ON EQUATIONS OF THE FIFTH DEGREE

By

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On Equations of the Fifth Degree: and especially on a certain System of Expressions connected with those Equations, which Professor Badano has lately proposed. By* Sir William Rowan Hamilton, LL. D., P.R.I.A., F.R.A.S., *Honorary Member of the Royal Societies of Edinburgh and Dublin; Honorary or Corresponding Member of the Royal or Imperial Academies of St. Petersburgh, Berlin, and Turin, of the American Society of Arts and Sciences, and of other Scientific Societies at home and abroad; Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

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1. Lagrange has shown that if α be a given root of the equation

$$\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1 = 0,$$

n being a prime factor of m, and if μ denote for abridgment the quotient

$$\mu = \frac{1\,.\,2\,.\,3\,\ldots\,m}{\left(1\,.\,2\,.\,3\,\ldots\,\dfrac{m}{n}\right)^n};$$

then the function

$$t = x' + \alpha x'' + \alpha^2 x''' + \ldots + \alpha^{m-1} x^{(m)}$$

has only μ different values, corresponding to all possible changes of arrangement of the m quantities $x', x'', \ldots x^{(m)}$, which may be considered as the roots of a given equation of the m^{th} degree,

$$x^m - \mathrm{A}x^{m-1} + \mathrm{B}x^{m-2} - \mathrm{C}x^{m-3} + \ldots = 0;$$

and that if the development of the n^{th} power of this function t be reduced, by the help of the equation

$$\alpha^n = 1,$$

(and not by the equation $\alpha^{n-1} + \&\text{c.} = 0$,) to the form

$$t^n = \xi^{(0)} + \alpha\xi' + \alpha^2\xi'' + \ldots + \alpha^{n-1}\xi^{(n-1)},$$

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then this power t^n itself has only $\dfrac{\mu}{n}$ different values, and the term $\xi^{(0)}$ has only $\dfrac{\mu}{n(n-1)}$ such values, or is a root of an equation of the degree

$$\frac{1\,.\,2\,.\,3\,\ldots\,m}{n(n-1)\left(1\,.\,2\,.\,3\,\ldots\,\dfrac{m}{n}\right)^n},$$

of which equation the coefficients are rational functions of the given coefficients A, B, C, &c.; while $\xi', \xi'', \ldots \xi^{(n-1)}$ are the roots of an equation of the degree $n-1$, of which the coefficients can be expressed rationally in terms of $\xi^{(0)}$ and of the same original coefficients A, ... of the given equation in x.

2. For example, if there be given an equation of the sixth degree,

$$x^6 - \mathrm{A}x^5 + \mathrm{B}x^4 - \mathrm{C}x^3 + \mathrm{D}x^2 - \mathrm{E}x + \mathrm{F} = 0,$$

of which the roots are denoted by x', x'', x''', x^{IV}, x^V, x^{VI}, and if we form the function

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^V + \alpha^5 x^{VI},$$

in which $\alpha = -1$; we shall then have

$$m = 6, \quad n = 2, \quad \mu = \frac{720}{36} = 20, \quad \frac{\mu}{n} = 10, \quad \frac{\mu}{n(n-1)} = 10;$$

and the function t will have twenty different values, but its square will have only ten. And if, by using only the equation $\alpha^2 = 1$, and not the equation $\alpha = -1$, we reduce the development of this square to the form

$$t^2 = \xi^{(0)} + \alpha\xi',$$

the term $\xi^{(0)}$ will itself be a ten-valued function of the six quantities $x', \ldots x^{VI}$; and ξ' will be a rational function of $\xi^{(0)}$ and A, namely,

$$\xi' = \mathrm{A}^2 - \xi^{(0)}.$$

3. Again, if with the same meanings of $x', \ldots x^{VI}$, we form t by the same expression as before, but suppose α to be a root of the equation

$$\alpha^2 + \alpha + 1 = 0,$$

then

$$m = 6, \quad n = 3, \quad \mu = \frac{720}{8} = 90, \quad \frac{\mu}{n} = 30, \quad \frac{\mu}{n(n-1)} = 15;$$

so that the function t will now have 90 different values, but its cube will have only 30; and if that cube be reduced, by the equation $\alpha^3 = 1$, to the form

$$t^3 = \xi^{(0)} + \alpha\xi' + \alpha^2\xi'',$$

then $\xi^{(0)}$ will be a root of an equation of the fifteenth degree, while ξ' and ξ'' will be the roots of a quadratic equation, the coefficients of this last equation being rational functions of $\xi^{(0)}$, and of the given coefficients A, &c.

4. And if, in like manner, we consider the case

$$m = 5, \quad n = 5, \quad \mu = 120, \quad \frac{\mu}{n} = 24, \quad \frac{\mu}{n(n-1)} = 6,$$

so that $x', \ldots x^V$ are the roots of a given equation of the fifth degree

$$x^5 - \mathrm{A}x^4 + \mathrm{B}x^3 - \mathrm{C}x^2 + \mathrm{D}x - \mathrm{E} = 0,$$

and

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^V,$$

in which α is a root of the equation

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0,$$

then the function t has itself 120 different values, but its fifth power has only 24; and if this fifth power be put under the form

$$t^5 = \xi^{(0)} + \alpha\xi' + \alpha^2\xi'' + \alpha^3\xi''' + \alpha^4\xi^{IV},$$

by the help of the equation $\alpha^5 = 1$, then $\xi^{(0)}$ is a root of an equation of the sixth degree, of which the coefficients are rational functions of A, B, C, D, E, while ξ', ξ'', ξ''', ξ^{IV}, are the roots of an equation of the fourth degree, of which the coefficients are rational functions of the same given coefficients A, &c., and of $\xi^{(0)}$.

5. LAGRANGE has shown that these principles explain the success of the known methods for resolving quadratic, cubic, and biquadratic equations; but that they tend to discourage the hope of resolving any general equation above the fourth degree, by any similar method. And in fact it has since* been shown to be impossible to express any root of any general equation, of the fifth or any higher degree, as a function of the coefficients of that equation, by any finite combination of radicals and rational functions. Yet it appears to be desirable to examine into the validity and import of an elegant system of radical expressions which have lately been proposed by Professor BADANO of Genoa, for the twenty-four values of LAGRANGE's function t^5 referred to in the last article; and to inquire whether these new expressions are adapted to assist in the solution of equations of the fifth degree, or why they fail to do so.

6. In order to understand more easily and more clearly the expressions which are thus to be examined, it will be advantageous to begin by applying the method by which they are obtained to equations of lower degrees. And first it is evident that the general quadratic equation

$$x^2 - \mathrm{A}x + \mathrm{B} = 0,$$

* See a paper by the present writer, "On the Argument of Abel," &c., in the Second Part of the Eighteenth Volume of the Transactions of this Academy.

has its roots expressed as follows:

$$x' = \alpha + \beta, \quad x'' = \alpha - \beta;$$

α not here denoting any root of unity, but a rational function of the coefficients of the given equation (namely $\frac{1}{2}\text{A}$), and β^2 being another rational function of those coefficients (namely $\frac{1}{4}\text{A}^2 - \text{B}$); because by the general principles of article 1., when $m = 2$ and $n = 2$, we have $\frac{\mu}{n} = 1$, so that the function $(x' - x'')^2$ is symmetric, as indeed it is well known to be.

7. Proceeding to the cubic equation

$$x^3 - \text{A}x^2 + \text{B}x - \text{C} = 0,$$

and seeking the values of the function

$$t^3 = (x' + \theta x'' + \theta^2 x''')^3,$$

in which θ is such that

$$\theta^2 + \theta + 1 = 0,$$

we know first, by the same general principles, that the number of these values is two, because $\frac{\mu}{n} = 2$, when $m = 3$, $n = 3$. And because these values will not be altered by adding any common term to the three roots x', x'', x''', it is permitted to treat the sum of these three roots as vanishing, or to assume that

$$x' + x'' + x''' = 0;$$

that is, to reduce the cubic equation to the form

$$x'^3 + px' + q = 0.$$

In other words, the function

$$t^3 = (x_1 + \theta x_2 + \theta^2 x_3)^3,$$

in which x_1, x_2, x_3 are the three roots of the equation with coefficients A, B, C, will depend on those coefficients, only by depending on p and q, if these two quantities be chosen such that we shall have identically

$$x^3 - \text{A}x^2 + \text{B}x - \text{C} = (x - \tfrac{1}{3}\text{A})^3 + p(x - \tfrac{1}{3}\text{A}) + q.$$

8. This being perceived, and x'' and x''' being seen to be the two roots of the quadratic equation

$$x''^2 + x'x'' + x'^2 + p = 0,$$

which is obtained by dividing the cubic

$$x''^3 + px'' - x'^3 - px' = 0,$$

by the linear factor $x'' - x'$; we may, by the theory of quadratics, assume the expressions

$$x'' = \alpha + \beta, \quad x''' = \alpha - \beta,$$

provided that we make

$$\alpha = -\tfrac{1}{2}x', \quad \beta^2 = -\tfrac{3}{4}x'^2 - p,$$

that is, provided that we establish the identity

$$(x'' - \alpha)^2 - \beta^2 = x''^2 + x'x'' + x'^2 + p.$$

And, substituting for x', x'', x''', their values as functions of α and β, and reducing by the equation $\theta^2 + \theta + 1 = 0$, we find

$$t^3 = \{-3\alpha + (\theta - \theta^2)\beta\}^3 = \alpha' + \beta';$$

in which

$$\alpha' = -27\alpha(\alpha^2 - \beta^2), \quad \beta'^2 = -27\beta^2(9\alpha^2 - \beta^2)^2.$$

But α and β^2 are rational functions of x' and p; and substituting their expressions as such, we find corresponding expressions for α' and β'^2, namely,

$$\alpha' = \tfrac{27}{2}x'(x'^2 + p), \quad \beta'^2 = \tfrac{27}{4}(3x'^2 + 4p)(3x'^2 + p)^2.$$

9. Finally, x' is such that

$$x'^3 + px' = -q;$$

and it is found on trial to be possible by this condition to eliminate x' from the expressions for α' and β'^2, obtained at the end of the last article, and so to arrive at these other expressions, which are rational functions of p and q:

$$\alpha' = -\tfrac{27}{2}q, \quad \beta'^2 = \tfrac{27}{4}(27q^2 + 4p^3).$$

In this manner then it might have been discovered, what has long been otherwise known, that the function t^3 is a root of the auxiliary quadratic equation

$$(t^3)^2 + 27q(t^3) - 27p^3 = 0.$$

And because the same method gives

$$(x' + \theta x'' + \theta^2 x''')(x' + \theta^2 x'' + \theta x''') = 9\alpha^2 + 3\beta^2 = -3p,$$

we should obtain the known expressions for the three roots of the cubic equation

$$x'^3 + px' + q = 0,$$

under the forms:

$$x' = \frac{t}{3} - \frac{p}{t}, \quad x'' = \frac{\theta^2 t}{3} - \frac{\theta p}{t}, \quad x''' = \frac{\theta t}{3} - \frac{\theta^2 p}{t};$$

which are immediately verified by observing that

$$\theta^3 = 1, \quad \left(\frac{t}{3}\right)^3 - \left(\frac{p}{t}\right)^3 = -q.$$

The foregoing method therefore succeeds completely for equations of the third degree.

10. In the case of the biquadratic equation, deprived for simplicity of its second term, namely,

$$x'^4 + px'^2 + qx' + r = 0,$$

so that the sum of the four roots vanishes,

$$x' + x'' + x''' + x^{IV} = 0,$$

we may consider x'', x''', x^{IV}, as roots of the cubic equation

$$x''^3 + x'x''^2 + (x'^2 + p)x'' + x'^3 + px' + q = 0;$$

and this may be put under the form

$$(x'' - \alpha)^3 - 3\eta(x'' - \alpha) - 2\epsilon = 0,$$

of which the roots (by the theory of cubic equations) may be expressed as follows:

$$x'' = \alpha + \beta + \gamma, \quad x''' = \alpha + \theta\beta + \theta^2\gamma, \quad x^{IV} = \alpha + \theta^2\beta + \theta\gamma,$$

β, γ, and θ being such as to satisfy the conditions

$$\beta^3 + \gamma^3 = 2\epsilon, \quad \beta\gamma = \eta, \quad \theta^2 + \theta + 1 = 0.$$

Comparing the two forms of the cubic equation in x'', we find the relations

$$x' = -3\alpha, \quad x'^2 + p = 3(\alpha^2 - \eta), \quad x'^3 + px' + q = -\alpha^3 + 3\alpha\eta - 2\epsilon;$$

which give

$$\alpha = -\tfrac{1}{3}x', \quad \eta = -\tfrac{1}{9}(2x'^2 + 3p), \quad \epsilon = -\tfrac{1}{54}(20x'^3 + 18px' + 27q).$$

Thus, any rational function of the four roots of the given biquadratic can be expressed rationally in terms of α, β, γ; while α, $\beta\gamma$, and $\beta^3 + \gamma^3$, are rational functions of x', p, q; and the function $x'^4 + px'^2 + qx'$ may be changed, wherever it occurs, to the given quantity $-r$.

