* property of the *addition* of lines among themselves.

10. The equation (21) expresses sufficiently that the point C is situated *somewhere* upon the indefinite straight line which passes through the two points A and B, or that the *three* points A B C are *collinear*; and it expresses *nothing more* than this relation of collinearity, if we conceive the number a to remain undetermined. The formula (21) may therefore, with this use of an indeterminate numerical coefficient a, be regarded as the *equation of an indefinite right line in space*; *one such equation* being *sufficient* in this mode of dealing with the subject. This equation (21) may however be made to assume a more symmetric form, by introducing the consideration of an arbitrary fourth point D, supposed to be situated anywhere in space, with which the three collinear points A B C shall be compared. For thus, by writing the equation successively under the following forms,

$$(\mathrm{C} - \mathrm{D}) - (\mathrm{A} - \mathrm{D}) = a(\mathrm{B} - \mathrm{D}) - a(\mathrm{A} - \mathrm{D}) \tag{24}$$

$$\text{C} - \text{D} = a(\text{B} - \text{D}) + (1 - a)(\text{A} - \text{D}) \qquad (25),$$

$$(b - ba)(\text{A} - \text{D}) + ba(\text{B} - \text{D}) - b(\text{C} - \text{D}) = 0 \qquad (26),$$

$$l(\text{A} - \text{D}) + m(\text{B} - \text{D}) + n(\text{C} - \text{D}) = 0 \qquad (27),$$

in each of the two last of which forms the symbol for zero is understood to denote a null line, we see, by comparing these two forms, that the coefficients l, m, n, in the form (27), are connected among themselves by the equation of condition

$$l + m + n = 0 \qquad (28);$$

and, conversely, that under this last condition the formula (27) expresses that the three points A B C are ranged upon one common right line. In fact, when the condition (28) is satisfied, we can eliminate the coefficient l by it from (27), and so obtain the equation

$$m(\text{B} - \text{A}) + n(\text{C} - \text{A}) = 0 \qquad (29),$$

which evidently agrees with the form (21), and conducts to similar consequences.

11. Let E be a new point situated anywhere upon the indefinite straight line CD; and therefore satisfying an equation of the form

$$p(\text{E} - \text{D}) = n(\text{C} - \text{D}) \qquad (30),$$

where n may denote the same coefficient as in (27). Eliminating this coefficient n, the symbol C for the point of intersection of the two indefinite straight lines AB, DE, disappears; and there results, as the expression of the condition that *some* such point C shall exist, or that the *four points* A B D E shall be situated in *one common plane*, an equation of the following form,

$$l(\text{A} - \text{D}) + m(\text{B} - \text{D}) + p(\text{E} - \text{D}) = 0 \qquad (31);$$

which seems to resemble the equation (27), but differs from it in this important respect, that the sum of the three new coefficients $l\,m\,p$ does not now generally vanish, as the sum of the three old coefficients $l\,m\,n$ did vanish, in virtue of the condition (28). And by comparing the *equation of coplanarity* (31) with the *condition of collinearity* (28), we may now see that this last mentioned condition (28), in combination with the equation (27), expressed that the three given points A B C are *coplanar with any arbitrary fourth point* D; which can only be by those *three* points A B C being *collinear* with each other. In fact, under the condition (28), we have

$$l(\text{D} - \text{D}') + m(\text{D} - \text{D}') + n(\text{D} - \text{D}') = 0 \qquad (32),$$

by adding which to (27) we obtain the same result, namely

$$l(\text{A} - \text{D}') + m(\text{B} - \text{D}') + n(\text{C} - \text{D}') = 0 \qquad (33),$$

as if we had simply changed the symbol of the fourth point D to that of any arbitrary fifth point D$'$. By introducing the symbol O of a new and arbitrary point of space, with which the four coplanar points A B D E may be compared, through drawing lines from it to them, the equation of coplanarity (31) assumes the form

$$l(\mathrm{A} - \mathrm{O}) + m(\mathrm{B} - \mathrm{O}) + n(\mathrm{D} - \mathrm{O}) + p(\mathrm{E} - \mathrm{O}) = 0 \qquad (34);$$

where n is a new coefficient, connected with the others by the condition

$$l + m + n + p = 0 \qquad (35).$$

These remarks, and a few others which shall be offered in some following articles, may be of use, as serving to illustrate and exemplify an unusual mode of *notation** in geometry; but they can only be regarded as *preparatory* to the theory of the *quaternions*, because that theory, in its *geometrical* aspect, depends essentially on the conception of the *multiplication* and *division* of *one* directed line *by another line* of the same kind, and not merely by a numerical *coefficient*: a QUATERNION (in the author's system) being always equal to such a *product or quotient of two directed lines in space.*

[*To be continued.*]

* The notation $\mathrm{B} - \mathrm{A}$ for a *directed right line in space*, was proposed in a note to the first article of the paper on Symbolical Geometry, printed in the present *Journal* (towards the end of 1845): and it had been long familiar to the writer, as an extension to *space* of a similar notation relatively to *time*, which had been published by him in the year 1835, to express a *time-step*, or directed interval in time, from any *one moment* (not number) denoted by A, to any *other moment* of time denoted by B. (See the Essay on Algebra as the Science of Pure Time, in the 1st part of the XVIIth volume of the *Transactions of the Royal Irish Academy*). With respect to the mere fact of *distinguishing* between two elementary geometrical symbols, AB and BA, as denoting two *opposite lines*, the present author cheerfully acknowledges that this simple and natural distinction has often been noticed and employed by other writers on Geometry.