RESEARCHES ON THE DYNAMICS OF LIGHT

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The Chair having been taken, *pro tempore*, by his Grace the Archbishop of Dublin, V. P., the President continued his account of his researches in the theory of light.

As a specimen of the problems which he had lately considered and resolved, the following question was stated:—An indefinite series of equal and equally distant particles, $\ldots m_{-1}, m_0, m_1, \ldots$, situated in the axis of x, at the points $\ldots -1, 0, +1, \ldots$, being supposed to receive, at the time 0, any very small transversal displacements $\ldots y_{-1,0}, y_{0,0}, y_{1,0}, \ldots$, and any very small transversal velocities $\ldots y'_{-1,0}, y'_{0,0}, y'_{1,0}, \ldots$, it is required to determine their displacements $\ldots y_{-1,t}, y_{0,t}, y_{1,t}, \ldots$ for any other time t; each particle being supposed to attract the one which immediately precedes or follows it in the series, with an energy $= a^2$, and to have no sensible influence on any of the more distant particles. This problem may be considered as equivalent to that of integrating generally the equation in mixed differences,

$$y''_{x,y} = a^2(y_{x+1,t} - 2y_{x,t} - y_{x-1,t}); \tag{1}$$

which may also be thus written:

$$\left(\frac{d}{dt}\right)^2 y_{x,t} = \frac{(a\Delta_x)^2}{1+\Delta_x} y_{x,t}. \tag*{(1)$'$}$$

The general integral required, may be thus written:

$$y_{x,t} = \left\{1 - \frac{a^2\Delta_x^2}{1+\Delta_x}\left(\int_0^t dt\right)^2\right\}^{-1}(y_{x,0} + ty'_{x,0}); \tag{2}$$

an expression which may be developed into the sum of two series, as follows,

$$\begin{aligned} y_{x,t} = y_{x,0} &+ \frac{a^2t^2}{1\,.\,2}\Delta_x^2 y_{x-1,0} + \frac{a^4t^4}{1\,.\,2\,.\,3\,.\,4}\Delta_x^4 y_{x-2,0} + \&\text{c.} \\ &+ ty'_{x,0} + \frac{a^3t^3}{1\,.\,2\,.\,3}\Delta_x^2 y'_{x-1,0} + \frac{a^5t^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}\Delta_x^4 y'_{x-2,0} + \&\text{c.}; \end{aligned} \tag*{(2)$'$}$$

and may be put under this other form,

$$\begin{aligned} y_{x,t} = \frac{2}{\pi}\Sigma_{(l)}\,{}_{-\infty}^{\infty} y_{x+l,0} \int_0^{\frac{\pi}{2}} d\theta\,\cos(2l\theta)\cos(2at\sin\theta) \\ + \frac{1}{a\pi}\Sigma_{(l)}\,{}_{-\infty}^{\infty} y'_{x+l,0} \int_0^{\frac{\pi}{2}} d\theta\,\cos(2l\theta)\,\mathrm{cosec}\,\theta\sin(2at\sin\theta); \end{aligned} \tag*{(2)$''$}$$

the first line of $(2)'$ or $(2)''$ expressing the effect of the initial displacements, and the second line expressing the effect of the initial velocities, for all possible suppositions respecting these initial data, or for all possible forms of the two arbitrary functions $y_{x,0}$ and $y'_{x,0}$.

Supposing now that these arbitrary forms or initial conditions are such, that

$$y_{x,0} = \eta \operatorname{vers} 2x\frac{\pi}{n}, \quad \text{and} \quad y'_{x,0} = -2a\eta \sin\frac{\pi}{n} \sin 2x\frac{\pi}{n}, \tag{3}$$

for all values of the integer x between the limits 0 and $-in$, n and i being positive and large, but finite integer numbers, and that for all other values of x the functions $y_{x,0}$ and $y'_{x,0}$ vanish: which is equivalent to supposing that at at the origin of t, and for a large number i of wave-lengths (each $= n$) behind the origin of x, the displacements and velocities of the particles are such as to agree with the following law of undulatory vibration,

$$y_{x,t} = \eta \operatorname{vers}\left(2x\frac{\pi}{n} - 2at\sin\frac{\pi}{n}\right), \tag{3$'$}$$

but that all the other particles are, at that moment, at rest: it is required to determine the motion which will ensue, as a consequence of these initial conditions. The solution is expressed by the following formula, which is a rigorous deduction from the equation in mixed differences (1):

$$y_{x,t} = \frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\sin in\theta}{\sin\theta} \frac{\cos(2x\theta + in\theta - 2at\sin\theta)'}{\cos\theta - \cos\dfrac{\pi}{n}}\,d\theta; \tag{4}$$

an expression which tends indefinitely to become

$$y_{x,t} = \frac{\eta}{2}\operatorname{vers}\left(2x\frac{\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\sin(2x\theta - 2at\sin\theta)}{\sin\theta\left(\cos\theta - \cos\dfrac{\pi}{n}\right)}\,d\theta, \tag{4$'$}$$

as the number i increases without limit. The approximate values are discussed, which these rigorous integrals acquire, when the value of t is large. It is found that a vibration, of which the phase and the amplitude agree with the law $(3)'$, is propagated forward, but not backward, so as to agitate successively new and more distant particles, (and to leave successively others at rest, if i be finite,) with a velocity of progress which is expressed by $a\cos\frac{\pi}{n}$, and which is therefore less, by a finite though small amount, than the velocity of passage $a\frac{n}{\pi}\sin\frac{\pi}{n}$ of any given phase, from one vibrating particle to another within that extent of the series which is already fully agitated. In other words, the communicated vibration does not attain a sensible amplitude, until a finite interval of time has elapsed from the moment when one should expect it to begin, judging only by the law of the propagation of phase through an indefinite series of particles, which are all in vibration already. A small disturbance, distinct from the vibration $(3)'$, is also propagated, backward as well as forward, with a velocity $= a$, independent of the length of the wave. And all these propagations are accompanied with a small degree of terminal diffusion, which, after a very long time, renders all the displacements insensible, if the number i, however large, be finite, that is, if the vibration be originally limited to any finite number of particles.