11. With these preparations it is easy to express, as follows, the function

$$(x' - x'' + x''' - x^{IV})^2,$$

which the general theorems of LAGRANGE, already mentioned, lead us to consider. Denoting it by $4z$, we have

$$z = (-2\alpha + \theta\beta + \theta^2\gamma)^2 = \alpha' + \theta\beta' + \theta^2\gamma';$$

in which

$$\alpha' = 4\alpha^2 + 2\beta\gamma, \quad \beta' = \gamma^2 - 4\alpha\beta, \quad \gamma' = \beta^2 - 4\alpha\gamma:$$

and the three values of z are the three roots of the cubic equation

$$(z-\alpha')^3 - 3\eta'(z-\alpha') - 2\epsilon' = 0;$$

in which

$$\begin{aligned}
\alpha' &= 4\alpha^2 + 2\eta, \\
\eta' &= \beta'\gamma' = \eta^2 + 16\alpha^2\eta - 8\alpha\epsilon, \\
\epsilon' &= \tfrac{1}{2}(\beta'^3 + \gamma'^3) = 2\epsilon^2 - \eta^3 - 12\alpha\epsilon\eta + 48\alpha^2\eta^2 - 64\alpha^3\epsilon.
\end{aligned}$$

Substituting for α, η, ϵ, their values, as functions of x', p, q, we find

$$\begin{aligned}
\alpha' &= -\tfrac{2}{3}p; \\
\eta' &= \tfrac{1}{9}(-12x'^4 - 12px'^2 - 12qx' + p^2); \\
\epsilon' &= \tfrac{1}{54}(72px'^4 + 72p^2x'^2 + 72pqx' + 27q^2 + 2p^3);
\end{aligned}$$

and eliminating x', by the condition

$$x'^4 + px'^2 + qx' = -r,$$

we obtain

$$\begin{aligned}
\eta' &= \tfrac{1}{9}(12r + p^2); \\
\epsilon' &= \tfrac{1}{54}(-72pr + 27q^2 + 2p^3).
\end{aligned}$$

The auxiliary cubic in z becomes therefore

$$(z + \tfrac{2}{3}p)^3 - \tfrac{1}{3}(12r + p^2)(z + \tfrac{2}{3}p) + \tfrac{1}{27}(72pr - 27q^2 - 2p^3) = 0;$$

that is

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0;$$

and if its three roots be denoted by z', z'', z''', in an order such that we may write

$$\begin{aligned}
z' &= \tfrac{1}{4}(x' + x'' - x''' - x^{IV})^2 = \alpha' + \beta' + \gamma', \\
z'' &= \tfrac{1}{4}(x' - x'' + x''' - x^{IV})^2 = \alpha' + \theta\beta' + \theta^2\gamma', \\
z''' &= \tfrac{1}{4}(x' - x'' - x''' + x^{IV})^2 = \alpha' + \theta^2\beta' + \theta\gamma',
\end{aligned}$$

we may express the four roots of the biquadratic equation under known forms, by means of the square roots of z', z'', z''', as follows:

$$\begin{aligned}
x' &= +\tfrac{1}{2}\sqrt{z'} + \tfrac{1}{2}\sqrt{z''} + \tfrac{1}{2}\sqrt{z'''}, \\
x'' &= +\tfrac{1}{2}\sqrt{z'} - \tfrac{1}{2}\sqrt{z''} - \tfrac{1}{2}\sqrt{z'''}, \\
x''' &= -\tfrac{1}{2}\sqrt{z'} + \tfrac{1}{2}\sqrt{z''} - \tfrac{1}{2}\sqrt{z'''}, \\
x^{IV} &= -\tfrac{1}{2}\sqrt{z'} - \tfrac{1}{2}\sqrt{z''} + \tfrac{1}{2}\sqrt{z'''}.
\end{aligned}$$

It may be noticed also that the present method gives for the product of these three square roots, the expression:

$$\begin{aligned}\sqrt{z'} \,.\, \sqrt{z''} \,.\, \sqrt{z'''} &= \tfrac{1}{8}(x' + x'' - x''' - x^{IV})(x' - x'' + x''' - x^{IV})(x' - x'' - x''' + x^{IV}) \\ &= (-2\alpha + \beta + \gamma)(-2\alpha + \theta\beta + \theta^2\gamma)(-2\alpha + \theta^2\beta + \theta\gamma) \\ &= -8\alpha^3 + 6\alpha\eta + 2\epsilon = -q;\end{aligned}$$

a result which may be verified by observing that, by the expressions given above for α', η', ϵ', in terms of α, η, ϵ, we have the relation

$$z'z''z''' = \alpha'^3 - 3\alpha'\eta' + 2\epsilon' = (-8\alpha^3 + 6\alpha\eta + 2\epsilon)^2.$$

12. In this manner, then, it might have been discovered that the four roots x_1, x_2, x_3, x_4, of the general biquadratic equation

$$x^4 - \text{A}x^3 + \text{B}x^2 - \text{C}x + \text{D} = 0,$$

are the four values of an expression of the form $\alpha + \beta + \gamma + \delta$, in which, α, $\beta^2 + \gamma^2 + \delta^2$, $\beta\gamma\delta$, and $\beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2$, are rational functions of the coefficients A, B, C, D, and may be determined as such by comparison with the identical equation

$$\begin{aligned}(\alpha + \beta + \gamma + \delta - \alpha)^4 - 2(\beta^2 + \gamma^2 + \delta^2)(\alpha + \beta + \gamma + \delta - \alpha)^2 + (\beta^2 + \gamma^2 + \delta^2)^2 \\ = 8\beta\gamma\delta(\alpha + \beta + \gamma + \delta - \alpha) + 4(\beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2),\end{aligned}$$

of which each member is an expression for the square of $2(\beta\gamma + \gamma\delta + \delta\beta)$. It might have been perceived also that any three quantities, such as here β^2, γ^2, δ^2, which are the three roots of a given cubic equation, may be considered as the three values of an expression of the form $\alpha' + \beta' + \gamma'$, in which, α', $\beta'\gamma'$, and $\beta'^3 + \gamma'^3$ are rational functions of the coefficients of that given equation, and may have their forms determined by comparison with the identity,

$$(\alpha' + \beta' + \gamma' - \alpha')^3 - 3\beta'\gamma'(\alpha' + \beta' + \gamma' - \alpha') - \beta'^3 - \gamma'^3 = 0.$$

And finally that any two quantities which, as here β'^3 and γ'^3, are the two roots of a given quadratic equation, are also the two values of an expression of the form $\alpha'' + \beta''$, in which α'' and β''^2 may be determined by comparing the given equation with the following identical form,

$$(\alpha'' + \beta'' - \alpha'')^2 - \beta''^2 = 0.$$

Let us now endeavour to apply similar methods of expression to a system of five arbitrary quantities, or to an equation of the fifth degree.

13. Let, therefore, x_1, x_2, x_3, x_4, x_5, be the five roots of the equation

$$x^5 - \text{A}x^4 + \text{B}x^3 - \text{C}x^2 + \text{D}x - \text{E} = 0, \tag{1}$$

and let x', x'', x''', x^{IV}, x^V, be the five roots of the same equation when deprived of its second term, or put under the form

$$x'^5 + px'^3 + qx'^2 + rx' + s = 0, \tag{2}$$

so that

$$x' + x'' + x''' + x^{IV} + x^V = 0, \tag{3}$$

and

$$x_1 = x' + \tfrac{1}{5}\mathrm{A}, \quad x_2 = x'' + \tfrac{1}{5}\mathrm{A}, \quad \&\text{c.} \tag{4}$$

Dividing the equation of the fifth degree

$$x''^5 - x'^5 + p(x''^3 - x'^3) + q(x''^2 - x'^2) + r(x'' - x') = 0, \tag{5}$$

by the linear factor $x'' - x'$, we obtain the biquadratic

$$x''^4 + x'x''^3 + (x'^2 + p)x''^2 + (x'^3 + px' + q)x'' + x'^4 + px'^2 + qx' + r = 0, \tag{6}$$

of which the four roots are x'', x''', x^{IV}, x^V. Hence, by the theory of biquadratic equations, we may employ the expressions:

$$x'' = \alpha + \beta + \gamma + \delta, \quad x''' = \alpha + \beta - \gamma - \delta, \quad x^{IV} = \alpha - \beta + \gamma - \delta, \quad x^V = \alpha - \beta - \gamma + \delta; \tag{7}$$

provided that α, β, γ, δ are such as to satisfy, independently of x'', the condition:

$$\left.\begin{aligned} &(x'' - \alpha)^4 - 2(\beta^2 + \gamma^2 + \delta^2)(x'' - \alpha)^2 - 8\beta\gamma\delta(x'' - \alpha) \\ &\qquad + \beta^4 + \gamma^4 + \delta^4 - 2(\beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2) \\ &= x''^4 + x'x''^3 + (x'^2 + p)x''^2 + (x'^3 + px' + q)x'' + x'^4 + px'^2 + qx' + r; \end{aligned}\right\} \tag{8}$$

which decomposes itself into the four following:

$$\left.\begin{aligned} &- 4\alpha = x'; \\ &+ 6\alpha^2 - 2(\beta^2 + \gamma^2 + \delta^2) = x'^2 + p; \\ &- 4\alpha^3 + 4\alpha(\beta^2 + \gamma^2 + \delta^2) - 8\beta\gamma\delta = x'^3 + px' + q; \\ &+ \alpha^4 - 2\alpha^2(\beta^2 + \gamma^2 + \delta^2) + 8\alpha\beta\gamma\delta + (\beta^2 + \gamma^2 + \delta^2)^2 - 4(\beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2) \\ &\qquad\qquad = x'^4 + px'^2 + qx' + r; \end{aligned}\right\} \tag{9}$$

and, therefore, conducts to expressions for α, $\beta^2 + \gamma^2 + \delta^2$, $\beta\gamma\delta$, and $\beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2$, as rational functions of x', p, q, r. Again, by the theory of cubic equations, we may write:

$$\beta^2 = \epsilon + \kappa + \lambda, \quad \gamma^2 = \epsilon + \theta\kappa + \theta^2\lambda, \quad \delta^2 = \epsilon + \theta^2\kappa + \theta\lambda, \tag{10}$$

in which θ is a root of the equation

$$\theta^2 + \theta + 1 = 0, \tag{11}$$

while ϵ, $\kappa\lambda$, and $\kappa^3 + \lambda^3$ are symmetric function of β^2, γ^2, δ^2. Making, for abridgment,

$$\beta\gamma\delta = \eta, \quad \kappa\lambda = \iota, \tag{12}$$

we have, by (10) and (11),

$$\kappa^3 + \lambda^3 = \eta^2 + \epsilon^3 + 3\epsilon\iota, \tag{13}$$

and

$$\beta^2 + \gamma^2 + \delta^2 = 3\epsilon, \quad \beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\beta^2 = 3(\epsilon^2 - \iota); \tag{14}$$

and therefore, by (9),

$$\left.\begin{aligned} -4\alpha = x'; \quad 6(\alpha^2 - \epsilon) &= x'^2 + p; \\ -4\alpha^3 + 12\alpha\epsilon - 8\eta &= x'^3 + px' + q; \\ \alpha^4 - 6\alpha^2\epsilon + 8\alpha\eta - 3\epsilon^2 + 12\iota &= x'^4 + px'^2 + qx' + r; \end{aligned}\right\} \tag{15}$$

conditions which give

$$\left.\begin{aligned} \alpha &= -\tfrac{1}{4}x'; \\ \epsilon &= -\tfrac{1}{48}(5x'^2 + 8p); \\ \eta &= -\tfrac{1}{64}(5x'^3 + 4px' + 8q); \\ \iota &= +\tfrac{1}{144}(10x'^4 + 11px'^2 + 9qx' + p^2 + 12r). \end{aligned}\right\} \tag{16}$$

Thus, α, ϵ, η and ι, on the one hand, are rational functions of x', p, q, r; and, on the other hand, x', x'', x''', x^{IV}, x^{V} may be considered as functions, although not entirely rational, of α, ϵ, η, ι. In fact, if these four last quantities (denoted to help the memory by four Greek vowels) be supposed to be given, and if, by extraction of a square root and a cube root, a value of κ be found, which satisfies the auxiliary equation

$$\kappa^6 - (\eta^2 - \epsilon^3 + 3\epsilon\iota)\kappa^3 + \iota^3 = 0, \tag{17}$$

and then a corresponding value of λ by the condition $\kappa\lambda = \iota$, we shall have $\pm\beta$ by extraction of another square root, since $\beta^2 = \epsilon + \kappa + \lambda$; and may afterwards, by extraction of a third square root, either find $\pm\gamma$ from the expression $\gamma^2 = \epsilon + \theta\kappa + \theta^2\lambda$, and deduce δ from the product $\beta\gamma\delta = \eta$, or else find $\pm(\gamma + \delta)$ from the expression

$$(\gamma + \delta)^2 = 2\epsilon - \kappa - \lambda + \frac{2\eta}{\beta}; \tag{18}$$

and may then treat x'', x''', x^{IV}, x^{V}, as the four values of $\alpha + \beta + \gamma + \delta$, while $x' = -4\alpha$. Hence any function whatever of the five roots of the general equation (1) of the fifth degree may be considered as a function of the five quantities A, α, ϵ, η, ι; and if, in the expression of that function, the values (16) be substituted for α, ϵ, η, ι, so as to introduce in their stead the quantities x', p, q, r, it is permitted to make any simplifications of the result which can be obtained from the relation (2), by changing $x'^5 + px'^3 + qx'^2 + rx'$, wherever it occurs, to the known quantity $-s$.

14. Consider then the twentyfour-valued function, referred to in a former article, and suggested (as LAGRANGE has shown) by the analogy of equations of lower degrees; namely, t^5, in which

$$t = x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5, \tag{19}$$

and

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0; \tag{20}$$

ω here (and not α) denoting an imaginary fifth root of unity, so that

$$\omega^5 = 1. \tag{21}$$

Observing, that by (4) and (20), x_1, &c. may be changed in (19) to x', &c.; and distinguishing among themselves the 120 values of the function t by employing the notation

$$t_{abcde} = \omega^5 x^{(a)} + \omega^4 x^{(b)} + \omega^3 x^{(c)} + \omega^2 x^{(d)} + \omega^1 x^{(e)}, \tag{22}$$

which gives, for example,

$$t_{12345} = x' + \omega^4 x'' + \omega^3 x''' + \omega^2 x^{IV} + \omega x^{V}; \tag{23}$$

we shall have, on substituting for x' its value -4α, and for x'', x''', x^{IV}, x^{V}, their values (7), the system of twenty-four expressions following:

$$\left.\begin{array}{l}
t_{12345} = -5\alpha + \mathrm{B}\beta + \mathrm{C}\gamma + \mathrm{D}\delta;\\
t_{13254} = -5\alpha + \mathrm{B}\beta - \mathrm{C}\gamma - \mathrm{D}\delta;\\
t_{14523} = -5\alpha - \mathrm{B}\beta + \mathrm{C}\gamma - \mathrm{D}\delta;\\
t_{15432} = -5\alpha - \mathrm{B}\beta - \mathrm{C}\gamma + \mathrm{D}\delta;
\end{array}\right\} \tag{24}$$

$$\left.\begin{array}{l}
t_{12453} = -5\alpha + \mathrm{B}\gamma + \mathrm{C}\delta + \mathrm{D}\beta;\\
t_{14235} = -5\alpha + \mathrm{B}\gamma - \mathrm{C}\delta - \mathrm{D}\beta;\\
t_{15324} = - \quad - \quad + \quad -\\
t_{13542} = - \quad - \quad - \quad +
\end{array}\right\} \tag{25}$$

$$\left.\begin{array}{l}
t_{12534} = -5\alpha + \mathrm{B}\delta + \mathrm{C}\beta + \mathrm{D}\gamma;\\
t_{15243} = - \quad + \quad - \quad -\\
t_{13425} = - \quad - \quad + \quad -\\
t_{14352} = - \quad - \quad - \quad +
\end{array}\right\} \tag{26}$$

$$\left.\begin{array}{l}
t_{12354} = -5\alpha + \mathrm{B}\beta + \mathrm{C}\delta + \mathrm{D}\gamma;\\
t_{13245} = - \quad + \quad - \quad -\\
t_{15423} = - \quad - \quad + \quad -\\
t_{14532} = - \quad - \quad - \quad +
\end{array}\right\} \tag{27}$$

$$\left.\begin{aligned}t_{12543} &= -5\alpha + \text{B}\delta + \text{C}\gamma + \text{D}\beta;\\ t_{15234} &= - \quad + \quad - \quad -\\ t_{14325} &= - \quad - \quad + \quad -\\ t_{13452} &= - \quad - \quad - \quad +\end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned}t_{12435} &= -5\alpha + \text{B}\gamma + \text{C}\beta + \text{D}\delta;\\ t_{14253} &= - \quad + \quad - \quad -\\ t_{13524} &= - \quad - \quad + \quad -\\ t_{15342} &= - \quad - \quad - \quad +\end{aligned}\right\} \tag{29}$$

in which we have made, for abridgment,

$$\left.\begin{aligned}\text{B} &= \omega^4 + \omega^3 - \omega^2 - \omega,\\ \text{C} &= \omega^4 - \omega^3 + \omega^2 - \omega,\\ \text{D} &= \omega^4 - \omega^3 - \omega^2 + \omega.\end{aligned}\right\} \tag{30}$$

But also, by (22) and (21),

$$t_{bcdea} = \omega t_{abcde}, \quad t^5_{bcdea} = t^5_{abcde}; \tag{31}$$

making then

$$t^5_{1abcd} = \text{T}_{abcd}, \tag{32}$$

the twenty-four values of the function t^5 will be those of the function T which arise from arranging in all possible ways the four indices 2, 3, 4, 5; that is, they are fifth powers of the twenty-four expressions (24) … (29). It is required, therefore, to develop these fifth powers, and to examine into their composition.

15. For this purpose it is convenient first to consider those parts of any one such power, which are common to the three other powers of the same group, (24) or (35), &c., and, therefore, to introduce the consideration of six new functions, determined by the following definition:

$$\text{V}_{abc} = \tfrac{1}{4}(\text{T}_{2abc} + \text{T}_{a2cb} + \text{T}_{bc2a} + \text{T}_{cba2}); \tag{33}$$

which gives, for example,

$$\left.\begin{aligned}\text{V}_{345} = (-5\alpha)^5 &+ 60(-5\alpha)^2\text{BCD}\beta\gamma\delta\\ &+ 10\{(-5\alpha)^3 + 2\text{BCD}\beta\gamma\delta\}(\text{B}^2\beta^2 + \text{C}^2\gamma^2 + \text{D}^2\delta^2)\\ &+ 5(-5\alpha)(\text{B}^4\beta^4 + \text{C}^4\gamma^4 + \text{D}^4\delta^4 + 6\text{B}^2\text{C}^2\beta^2\gamma^2 + 6\text{C}^2\text{D}^2\gamma^2\delta^2 + 6\text{D}^2\text{B}^2\delta^2\beta^2);\end{aligned}\right\} \tag{34}$$

this being (as is evident on inspection) the part common to the four functions T_{2345}, T_{3254}, T_{4523}, T_{5432}, or to the fifth powers of the four expressions in the group (24). By changing β, γ, δ, first to γ, δ, β, and afterwards to δ, β, γ, the expression (34) for V_{345} will be changed successively to those for V_{453} and V_{534}, which, therefore, it is unnecessary to write; and V_{354}, V_{543}, V_{435}, may be formed, respectively, from V_{345}, V_{453}, V_{534}, by interchanging γ and δ. Or, after substituting in (34) for β^2, γ^2, δ^2, their values (10), and writing η for $\beta\gamma\delta$, it will only be necessary to multiply κ by θ, and λ by θ^2, wherever they occur, in order to change V_{345} to V_{453}; and to repeat this process, in order to change V_{453} to V_{534}: while V_{345}, V_{453}, V_{534} will be changed, respectively, to V_{354}, V_{543}, V_{435}, by interchanging θ and θ^2, or κ and λ.

16. In this manner it is not difficult to perceive that we may write

$$\left.\begin{aligned} \mathrm{V}_{345} &= g + h + i, \\ \mathrm{V}_{453} &= g + \theta h + \theta^2 i, \\ \mathrm{V}_{534} &= g + \theta^2 h + \theta i, \end{aligned}\right\} \tag{35}$$

and

$$\left.\begin{aligned} \mathrm{V}_{354} &= g' + h' + i', \\ \mathrm{V}_{543} &= g' + \theta h' + \theta^2 i', \\ \mathrm{V}_{435} &= g' + \theta^2 h' + \theta i', \end{aligned}\right\} \tag{36}$$

in which,

$$\left.\begin{aligned} g = g' = (-5\alpha)^5 &+ 60(-5\alpha)^2\eta\mathrm{BCD} + 10\{(-5\alpha)^3 + 2\eta\mathrm{BCD}\}\epsilon(\mathrm{B}^2 + \mathrm{C}^2 + \mathrm{D}^2) \\ &+ 5(-5\alpha)\epsilon^2(\mathrm{B}^4 + \mathrm{C}^4 + \mathrm{D}^4 + 6\mathrm{C}^2\mathrm{D}^2 + 6\mathrm{D}^2\mathrm{B}^2 + 6\mathrm{B}^2\mathrm{C}^2) \\ &+ 10(-5\alpha)\iota(\mathrm{B}^4 + \mathrm{C}^4 + \mathrm{D}^4 - 3\mathrm{C}^2\mathrm{D}^2 - 3\mathrm{D}^2\mathrm{B}^2 - 3\mathrm{B}^2\mathrm{C}^2); \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} h &= k\kappa + l\lambda^2, \quad & i &= \kappa'\lambda + l'\kappa^2; \\ h' &= k\lambda + l\kappa^2, \quad & i' &= k'\kappa + l'\lambda^2; \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} k = 10\{&(-5\alpha)^3 + 2\eta\mathrm{BCD}\}(\mathrm{B}^2 + \theta\mathrm{C}^2 + \theta^2\mathrm{D}^2) \\ &+ 10(-5\alpha)\epsilon(\mathrm{B}^4 + \theta\mathrm{C}^4 + \theta^2\mathrm{D}^4 - 3\mathrm{C}^2\mathrm{D}^2 - 3\theta\mathrm{D}^2\mathrm{B}^2 - 3\theta^2\mathrm{B}^2\mathrm{C}^2); \\ l = 5(&-5\alpha)(\mathrm{B}^4 + \theta\mathrm{C}^4 + \theta^2\mathrm{D}^4 + 6\mathrm{C}^2\mathrm{D}^2 + 6\theta\mathrm{D}^2\mathrm{B}^2 + 6\theta^2\mathrm{B}^2\mathrm{C}^2); \end{aligned}\right\} \tag{39}$$

and k', l' are formed from k, l, by interchanging θ and θ^2. Hence also, by the same properties of ϵ, η, ι, which were employed in deducing these equations, we have:

$$\left.\begin{aligned} hh' &= k^2\iota + l^2\iota^2 + kl(\eta^2 - \epsilon^3 + 3\epsilon\iota); \\ h^3 + h'^3 &= 2(3k^2 - l^2\iota)l\iota^2 + (k^2 + 3l^2\iota)k(\eta^2 - \epsilon^3 + 3\epsilon\iota) + l^3(\eta^2 - \epsilon^3 + 3\epsilon\iota)^2; \end{aligned}\right\} \tag{40}$$

and ii', $i^3 + i'^3$ have corresponding expressions, obtained by accenting k and l.

17. If then we make

$$g = \mathrm{H}_1 + \sqrt{\mathrm{H}_2}, \quad g' = \mathrm{H}_1 - \sqrt{\mathrm{H}_2}; \tag{41}$$

$$h^3 + h'^3 = 2\mathrm{H}_3; \quad h^3 - h'^3 = 2\sqrt{\mathrm{H}_4}; \tag{42}$$

$$i'^3 + i^3 = 2\mathrm{H}_5, \quad i'^3 - i^3 = 2\sqrt{\mathrm{H}_6}; \tag{43}$$

we see that the six functions V may be expressed by the help of square-roots and cube-roots, in terms of these six quantities H, by means of the following formulæ:

$$\left.\begin{aligned} \mathrm{V}_{345} &= \mathrm{H}_1 + \sqrt{\mathrm{H}_2} + \sqrt[3]{\mathrm{H}_3 + \sqrt{\mathrm{H}_4}} + \sqrt[3]{\mathrm{H}_5 - \sqrt{\mathrm{H}_6}}; \\ \mathrm{V}_{453} &= \mathrm{H}_1 + \sqrt{\mathrm{H}_2} + \theta\sqrt[3]{\mathrm{H}_3 + \sqrt{\mathrm{H}_4}} + \theta^2\sqrt[3]{\mathrm{H}_5 - \sqrt{\mathrm{H}_6}}; \\ \mathrm{V}_{534} &= \mathrm{H}_1 + \sqrt{\mathrm{H}_2} + \theta^2\sqrt[3]{\mathrm{H}_3 + \sqrt{\mathrm{H}_4}} + \theta\sqrt[3]{\mathrm{H}_5 - \sqrt{\mathrm{H}_6}}; \end{aligned}\right\} \tag{a}$$

and

$$\left.\begin{aligned}
\mathrm{V}_{354} &= \mathrm{H}_1 - \sqrt{\mathrm{H}_2} + \sqrt[3]{\mathrm{H}_3 - \sqrt{\mathrm{H}_4}} + \sqrt[3]{\mathrm{H}_5 + \sqrt{\mathrm{H}_6}};\\
\mathrm{V}_{543} &= \mathrm{H}_1 - \sqrt{\mathrm{H}_2} + \theta\sqrt[3]{\mathrm{H}_3 - \sqrt{\mathrm{H}_4}} + \theta^2\sqrt[3]{\mathrm{H}_5 + \sqrt{\mathrm{H}_6}};\\
\mathrm{V}_{435} &= \mathrm{H}_1 - \sqrt{\mathrm{H}_2} + \theta^2\sqrt[3]{\mathrm{H}_3 - \sqrt{\mathrm{H}_4}} + \theta\sqrt[3]{\mathrm{H}_5 + \sqrt{\mathrm{H}_6}}:
\end{aligned}\right\} \tag{b}$$

which have accordingly, with some slight differences of notation, been assigned by Professor BADANO, as among the results of his method of treating equations of the fifth degree. We see too, that the six quantities $\mathrm{H}_1, \ldots \mathrm{H}_6$, (of which indeed the second, namely H_2, vanishes), are rational functions of α, ϵ, η, ι; and therefore, by article 13., of x', p, q, r. But it is necessary to examine whether it be true, as Professor BADANO appears to think (guided in part, as he himself states, by the analogy of equations of lower degrees), that these quantities H are all rational functions of the coefficients p, q, r, s, of the equation (2) of the fifth degree; or, in other words, to examine whether it be possible to eliminate from the expressions of those six quantities H, the unknown root x' of that equation, by its means, in the same way as it was found possible, in articles 11. and 9. of the present paper, to eliminate from the correspondent expressions, the roots of the biquadratic and cubic equations which it was there proposed to resolve. For, if it shall be found that any one of the six quantities $\mathrm{H}_1, \ldots \mathrm{H}_6$, which enter into the formulæ (a) and (b), depends essentially, and not merely in appearance, on the unknown root x'; so as to change its value when that root is changed to another, such as x'', which satisfies the same equation (2): it will then be seen that these formulæ, although true, give no assistance towards the general solution of the equation of the fifth degree.

18. The auxiliary quantities ω, B, C, D, being such that, by their definitions (20) and (30),

$$\left.\begin{aligned}
-1 + \mathrm{B} + \mathrm{C} + \mathrm{D} &= 4\omega^4,\\
-1 + \mathrm{B} - \mathrm{C} - \mathrm{D} &= 4\omega^3,\\
-1 - \mathrm{B} + \mathrm{C} - \mathrm{D} &= 4\omega^2,\\
-1 - \mathrm{B} - \mathrm{C} + \mathrm{D} &= 4\omega,
\end{aligned}\right\} \tag{44}$$

while ω, ω^2, ω^3, ω^4 are the four imaginary fifth roots of unity, we shall have, by the theory of biquadratics already explained, the following identical equation:

$$\left.\begin{aligned}
&\{(x+1)^2 - (\mathrm{B}^2 + \mathrm{C}^2 + \mathrm{D}^2)\}^2 - 8\mathrm{BCD}(x+1) - 4(\mathrm{B}^2\mathrm{C}^2 + \mathrm{C}^2\mathrm{D}^2 + \mathrm{D}^2\mathrm{B}^2)\\
&\qquad = \{(x+1)^2 + 5\}^2 + 40(x+1) + 180,
\end{aligned}\right\} \tag{45}$$

the second member being equivalent to

$$x^4 + 4x^3 + 4^2x^2 + 4^3x + 4^4;$$

we find, therefore, that

$$\mathrm{B}^2 + \mathrm{C}^2 + \mathrm{D}^2 = -5; \quad \mathrm{BCD} = -5; \quad \mathrm{B}^2\mathrm{C}^2 + \mathrm{C}^2\mathrm{D}^2 + \mathrm{D}^2\mathrm{B}^2 = -45; \tag{46}$$

and, consequently,

$$\mathrm{B}^4 + \mathrm{C}^4 + \mathrm{D}^4 = 115. \tag{47}$$

Hence, by (37), the common value of g and g', considered as a function of α, ϵ, η, ι, is:

$$g = g' = 125(-25\alpha^5 + 50\alpha^3\epsilon - 60\alpha^2\eta + 31\alpha\epsilon^2 - 100\alpha\iota + 4\epsilon\eta); \tag{48}$$

and if in this we substitute, for the quantities α, ϵ, η, ι, their values (16), or otherwise eliminate those quantities by the relations (15), and attend to the definitions (41) of the quantities H_1 and H_2, we find:

$$\mathrm{H}_1 = \frac{125}{12}(25x'^5 + 25px'^3 + 25qx'^2 + 25rx' + pq); \tag{49}$$

and, as was said already,

$$\mathrm{H}_2 = 0. \tag{50}$$

It is therefore true, of *these* two quantities H, that they are independent of the root x' of the proposed equation of the fifth degree, or remain unchanged when that root is changed to another, such as x'', which satisfies the same equation: since it is possible to eliminate x' from the expression (49) by means of the proposed equation (2), and so to obtain H_1 as a rational function of the coefficients of that equation, namely,

$$\mathrm{H}_1 = \frac{125}{12}(pq - 25s). \tag{51}$$

Indeed, it was evident *à priori* that H_1 must be found to be equal to *some* rational function of those four coefficients, p, q, r, s, or some symmetric function of the five roots of the equation (2); because it is, by its definition, the sixth part of the sum of the six functions V, and, therefore, the twenty-fourth part of the sum of the twenty-four different values of the function T; or finally the mean of all the different values which the function t^5 can receive, by all possible changes of arrangement of the five roots, $x', \dots x^V$, or $x_1, \dots x_5$, among themselves. The evanescence of H_2 shows farther, that, in the arrangement assigned above, the sum of the three first of the six functions V, or the sum of the twelve first of the twenty-four functions T, is equal to the sum of the other three, or of the other twelve of these functions. But we shall find that it would be erroneous to conclude, from the analogy of these results, even when combined with the corresponding results for equations of inferior degrees, that the other four quantities H, which enter into the formulæ (a) and (b), can likewise be expressed as rational functions of the coefficients of the equation of the fifth degree.

19. The auxiliary quantities B^2, C^2, D^2, being seen, by (46), to be the three roots z_1, z_2, z_3, of the cubic equation

$$z^3 + 5z^2 - 45z - 25 = 0, \tag{52}$$

which decomposes itself into one of the first and another of the second degree, namely,

$$z - 5 = 0, \quad z^2 + 10z + 5 = 0; \tag{53}$$

we see that one of the three quantities B, C, D, must be real, and $= \pm\sqrt{5}$, while the other two must be imaginary. And on referring to the definitions (30), and remembering that ω is an imaginary fifth root of unity, so that ω^4 and ω^3 are the reciprocals of ω and ω^2, we easily perceive that the real one of the three is D, and that the following expressions hold good:

$$\mathrm{B}^2 = -5 - 2\mathrm{D}; \quad \mathrm{C}^2 = -5 + 2\mathrm{D}; \quad \mathrm{D}^2 = 5; \tag{54}$$

with which we may combine, whenever it may be necessary or useful, the relation

$$\mathrm{BC} = -\mathrm{D}. \tag{55}$$

If then we make, for abridgment,

$$\zeta = (\theta - \theta^2)\mathrm{D} = (\theta - \theta^2)(\omega^4 - \omega^3 - \omega^2 - \omega), \tag{56}$$

θ being still the same imaginary cubic root of unity as before, so that

$$\zeta^2 = -15; \tag{57}$$

we shall have, in (39),

$$\left.\begin{aligned} &\mathrm{D}^2 + \theta\mathrm{B}^2 + \theta^2\mathrm{C}^2 = 10 - 2\zeta, \\ &\mathrm{D}^4 + \theta\mathrm{B}^4 + \theta^2\mathrm{C}^4 = -20 + 20\zeta, \\ &\mathrm{B}^2\mathrm{C}^2 + \theta\mathrm{C}^2\mathrm{D}^2 + \theta^2\mathrm{D}^2\mathrm{B}^2 = 30 + 10\zeta; \end{aligned}\right\} \tag{58}$$

and, consequently (because $\mathrm{BCD} = -5$),

$$\left.\begin{aligned} \theta k &= -100(5 - \zeta)(25\alpha^3 + 2\eta) + 500(11 + \zeta)\alpha\epsilon; \\ \theta l &= -2000(2 + \zeta)\alpha; \end{aligned}\right\} \tag{59}$$

while $\theta^2 k'$ and $\theta^2 l'$ are formed from θk and θl, by changing the signs of ζ. It is easy, therefore, to see, by the remarks already made, and by the definitions (42) and (43), that the quantities H_3, H_4, H_5, H_6, when expressed as rational functions of α, ϵ, η, ι, or of x', p, q, r, will not involve either of the two imaginary roots of unity, θ and ω, except so far as they may involve the combination ζ of those roots, or the radical $\sqrt{-15}$; and that H_5 will be formed from H_3, and H_6 from H_4, by changing the sign of this radical. We shall now proceed to study, in particular, the composition of the quantity H_4; because, although this quantity, when expressed by means of x', p, q, r, is of the thirtieth dimension relatively to x', (p, q, and r being considered as of the second, third, and fourth dimensions, respectively), while H_3 rises no higher than the fifteenth dimension; yet we shall find it possible to decompose H_4 into two factors, of which one is the twelfth dimension, and has a very simple meaning, being the product of the squares of the differences of the four roots x'', x''', x^{IV}, x^{V}; while the other factor of H_4 is an exact square, of a function of the ninth dimension. We shall even see it to be possible to decompose this last function into three factors, which are each as low as the third dimension, and are rational functions of the five roots of the original equation of the fifth degree; whereas it does not appear that H_3, when regarded as a function of the same five roots, can be decomposed into more than three rational factors, nor that any of these can be depressed below the fifth dimension.

20. Confining ourselves then for the present to the consideration of H_4, we have, by (42) and (38), the following expression for the square-root of that quantity:

$$\sqrt{\text{H}_4} = \tfrac{1}{2}(\kappa^3 - \lambda^3)\{k^3 - 3kl^2\kappa\lambda - l^3(\kappa^3 + \lambda^3)\}; \tag{60}$$

and, therefore, by (59), and by the same relations between κ, λ, and ϵ, η, ι, which were used in deducing the formulæ of the sixteenth article, we obtain the following expression for the quantity H_4 itself, considered as a function of α, ϵ, η, ι:

$$\text{H}_4 = 2^{10}5^{18}\{(\eta^2 - \epsilon^3 + 3\epsilon\iota)^2 - 4\iota^3\}\text{L}^2; \tag{61}$$

in which we have made, for abridgment,

$$\text{L} = \mu^3 - 3\iota\mu\nu^2 + (\eta^2 - \epsilon^3 + 3\epsilon\iota)\nu^3, \tag{62}$$

and

$$\mu = (-5 + \zeta)(5\alpha^3 + \tfrac{2}{5}\eta) + (11 + \zeta)\alpha\epsilon, \quad \nu = 4(2 + \zeta)\alpha. \tag{63}$$

Now, without yet entering on the actual process of substituting, in the expression (61), the values (16) for α, ϵ, η, ι; or of otherwise eliminating those four quantities by means of the equations (15), in order to express H_4 as a function of x', p, q, r, from which x' is afterwards to be eliminated, as far as possible, by the equation of the fifth degree; we see that, in agreement with the remarks made in the last article, this expression (61) contains (besides its numerical coefficient) one factor, namely,

$$(\eta^2 - \epsilon^3 + 3\epsilon\iota)^2 - 4\iota^3 = (\kappa^3 - \lambda^3)^2, \tag{64}$$

which is of the twelfth dimension; and another, namely, L^2, which is indeed itself of the eighteenth, but is the square of a function (62), which is only of the ninth dimension: because α, ϵ, η, ι, are to be considered as being respectively of the first, second, third, and fourth dimensions; and, therefore, μ is to be regarded as being of the third, and ν of the first dimension.

21. Again, on examining the factor (64), we see that it is the square of another function, namely $\kappa^3 - \lambda^3$, which is itself of the sixth dimension, and is rational with respect to x'', x''', x^{IV}, x^V, though not with respect to α, ϵ, η, ι, nor with respect to x', p, q, r. This function $\kappa^3 - \lambda^3$ may even be decomposed into six linear factors; for first, we have, by (11),

$$\kappa^3 - \lambda^3 = (\kappa - \lambda)(\kappa - \theta\lambda)(\kappa - \theta^2\lambda); \tag{65}$$

and secondly, by (10),

$$3\kappa = \beta^2 + \theta^2\gamma^2 + \theta\delta^2, \quad 3\lambda = \beta^2 + \theta\gamma^2 + \theta^2\delta^2, \tag{66}$$

expressions which give

$$\left.\begin{aligned} \kappa - \lambda &= \tfrac{1}{3}(\theta - \theta^2)(\delta^2 - \gamma^2), \\ \kappa - \theta\lambda &= \tfrac{1}{3}(1 - \theta)(\beta^2 - \delta^2), \\ \kappa - \theta^2\lambda &= \tfrac{1}{3}(\theta^2 - 1)(\gamma^2 - \beta^2); \end{aligned}\right\} \tag{67}$$

but also, by (7),

$$\left.\begin{aligned}\delta^2-\gamma^2&=\tfrac{1}{4}(x''-x''')(x^V-x^{IV}),\\ \beta^2-\delta^2&=\tfrac{1}{4}(x''-x^{IV})(x'''-x^V),\\ \gamma^2-\beta^2&=\tfrac{1}{4}(x''-x^V)(x^{IV}-x''');\end{aligned}\right\}\tag{68}$$

and

$$(\theta-\theta^2)(1-\theta)(\theta^2-1)=(1-\theta)^3=-3(\theta-\theta^2);\tag{69}$$

therefore,

$$\kappa^3-\lambda^3=-2^{-6}3^{-2}(\theta-\theta^2)(x''-x''')(x''-x^{IV})(x''-x^V)(x'''-x^{IV})(x'''-x^V)(x^{IV}-x^V).\tag{70}$$

Thus, then, the square of the product of these six linear factors (70), and of the numerical coefficients annexed, is equal to the function (64), of the twelfth dimension, which itself entered as a factor into the expression (61) for H_4; and we see that this square is free from the imaginary radical θ, because, by (11),

$$(\theta-\theta^2)^2=-3;\tag{71}$$

and that it is a symmetric function of the four roots, x'', x''', x^{IV}, x^V, being proportional to the product of the squares of their differences, as was stated in article 19.: so that this square (though not its root) may be expressed, in virtue of the biquadratic equation (6), as a rational function of x', p, q, r; which followed also from its being expressible rationally, by (64), in terms of ϵ, η, ι.

22. Introducing now, in the expression (64), here referred to, the values (16), or the relations (15), we find, after reductions:

$$\left.\begin{aligned}\kappa^3+\lambda^3&=\eta^2-\epsilon^3+3\epsilon\iota\\ &=-2^{-6}3^{-3}\{25x'^6+75px'^4+(48p^2+45r)x'^2+27pqx'\\ &\quad-2p^3+72pr-27q^2\};\end{aligned}\right\}\tag{72}$$

$$\left.\begin{aligned}(\kappa^3+\lambda^3)^2&=(\eta^2-\epsilon^3+3\epsilon\iota)^2\\ &=2^{-12}3^{-6}\{625x'^{12}+3750px'^{10}+(8025p^2+2250r)x'^8+1350pqx'^7\\ &\quad+(7100p^3+10350pr-1350q^2)x'^6+4050p^2qx'^5\\ &\quad+(2004p^4+15120p^2r-4050pq^2+2025r^2)x'^4\\ &\quad+(2592p^3q+2430pqr)x'^3\\ &\quad+(-192p^5+6732p^3r-1863p^2q^2+6480pr^2-2430q^2r)x'^2\\ &\quad+(-108p^4q+3888p^2qr-1458pq^3)x'\\ &\quad+4p^6-288p^4r+108p^3q^2+5184p^2r^2-3888pq^2r+729q^4\};\end{aligned}\right\}\tag{73}$$

$$\left.\begin{aligned}
4\kappa^3\lambda^3 &= 4\iota^3 \\
&= 2^{-10}3^{-6}\{1000x'^{12} + 3300px'^{10} + 2700qx'^9 + (3930p^2 + 3600r)x'^8 \\
&\quad + 5940pqx'^7 + (1991p^3 + 7920pr + 2430q^2)x'^6 + (3807p^2q + 6480qr)x'^5 \\
&\quad + (393p^4 + 5076p^2r + 2673pq^2 + 4320r^2)x'^4 \\
&\quad + (594p^3q + 7128pqr + 729q^3)x'^3 \\
&\quad + (33p^5 + 792p^3r + 243p^2q^2 + 4752pr^2 + 2916q^2r)x'^2 \\
&\quad + (27p^4q + 648p^2qr + 3888qr^2)x' + p^6 + 36p^4r + 432p^2r^2 + 1728r^3\};
\end{aligned}\right\} \qquad (74)$$

and, finally,

$$\left.\begin{aligned}
(\kappa^3 - \lambda^3)^2 &= (\eta^2 - \epsilon^3 + 3\epsilon\iota)^2 - 4\iota^3 \\
&= -2^{-12}3^{-3}\{125x'^{12} + 350px'^{10} + 400qx'^9 + (285p^2 + 450r)x'^8 \\
&\quad + 830pqx'^7 + (32p^3 + 790pr + 410q^2)x'^6 + (414p^2q + 960qr)x'^5 \\
&\quad + (-16p^4 + 192p^2r + 546pq^2 + 565r^2)x'^4 \\
&\quad + (-8p^3q + 966pqr + 108q^3)x'^3 \\
&\quad + (12p^5 - 132p^3r + 105p^2q^2 + 464pr^2 + 522q^2r)x'^2 \\
&\quad + (8p^4q - 48p^2qr + 54pq^3 + 576qr^2)x' \\
&\quad + 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r + 256r^3 - 27q^4\}.
\end{aligned}\right\} \qquad (75)$$

23. This last result may be verified, or rather proved anew, and at the same time put under another form, which we shall find to be useful, by a process such as the following. The biquadratic equation (6), of which the roots are x'', x''', x^{IV}, x^V, shows that, whatever x may be,

$$\left.\begin{aligned}
&(x - x'')(x - x''')(x - x^{IV})(x - x^V) \\
&\qquad = x^4 + x'x^3 + x'^2x^2 + x'^3x + x'^4 + p(x^2 + x'x + x'^2) + q(x + x') + r;
\end{aligned}\right\} \qquad (76)$$

and, therefore, that

$$(x' - x'')(x' - x''')(x' - x^{IV})(x' - x^V) = 5x'^4 + 3px'^2 + 2qx' + r. \qquad (77)$$

If we then multiply the expression (75) by the square of this last function (77), we ought to obtain a symmetric function of all the five roots of the equation of the fifth degree, namely, the product of the ten squares of their differences, multiplied indeed by a numerical coefficient, namely, $-2^{-12}3^{-3}$, as appears from (70) and (71): and consequently an expression for this product itself, that is for

$$\left.\begin{aligned}
\mathrm{P} &= (x' - x'')^2(x' - x''')^2(x' - x^{IV})^2(x' - x^V)^2(x'' - x''')^2 \\
&\quad \times (x'' - x^{IV})^2(x'' - x^V)^2(x''' - x^{IV})^2(x''' - x^V)^2(x^{IV} - x^V)^2,
\end{aligned}\right\} \qquad (78)$$

must be obtained by multiplying the factor $125x'^{12}$ + &c. which is within the brackets in (75), by the square of $5x'^4 + 3px'^2 + 2qx' + r$, and then reducing by the condition that $x'^5 + px'^3 + qx'^2 + rx' = -s$. Accordingly this process gives:

$$\left.\begin{aligned}\mathrm{P} = {} & 3125s^4 - 3750pqs^3 \\ & + (108p^5 - 900p^3r + 825p^2q^2 + 2000pr^2 + 2250q^2r)s^2 \\ & - (72p^4qr - 16p^3q^3 - 560p^2qr^2 + 630pq^3r + 1600qr^3 - 108q^5)s \\ & + 16p^4r^3 - 4p^3q^2r^2 - 128p^2r^4 + 144pq^2r^3 + 256r^5 - 27q^4r^2;\end{aligned}\right\} \tag{79}$$

an expression for the product of the squares of the differences of the five roots of an equation of the fifth degree, which agrees with known results. And we see that with this meaning of P, we may write:

$$(\kappa^3 - \lambda^3)^2 = -2^{-12}3^{-3}\mathrm{P}(5x'^4 + 3px'^2 + 2qx' + r)^{-2}. \tag{80}$$

The expression (61) for H_4 becomes, therefore:

$$\mathrm{H}_4 = -2^{-2}3^{-3}5^{18}\mathrm{P}\left(\frac{\mu^3 - 3\iota\mu\nu^2 + (\eta^2 - \epsilon^3 + 3\epsilon\iota)\nu^3}{5x'^4 + 3px'^2 + 2qx' + r}\right)^2; \tag{81}$$

μ and ν having the meanings defined by (63).

24. With respect now to the factor L, which enters by its square into the expression (61), and is the numerator of the fraction which is squared in the form (81), we have, by (62), (63), and (57),

$$\left.\begin{aligned}\mathrm{L} = {} & \tfrac{4}{5}(15625\alpha^9 + 24375\alpha^7\epsilon + 3750\alpha^6\eta - 16125\alpha^5\epsilon^2 \\ & + 1500\alpha^5\iota + 3900\alpha^4\epsilon\eta + 7605\alpha^3\epsilon^3 - 8820\alpha^3\epsilon\iota - 6260\alpha^3\eta^2 \\ & - 1290\alpha^2\epsilon^2\eta + 120\alpha^2\eta\iota + 156\alpha\epsilon\eta^2 + 8\eta^3) \\ & + \tfrac{12}{25}\zeta(15625(\alpha^9 - \alpha^7\epsilon) + 3750\alpha^6\eta - 125\alpha^5\epsilon^2 + 15500\alpha^5\iota - 2500\alpha^4\epsilon\eta \\ & + 1125\alpha^3\epsilon^3 - 4500\alpha^3\epsilon\iota - 100\alpha^3\eta^2 - 10\alpha^2\epsilon^2\eta + 1240\alpha^2\eta\iota - 100\alpha\epsilon\eta^2 + 8\eta^3);\end{aligned}\right\} \tag{82}$$

and when we substitute for α, ϵ, η, ι, their values (16), we find, after reductions, a result which may be thus written:

$$2^6 5^2\mathrm{L} = 5\mathrm{L}' - \zeta\mathrm{L}''; \tag{83}$$

if we make, for abridgment,

$$\left.\begin{aligned}\mathrm{L}' = {} & 25px'^7 + 275qx'^6 + (135p^2 - 350r)x'^5 + 210pqx'^4 \\ & + (141p^3 - 500pr + 385q^2)x'^3 \\ & + (93p^2q - 20qr)x'^2 + 20pq^2x' - 4q^3; \\ \mathrm{L}'' = {} & 1750x'^9 + 2825px'^7 + 2100qx'^6 + (1120p^2 + 1825r)x'^5 + 1615pqx'^4 \\ & + (39p^3 + 1060pr + 500q^2)x'^3 + (109p^2q + 620qr)x'^2 + 68pq^2x' + 12q^3.\end{aligned}\right\} \tag{84}$$

With these meanings of L' and L'', the quantity H_4, considered as a rational function of x', p, q, r, may therefore be thus expressed:

$$\text{H}_4 = -2^{-14}3^{-3}5^{14}\text{P}\left(\frac{5\text{L}' - \zeta\text{L}''}{5x'^4 + 3px'^2 + 2qx' + r}\right)^2; \tag{85}$$

P being still the quantity (79), and ζ being still $= \sqrt{-15}$.

25. Depressing, next, as far as possible, the degrees of the powers of x', by means of the equation (2) of the fifth degree which x' must satisfy, we find:

$$\left.\begin{aligned} \text{L}' &= \text{L}'_0 + \text{L}'_1 x' + \text{L}'_2 x'^2 + \text{L}'_3 x'^3 + \text{L}'_4 x'^4; \\ \text{L}'' &= \text{L}''_0 + \text{L}''_1 x' + \text{L}''_2 x'^2 + \text{L}''_3 x'^3 + \text{L}''_4 x'^4; \end{aligned}\right\} \tag{86}$$

in which the coefficients are thus composed:

$$\left.\begin{aligned} \text{L}'_0 &= -110p^2s - 4q^3 + 350rs, \\ \text{L}'_1 &= -110p^2r + 20pq^2 - 275qs + 350r^2, \\ \text{L}'_2 &= -17p^2q - 25ps - 55qr, \\ \text{L}'_3 &= +31p^3 - 175pr + 110q^2, \\ \text{L}'_4 &= -90pq; \end{aligned}\right\} \tag{87}$$

and

$$\left.\begin{aligned} \text{L}''_0 &= -45p^2s + 12q^3 - 75rs; \\ \text{L}''_1 &= -45p^2r + 68pq^2 - 350ps - 75r^2; \\ \text{L}''_2 &= +64p^2q - 1075ps + 195qr; \\ \text{L}''_3 &= -6p^3 - 90pr + 150q^2; \\ \text{L}''_4 &= +190pq - 1750s. \end{aligned}\right\} \tag{88}$$

But because, after the completion of all these transformations and reductions, it is seen that the five quantities

$$5\text{L}'_0 - \zeta\text{L}''_0, \quad 5\text{L}'_1 - \zeta\text{L}''_1, \quad 5\text{L}'_2 - \zeta\text{L}''_2, \quad 5\text{L}'_3 - \zeta\text{L}''_3, \quad 5\text{L}'_4 - \zeta\text{L}''_4, \tag{89}$$

which become the coefficients of x'^0, x'^1, x'^2, x'^3, x'^4, in the numerator $5\text{L}' - \zeta\text{L}''$ of the fraction to be squared in the formula (85), are not proportional to the five other quantities

$$r, \quad 2q, \quad 3p, \quad 0, \quad 5, \tag{90}$$

which are the coefficients of the same five powers of x' in the denominator of the same fraction, it may be considered as already evident, at this stage of the investigation, that the root x' enters, not apparently, but also really, into the composition of the quantity H_4.

26. The foregoing calculations have been laborious, but they have been made and verified with care, and it is believed that the results may be relied on. Yet an additional light will be thrown upon the question, by carrying somewhat farther the analysis of the quantity or function H_4, and especially of the factor L; which, though itself of the ninth dimension relatively to the roots of the equation of the fifth degree, is yet, according to a remark made in the nineteenth article, susceptible of being decomposed into three less complicated factors; each of these last being rational with respect to the same five roots, and being only of the third dimension. In fact, we have, by (62), and by (11), (12), (13),

$$\mathrm{L} = (\mu + \kappa\nu + \lambda\nu)(\mu + \theta\kappa\nu + \theta^2\lambda\nu)(\mu + \theta^2\kappa\nu + \theta\lambda\nu); \tag{91}$$

that is, by (10),

$$\mathrm{L} = (\mu - \epsilon\nu + \beta^2\nu)(\mu - \epsilon\nu + \gamma^2\nu)(\mu - \epsilon\nu + \delta^2\nu); \tag{92}$$

in which, by the same equations, and by (63) and (57),

$$\left.\begin{aligned} \mu - \epsilon\nu &= (-5 + \zeta)(5\alpha^3 + \tfrac{2}{5}\beta\gamma\delta) + (1 - \zeta)\alpha(\beta^2 + \gamma^2 + \delta^2); \\ \nu &= (8 + 4\zeta)\alpha; \quad \zeta = \sqrt{-15}. \end{aligned}\right\} \tag{93}$$

Thus, L is seen to be composed of three factors,

$$\mathrm{L} = \mathrm{M}_1\mathrm{M}_2\mathrm{M}_3, \tag{94}$$

$$\mathrm{M}_1 = \mu - \epsilon\nu + \beta^2\nu, \quad \mathrm{M}_2 = \mu - \epsilon\nu + \gamma^2\nu, \quad \mathrm{M}_3 = \mu - \epsilon\nu + \delta^2\nu, \tag{95}$$

of which each is a rational, integral, and homogeneous function, of the third dimension, of the four quantities α, β, γ, δ, and, therefore, by (7), of the four roots x'', x''', x^{IV}, x^V, of the biquadratic equation (6); or finally, by (4), of the five roots x_1, x_2, x_3, x_4, x_5, of the original equation (1) of the fifth degree: because we have

$$x'' = x_2 - \tfrac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5), \quad \&\text{c.}; \tag{96}$$

or because

$$\left.\begin{aligned} 20\alpha &= x_2 + x_3 + x_4 + x_5 - 4x_1, \\ 4\beta &= x_2 + x_3 - x_4 - x_5, \\ 4\gamma &= x_2 - x_3 + x_4 - x_5, \\ 4\delta &= x_2 - x_3 - x_4 + x_5. \end{aligned}\right\} \tag{97}$$

And the first of these three factors of L may be expressed by the following equation:

$$100\mathrm{M}_1 = 5\mathrm{M}_1' - \zeta\mathrm{M}_1''; \tag{98}$$

in which,

$$\left.\begin{aligned} \mathrm{M}_1' = 4x_1^3 &- 3x_1^2(x_2 + x_3 + x_4 + x_5) - 2x_1(x_2^2 + x_3^2 + x_4^2 + x_5^2) \\ &- 2x_1(x_2x_3 + x_4x_5) + 6x_1(x_2 + x_3)(x_4 + x_5) \\ &+ 2\{x_2x_3(x_2 + x_3) + x_4x_5(x_4 + x_5)\} - 3\{x_2x_3(x_4 + x_5) + x_4x_5(x_2 + x_3)\}; \end{aligned}\right\} \tag{99}$$

and

$$\left.\begin{aligned}\mathrm{M}_1'' = 4x_1^3 &- 3x_1^2(x_2+x_3+x_4+x_5) + 2x_1(x_2^2+x_3^2+x_4^2+x_5^2)\\ &+ 14x_1(x_2x_3+x_4x_5) - 6x_1(x_2+x_3)(x_4+x_5)\\ &- 3\{x_2x_3(x_2+x_3)+x_4x_5(x_4+x_5)\} - \{x_2x_3(x_4+x_5)+x_4x_5(x_2+x_3)\}\\ &- \{x_2^3+x_3^3+x_4^3+x_5^3 - 2(x_2^2+x_3^2)(x_4+x_5) - 2(x_4^2+x_5^2)(x_2+x_3)\};\end{aligned}\right\} \tag{100}$$

while the second factor, M_2, can be formed from M_1 by merely interchanging x_3 and x_4; and the third factor M_3 from M_2, by interchanging x_4 and x_5.

27. If, now, we substitute the expression (94) for the numerator of the fraction which is to be squared in the formula (81), and transform also in like manner the denominator of the same fraction, by introducing the five original roots $x_1, \dots x_5$, through the equations (77) and (4), we find:

$$\mathrm{H}_4 = -\frac{2^{-2}3^{-3}5^{18}\mathrm{P}\mathrm{M}_1^2\mathrm{M}_2^2\mathrm{M}_3^2}{(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_1-x_5)^2}; \tag{101}$$

and we see that this quantity cannot be a symmetric function of those five roots, unless the product of the three factors M_1, M_2, M_3 be divisible by the product of the four differences $x_1 - x_2, \dots x_1 - x_5$. But this would require that at least some of those three factors M should be divisible by one of these four differences, for example by $x_1 - x_2$; which is not found to be true. Indeed, if any one of these factors, for example, M_1, were supposed to be divisible by any one difference, such as $x_1 - x_2$, it is easy to see, from its form, that it ought to be divisible also by each of the three other differences; because, in M_1, we may interchange x_2 and x_3, or x_4 and x_5, or may interchange x_2 and x_4, or x_2 and x_5, if we also interchange x_3 and x_5, or x_3 and x_4; but a rational and integral function of the third dimension cannot have four different linear divisors, without being identically equal to zero, which does not happen here. The same sort of reasoning may be applied to the expressions (95), combined with (93), for the three factors M_1, M_2, M_3, considered as functions, of the third dimension, of α, β, γ, δ; because if any one of these functions could be divisible by any one of the four following linear divisors,

$$\left.\begin{aligned}x_1 - x_2 &= -5\alpha - (\beta+\gamma+\delta),\\ x_1 - x_3 &= -5\alpha - (\beta-\gamma-\delta),\\ x_1 - x_4 &= -5\alpha - (-\beta+\gamma-\delta),\\ x_1 - x_5 &= -5\alpha - (-\beta-\gamma+\delta),\end{aligned}\right\} \tag{102}$$

it ought from its form to be divisible by all of them, which is immediately seen to be impossible. The conclusion of the twenty-fifth article is, therefore, confirmed anew; and we see, at the same time, by the theory of biquadratic equations, and by the meanings of ϵ, η, ι, that the denominator of the fraction which is to be squared, in the form (81) for H_4, may be expressed as follows:

$$\left.\begin{aligned}5x'^4 + 3px'^2 + 2qx' + r &= (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)\\ &= (5\alpha)^4 - 6\epsilon(5\alpha)^2 + 8\eta(5\alpha) - 3(\epsilon^2 - 4\iota);\end{aligned}\right\} \tag{103}$$

a result which may be otherwise proved by means of the relations (15).

28. The investigations in the preceding articles, respecting equations of the fifth degree, have been based on analogous investigations made previously with respect to biquadratic equations; because it was the theory of the equations last-mentioned which suggested to Professor BADANO the formulæ marked (a) and (b) in the seventeenth article of this paper. But if those formulæ had been suggested in any other way, or if they should be assumed as true by definition, and employed as such to fix the meanings of the quantities H which they involve; then, we might seek the values and composition of these quantities $\mathrm{H}_1, \ldots \mathrm{H}_6$, by means of the following converse formulæ, which (with a slightly abridged notation) have been given by the same author:

$$\left.\begin{aligned} \mathrm{H}_1 + \sqrt{\mathrm{H}_2} &= \tfrac{1}{3}(\mathrm{V}_{345} + \mathrm{V}_{453} + \mathrm{V}_{534}); \\ \mathrm{H}_3 + \sqrt{\mathrm{H}_4} &= \tfrac{1}{27}(\mathrm{V}_{345} + \theta^2\mathrm{V}_{453} + \theta\mathrm{V}_{534})^3; \\ \mathrm{H}_5 - \sqrt{\mathrm{H}_6} &= \tfrac{1}{27}(\mathrm{V}_{345} + \theta\mathrm{V}_{453} + \theta^2\mathrm{V}_{534})^3; \end{aligned}\right\} \tag{c}$$

and

$$\left.\begin{aligned} \mathrm{H}_1 - \sqrt{\mathrm{H}_2} &= \tfrac{1}{3}(\mathrm{V}_{354} + \mathrm{V}_{543} + \mathrm{V}_{435}); \\ \mathrm{H}_3 - \sqrt{\mathrm{H}_4} &= \tfrac{1}{27}(\mathrm{V}_{354} + \theta^2\mathrm{V}_{543} + \theta\mathrm{V}_{435})^3; \\ \mathrm{H}_5 + \sqrt{\mathrm{H}_6} &= \tfrac{1}{27}(\mathrm{V}_{354} + \theta\mathrm{V}_{543} + \theta^2\mathrm{V}_{435})^3. \end{aligned}\right\} \tag{d}$$

Let us, therefore, employ this other method to investigate the composition of H_4, by means of the equation

$$54\sqrt{\mathrm{H}_4} = (\mathrm{V}_{345} + \theta^2\mathrm{V}_{453} + \theta\mathrm{V}_{534})^3 - (\mathrm{V}_{354} + \theta^2\mathrm{V}_{543} + \theta\mathrm{V}_{435})^3; \tag{104}$$

determining still the six functions V by the definition (33), so that each shall still be the mean of four of the twenty-four functions T; and assigning still to these last functions the significations (32), or treating them as the fifth powers of twenty-four different values of LAGRANGE's function t, which has itself 120 values: but expressing now these values of t by the notation

$$t_{abcde} = \omega^5 x_a + \omega^4 x_b + \omega^3 x_c + \omega^2 x_d + \omega x_e, \tag{105}$$

which differs from the notation (22) only by having lower instead of upper indices of x; and is designed to signify that we now employ (for the sake of a greater directness and a more evident generality) the five arbitrary roots x_1, &c., of the original equation (1), between which roots no relation is supposed to subsist, instead of the other roots x', &c., of the equation (2), which equation was supposed to have been so prepared that the sum of its roots should be zero.

29. Resuming, then, the calculations on this plan, and making for abridgment

$$\mathrm{A} = x_a + x_b + x_c + x_d + x_e, \tag{106}$$

so that $-\mathrm{A}$ is still the coefficient of the fourth power of x in the equation of the fifth degree; making also

$$\mathrm{W}_{abcde} = x_a^4 x_b + 2x_a^3 x_d^2 + 4x_a^3 x_c x_e + 6x_a^2 x_b^2 x_e + 12x_a^2 x_b x_c x_d, \tag{107}$$

and

$$\text{X}_{bcde} = 5(\text{W}_{abcde} + \text{W}_{bcdea} + \text{W}_{cdeab} + \text{W}_{deabc} + \text{W}_{eabcd}); \tag{108}$$

we find (because $\omega^5 = 1$), for the fifth power of the combination (105) of the five roots x, the expression:

$$t^5_{abcde} = \text{A}^5 + (\omega^4 - 1)\text{X}_{bcde} + (\omega^3 - 1)\text{X}_{cebd} + (\omega - 1)\text{X}_{edcb} + (\omega^2 - 1)\text{X}_{dbec}; \tag{109}$$

and therefore, for the six functions V, with the same meanings of those functions as before, the formula:

$$\left.\begin{aligned} \text{V}_{cde} &= \tfrac{1}{4}(t^5_{12cde} + t^5_{1c2ed} + t^5_{1de2c} + t^5_{1edc2}) \\ &= \text{A}^5 + (\omega + \omega^4 - 2)\text{Y}_{cde} + (\omega^2 + \omega^3 - 2)\text{Y}_{dce}; \end{aligned}\right\} \tag{110}$$

in which,

$$4\text{Y}_{cde} = \text{X}_{2cde} + \text{X}_{c2ed} + \text{X}_{de2c} + \text{X}_{edc2}. \tag{111}$$

If then we make

$$\left.\begin{aligned} \text{Y}_{345} &= \text{Y}'_5 + \text{Y}''_5, & \text{Y}_{435} &= \text{Y}'_5 - \text{Y}''_5, \\ \text{Y}_{453} &= \text{Y}'_3 + \text{Y}''_3, & \text{Y}_{543} &= \text{Y}'_3 - \text{Y}''_3, \\ \text{Y}_{534} &= \text{Y}'_4 + \text{Y}''_4, & \text{Y}_{354} &= \text{Y}'_4 - \text{Y}''_4; \end{aligned}\right\} \tag{112}$$

we shall have, by (20) and (30), the following system of expressions for the functions V:

$$\left.\begin{aligned} \text{V}_{345} &= \text{A}^5 - 5\text{Y}'_5 + \text{D}\text{Y}''_5; \\ \text{V}_{453} &= \text{A}^5 - 5\text{Y}'_3 + \text{D}\text{Y}''_3; \\ \text{V}_{534} &= \text{A}^5 - 5\text{Y}'_4 + \text{D}\text{Y}''_4; \end{aligned}\right\} \tag{113}$$

and

$$\left.\begin{aligned} \text{V}_{354} &= \text{A}^5 - 5\text{Y}'_4 - \text{D}\text{Y}''_4; \\ \text{V}_{543} &= \text{A}^5 - 5\text{Y}'_3 - \text{D}\text{Y}''_3; \\ \text{V}_{435} &= \text{A}^5 - 5\text{Y}'_5 - \text{D}\text{Y}''_5; \end{aligned}\right\} \tag{114}$$

D being still $= \omega^4 - \omega^3 - \omega^2 + \omega$, so that D^2 is still $= 5$. We have also the equation:

$$\left.\begin{aligned} &\quad \text{X}_{2345} + \text{X}_{3254} + \text{X}_{4523} + \text{X}_{5432} \\ &+ \text{X}_{2453} + \text{X}_{4235} + \text{X}_{5324} + \text{X}_{3542} \\ &+ \text{X}_{2534} + \text{X}_{5243} + \text{X}_{3425} + \text{X}_{4352} \\ = &\quad \text{X}_{2354} + \text{X}_{3245} + \text{X}_{5423} + \text{X}_{4532} \\ &+ \text{X}_{2543} + \text{X}_{5234} + \text{X}_{4325} + \text{X}_{3452} \\ &+ \text{X}_{2435} + \text{X}_{4253} + \text{X}_{3524} + \text{X}_{5342}; \end{aligned}\right\} \tag{115}$$

because the first member may be converted into the second by interchanging any two of the four roots x_2, x_3, x_4, x_5, on which (and on x_1), the functions X depend, and therefore the difference of these two members must be equal to zero; since, being at highest of the fifth dimension, it cannot otherwise be divisible by the function

$$\varpi = (x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)(x_3 - x_5)(x_4 - x_5), \tag{116}$$

which is the product of the six differences of the four roots just mentioned, and is itself of the sixth dimension. We may therefore combine with the expressions (113) and (114) the relations:

$$\mathrm{Y}_{345} + \mathrm{Y}_{453} + \mathrm{Y}_{534} = \mathrm{Y}_{354} + \mathrm{Y}_{543} + \mathrm{Y}_{435}; \tag{117}$$

and

$$\mathrm{Y}''_3 + \mathrm{Y}''_4 + \mathrm{Y}''_5 = 0. \tag{118}$$

30. With these preparations for the study of the functions V, or of any combination of those functions, let us consider in particular the first of the three following factors of the expression (104) for $54\sqrt{\mathrm{H}_4}$:

$$\left.\begin{array}{l} \mathrm{V}_{345} - \mathrm{V}_{354} + \theta^2(\mathrm{V}_{453} - \mathrm{V}_{543}) + \theta(\mathrm{V}_{534} - \mathrm{V}_{435}); \\ \mathrm{V}_{345} - \mathrm{V}_{543} + \theta^2(\mathrm{V}_{453} - \mathrm{V}_{435}) + \theta(\mathrm{V}_{534} - \mathrm{V}_{354}); \\ \mathrm{V}_{345} - \mathrm{V}_{435} + \theta^2(\mathrm{V}_{453} - \mathrm{V}_{354}) + \theta(\mathrm{V}_{534} - \mathrm{V}_{543}); \end{array}\right\} \tag{119}$$

θ being still an imaginary cube-root of unity. We find:

$$\left.\begin{array}{l} \mathrm{V}_{345} - \mathrm{V}_{354} = 5(\mathrm{Y}'_4 - \mathrm{Y}'_5) - \mathrm{D}\mathrm{Y}''_3; \\ \mathrm{V}_{534} - \mathrm{V}_{435} = -5(\mathrm{Y}'_4 - \mathrm{Y}'_5) - \mathrm{D}\mathrm{Y}''_3; \\ \mathrm{V}_{453} - \mathrm{V}_{543} = 2\mathrm{D}\mathrm{Y}''_3; \end{array}\right\} \tag{120}$$

expressions which show immediately that

$$\mathrm{V}_{345} + \mathrm{V}_{453} + \mathrm{V}_{534} = \mathrm{V}_{354} + \mathrm{V}_{543} + \mathrm{V}_{435}, \tag{121}$$

and, therefore, by (c) and (d), that

$$\mathrm{H}_2 = 0,$$

as was otherwise found before. Also,

$$2\theta^2 - \theta - 1 = (\theta - 1)(2\theta + 1) = -(1 - \theta)(\theta - \theta^2); \tag{122}$$

and, consequently, by (120), the first of the three factors (119) is equivalent to the product of the two following:

$$1 - \theta, \quad 5(\mathrm{Y}'_4 - \mathrm{Y}'_5) - \zeta\mathrm{Y}''_3; \tag{123}$$

in which, as before,

$$\eta = (\theta - \theta^2)\mathrm{D} = \sqrt{-15}.$$

But, by (112) and (117),

$$2(\mathrm{Y}'_4 - \mathrm{Y}'_5) = \mathrm{Y}_{534} - \mathrm{Y}_{435} - (\mathrm{Y}_{345} - \mathrm{Y}_{354}) = 2(\mathrm{Y}_{534} - \mathrm{Y}_{435}) + \mathrm{Y}_{453} - \mathrm{Y}_{543}, \tag{124}$$

and

$$2\text{Y}_3'' = \text{Y}_{453} - \text{Y}_{543}; \tag{125}$$

so that the first factor (119) may be put under the form:

$$\tfrac{1}{2}(1-\theta)\{10(\text{Y}_{534} - \text{Y}_{435}) + (5-\zeta)(\text{Y}_{453} - \text{Y}_{543})\}. \tag{126}$$

Besides, by (111), the three differences

$$\text{Y}_{cde} - \text{Y}_{ced}, \quad \text{Y}_{cde} - \text{Y}_{edc}, \quad \text{Y}_{cde} - \text{Y}_{dce}, \tag{127}$$

are divisible, respectively, by the three products

$$(x_2 - x_c)(x_d - x_e), \quad (x_2 - x_d)(x_e - x_c), \quad (x_2 - x_e)(x_c - x_d); \tag{128}$$

and, therefore, the factor (126) is divisible by the product

$$(x_2 - x_3)(x_4 - x_5), \tag{129}$$

the quotient of this division being a rational and integral and homogeneous function of the five roots x, which is no higher than the third dimension, and which it is not difficult to calculate.

31. In this manner we are led to establish an equation of the form:

$$\text{V}_{345} - \text{V}_{354} + \theta^2(\text{V}_{453} - \text{V}_{543}) + \theta(\text{V}_{534} - \text{V}_{435}) = (1-\theta)(x_2 - x_3)(x_4 - x_5)\text{N}_1; \tag{130}$$

in which if we make

$$2\text{N}_1 = 10\text{N}_1' + (5-\zeta)\text{N}_1'', \tag{131}$$

we have

$$\text{N}_1' = \frac{\text{Y}_{534} - \text{Y}_{435}}{(x_2 - x_3)(x_4 - x_5)}, \quad \text{N}_1'' = \frac{\text{Y}_{453} - \text{Y}_{543}}{(x_2 - x_3)(x_4 - x_5)}. \tag{132}$$

Effecting the calculations indicated by these last formulæ, we find

$$\text{N}_1' = \tfrac{5}{4}(\text{M}_1'' - \text{M}_1'), \quad \text{N}_1'' = -\tfrac{5}{2}\text{M}_1'', \tag{133}$$

M_1' and M_1'' being determined by the equations (99) and (100); and, therefore, with the meaning (98) of M_1, we find the relation:

$$\text{N}_1 = -125\text{M}_1. \tag{134}$$

Thus, the first of the three factors (119) may be put under the form:

$$-125(1-\theta)(x_2 - x_3)(x_4 - x_5)\text{M}_1; \tag{135}$$

in deducing which, it is to be observed, that the first term, $x_a^4 x_b$, of the formula (107) for W_{abcde}, gives, by (108), the five following terms of X_{bcde}:

$$5x_a^4 x_b + 5x_b^4 x_c + 5x_c^4 x_d + 5x_d^4 x_e + 5x_e^4 x_a; \tag{136}$$

and these five terms of X give, respectively, by (111), the five following parts of Y_{cde}:

$$\left.\begin{array}{l} \frac{5}{4}x_1^4(x_2 + x_c + x_d + x_e), \\ \frac{5}{4}(x_2^4 x_c + x_c^4 x_2 + x_d^4 x_e + x_e^4 x_d), \\ \frac{5}{4}(x_c^4 x_d + x_2^4 x_e + x_e^4 x_2 + x_d^4 x_c), \\ \frac{5}{4}(x_4^4 x_e + x_e^4 x_d + x_2^4 x_c + x_c^4 x_2), \\ \frac{5}{4}(x_e^4 + x_d^4 + x_c^4 + x_2^4)x_1; \end{array}\right\} \tag{137}$$

which are to be combined with the other parts of Y, derived, in like manner, through X, from the other terms of W, and to be submitted to the processes indicated by the formulæ (132), in order to deduce the values (133) of N_1' and N_1'', and thence, by (131) and (98), the relation (134) between N_1 and M_1, which conducts, by (130), to the expression (135). For example, the first and last of the five parts (137) of Y, contribute nothing to either of the two quotients (132), because those parts are symmetric relatively to x_c, x_d, x_e; but the second part (137) contributes

$$-\tfrac{5}{4}(x_2^3 + x_2^2 x_d + x_2 x_d^2 + x_d^3 + x_e^3 + x_e^2 x_c + x_e x_c^2 + x_c^3), \tag{138}$$

to the quotient

$$\frac{\text{Y}_{cde} - \text{Y}_{edc}}{(x_2 - x_d)(x_e - x_c)}, \tag{139}$$

and

$$+\tfrac{5}{4}(x_2^3 + x_2^2 x_e + x_2 x_e^2 + x_e^3 + x_c^3 + x_c^2 x_d + x_c x_d^2 + x_d^3), \tag{140}$$

to the quotient

$$\frac{\text{Y}_{cde} - \text{Y}_{dce}}{(x_2 - x_e)(x_c - x_d)}; \tag{141}$$

this second part (137) of Y contributes therefore, by (132),

$$-\tfrac{5}{4}(x_2^3 + x_2^2 x_3 + x_2 x_3^2 + x_3^3 + x_4^3 + x_4^2 x_5 + x_4 x_5^2 + x_5^3), \tag{142}$$

to the quotient N_1', and the same quantity with its sign changed to the quotient N_1'': and the other parts of the same two quotients are determined in a similar manner.

32. The two other factors (119) may respectively be expressed as follows:

$$-125(1 - \theta^2)(x_2 - x_4)(x_3 - x_5)\text{M}_2, \tag{143}$$

and

$$-125(\theta - \theta^2)(x_2 - x_5)(x_3 - x_4)\text{M}_3; \tag{144}$$

in which, M_2 and M_3 are formed from M_1, as in the twenty-sixth article; because the second factor (119) may be formed from the first, by interchanging x_3 and x_4, and multiplying by $-\theta^2$; and the third factor may be formed from the second, by interchanging x_4 and x_5, and multiplying again by $-\theta^2$. If then we multiply the three expressions (135) (143) (144) for the three factors (119) together, and divide by three, we find:

$$18\sqrt{\mathrm{H}_4} = -5^9(\theta - \theta^2)\varpi \mathrm{M}_1\mathrm{M}_2\mathrm{M}_3; \tag{145}$$

ϖ denoting here the product (116) of the six differences of the four roots $x_2, \dots x_5$. The expression (101) for H_4 itself is therefore reproduced under the form:

$$\mathrm{H}_4 = -2^{-2}3^{-3}5^{18}\varpi^2\mathrm{M}_1^2\mathrm{M}_2^2\mathrm{M}_3^2; \tag{146}$$

and the conclusions of former articles are thus confirmed anew, by a method which is entirely different, in its conception and in its processes of calculation, from those which were employed before.

33. It may not, however, be useless to calculate, for some particular equation of the fifth degree, the numerical values of some of the most important quantities above considered, and so to illustrate and exemplify some of the chief formulæ already established. Consider therefore the equation:

$$x^5 - 5x^3 + 4x = 0; \tag{147}$$

of which the roots may be arranged in the order:

$$x_1 = 2, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = -1, \quad x_5 = -2; \tag{148}$$

and may (because their sum is zero) be also written thus:

$$x' = 2, \quad x'' = 1, \quad x''' = 0, \quad x^{IV} = -1, \quad x^{V} = -2. \tag{149}$$

Employing the notation (32), in combination with (22) or with (105), we have now:

$$\left.\begin{aligned} \mathrm{T}_{2345} &= (2 + \omega^4 - \omega^2 - 2\omega)^5; \\ \mathrm{T}_{3254} &= (2 + \omega^3 - 2\omega^2 - \omega)^5; \\ \mathrm{T}_{4523} &= (2 - \omega^4 - 2\omega^3 + \omega^2)^5; \\ \mathrm{T}_{5432} &= (2 - 2\omega^4 - \omega^3 + \omega)^5. \end{aligned}\right\} \tag{150}$$

But $\omega^5 = 1$; therefore,

$$\mathrm{T}_{5432} = (-2 - \omega^4 + \omega^2 + 2\omega)^5, \tag{151}$$

and

$$\mathrm{T}_{2345} + \mathrm{T}_{5432} = 0. \tag{152}$$

Again,

$$\mathrm{T}_{3254} = (1 - \omega^2)^5(2 - \omega)^5, \quad \mathrm{T}_{4523} = (1 - \omega^3)^5(2 - \omega^4)^5; \tag{153}$$

and if we make

$$(2-\omega)^5 = \text{E} - \text{O}, \quad (2+\omega)^5 = \text{E} + \text{O}, \tag{154}$$

we shall have

$$\text{E} = 32 + 80\omega^2 + 10\omega^4, \quad \text{O} = 80\omega + 40\omega^3 + \omega^5; \tag{155}$$

also,

$$(1-\omega^2)^5 = -5\omega^2(1-\omega^2)(1-\omega^2+\omega^4); \tag{156}$$

we find, therefore, by easy calculations,

$$\left.\begin{aligned} (1-\omega^2)^5\text{E} &= 300 + 430\omega - 110\omega^2 - 540\omega^3 - 80\omega^4, \\ (1-\omega^2)^5\text{O} &= 600 + 190\omega - 405\omega^2 - 395\omega^3 + 10\omega^4; \end{aligned}\right\} \tag{157}$$

and by subtracting the latter of these two products from the former, and afterwards changing ω to its reciprocal, we obtain:

$$\left.\begin{aligned} \text{T}_{3254} &= -300 + 240\omega + 295\omega^2 - 145\omega^3 - 90\omega^4, \\ \text{T}_{4523} &= -300 + 240\omega^4 + 295\omega^3 - 145\omega^2 - 90\omega. \end{aligned}\right\} \tag{158}$$

We have, therefore, by (20),

$$\text{T}_{3254} + \text{T}_{4523} = -750; \tag{159}$$

and, consequently, by (33) and (152),

$$\text{V}_{345} = -\frac{375}{2}. \tag{160}$$

34. In like manner, to compute, in this example, the second of the functions V, we have

$$\left.\begin{aligned} &\text{T}_{2453} = (2+\omega^4-\omega^3-2\omega^2)^5 = -\text{T}_{3542}; \\ &\text{T}_{4235} = (1-\omega)^5(2+\omega^3)^5, \quad \text{T}_{5324} = (1-\omega^4)^5(2+\omega^2)^5; \end{aligned}\right\} \tag{161}$$

adding then the two products (157) together, and afterwards changing ω to ω^3 and ω^2 successively, we find, by (154):

$$\left.\begin{aligned} \text{T}_{4235} &= 900 + 620\omega^3 - 515\omega - 935\omega^4 - 70\omega^2, \\ \text{T}_{5324} &= 900 + 620\omega^2 - 515\omega^4 - 935\omega - 70\omega^3; \end{aligned}\right\} \tag{162}$$

but, by (20), (30), and (54),

$$2(\omega+\omega^4) = -1 + \text{D}, \quad 2(\omega^2+\omega^3) = -1 - \text{D}, \quad \text{D}^2 = 5; \tag{163}$$

therefore,

$$\text{T}_{2453} + \text{T}_{3542} = 0, \quad \text{T}_{4235} + \text{T}_{5324} = 2250 - 1000\text{D}; \tag{164}$$

and

$$\text{V}_{453} = \tfrac{1}{2}(1125 - 500\text{D}). \tag{165}$$

35. To compute the third of the functions V, we have, in the present question, the relations:

$$\mathrm{T}_{2534} = -\mathrm{T}_{3254}, \quad \mathrm{T}_{5243} = -\mathrm{T}_{4235}, \quad \mathrm{T}_{3425} = -\mathrm{T}_{5324}, \quad \mathrm{T}_{4352} = -\mathrm{T}_{4523}; \tag{166}$$

and, therefore, by (159) and (164),

$$\mathrm{V}_{534} = -375 + 250\mathrm{D}. \tag{167}$$

For the fourth function V, we have, by processes entirely similar to the foregoing:

$$\left.\begin{array}{l} \mathrm{T}_{2354} = -(1-\omega^3)^5(2+\omega^4)^5, \quad \mathrm{T}_{4532} = -(1-\omega^2)^5(2+\omega)^5, \\ \mathrm{T}_{2354} + \mathrm{T}_{4532} = -2250 - 1000\mathrm{D}; \end{array}\right\} \tag{168}$$

$$\left.\begin{array}{l} \mathrm{T}_{3245} = -(1-\omega^4)^5(2-\omega^2)^5, \quad \mathrm{T}_{5423} = -(1-\omega)^5(2-\omega^3)^5, \\ \mathrm{T}_{3245} + \mathrm{T}_{5423} = +750; \end{array}\right\} \tag{169}$$

$$\mathrm{V}_{354} = -375 - 250\mathrm{D}. \tag{170}$$

For the fifth function V, we have the relations:

$$\mathrm{T}_{2543} = -\mathrm{T}_{2354}; \quad \mathrm{T}_{5234} = -\mathrm{T}_{4325}; \quad \mathrm{T}_{3452} = -\mathrm{T}_{4532}; \tag{171}$$

and, therefore, by (168),

$$\mathrm{V}_{543} = \tfrac{1}{2}(1125 + 500\mathrm{D}). \tag{172}$$

Finally, for the sixth function V, we have

$$\mathrm{T}_{2435} = -\mathrm{T}_{5423}; \quad \mathrm{T}_{4253} = -\mathrm{T}_{3524} : \quad \mathrm{T}_{5342} = -\mathrm{T}_{3245}; \tag{173}$$

and, therefore, by (169),

$$\mathrm{V}_{435} = -\frac{375}{2}. \tag{174}$$

The three first values of V may therefore be thus collected:

$$\frac{2}{125}\mathrm{V}_{345} = -3; \quad \frac{2}{125}\mathrm{V}_{453} = 9 - 4\mathrm{D}; \quad \frac{2}{125}\mathrm{V}_{534} = -6 + 4\mathrm{D}; \tag{175}$$

and the three last values, in an inverted order, may in like manner be expressed by the equations:

$$\frac{2}{125}\mathrm{V}_{435} = -3; \quad \frac{2}{125}\mathrm{V}_{543} = 9 + 4\mathrm{D}; \quad \frac{2}{125}\mathrm{V}_{354} = -6 - 4\mathrm{D}. \tag{176}$$

36. It is evident that these six values of V are of the forms (113) and (114), and that they verify, in the present case, the general relation (121). They show also, by (c) and (d) of article 28., that not only H_2, but H_1, vanishes in this example; the common value of the two sums (121), of the three first and three last values of V, being zero. Accordingly, if we compare the particular equation (147) with the general forms (1) and (2), we find the following values of the coefficients (B, C, D, E, not having here their recent meanings):

$$\mathrm{A} = 0, \quad \mathrm{B} = -5, \quad \mathrm{C} = 0, \quad \mathrm{D} = 4, \quad \mathrm{E} = 0, \tag{177}$$

and

$$p = -5, \quad q = 0, \quad r = 4, \quad s = 0; \tag{178}$$

and therefore the formula (51) gives here

$$\mathrm{H}_1 = 0. \tag{179}$$

We find also, with the same meanings of θ and ζ as in former articles:

$$\left.\begin{aligned} \frac{2}{125}(\mathrm{V}_{345} + \theta^2\mathrm{V}_{453} + \theta\mathrm{V}_{534}) &= 3(4\theta^2 - \theta) + 4\zeta; \\ \frac{2\theta^2}{125}(\mathrm{V}_{354} + \theta^2\mathrm{V}_{543} + \theta\mathrm{V}_{435}) &= 3(4\theta - \theta^2) + 4\zeta; \end{aligned}\right\} \tag{180}$$

and, therefore, by (c) and (d),

$$\left.\begin{aligned} 2^3 3^3 5^{-9}(\mathrm{H}_3 + \sqrt{\mathrm{H}_4}) &= \{3(4\theta^2 - \theta) + 4\zeta\}^3, \\ 2^3 3^3 5^{-9}(\mathrm{H}_3 - \sqrt{\mathrm{H}_4}) &= \{3(4\theta - \theta^2) + 4\zeta\}^3; \end{aligned}\right\} \tag{181}$$

equations which give, by (11) and (57):

$$\sqrt{\mathrm{H}_4} = 2^{-2}5^{10}(\theta - \theta^2)(23 + 3\zeta); \tag{182}$$

and

$$\mathrm{H}_4 = -2^{-3}3^1 5^{20}(197 + 69\zeta). \tag{183}$$

Let us now compare these last numerical results with the general formulæ found by other methods in earlier articles of this paper.

37. The method of the thirteenth article gives, in the present article,

$$\left.\begin{aligned} &\alpha = -\tfrac{1}{2}, \quad \beta = 1, \quad \gamma = \tfrac{1}{2}, \quad \delta = 0, \quad \epsilon = \tfrac{5}{12}, \quad \eta = 0, \\ &\kappa = \frac{4 + \theta^2}{12}, \quad \lambda = \frac{4 + \theta}{12}, \quad \iota = \kappa\lambda = \frac{13}{144}, \\ &\kappa^3 + \lambda^3 = \frac{35}{864}, \quad \tfrac{1}{2}(\kappa^3 - \lambda^3) = -2^{-5}3^{-1}(\theta - \theta^2); \end{aligned}\right\} \tag{184}$$

and therefore, by (59),

$$\left.\begin{array}{l}\dfrac{3\theta k}{250} = 5(1-\zeta), \quad \dfrac{3\theta l}{250} = 12(2+\zeta), \\ k^3 - 3kl^2\kappa\lambda - l^3(\kappa^3+\lambda^3) = -2^3 3^1 5^{10}(23+3\zeta);\end{array}\right\} \tag{185}$$

and, accordingly, if we multiply the last expressions (184) by the last expression (185), we are led, by the general formula (60), to the same result for $\sqrt{\mathrm{H}_4}$, and therefore for H_4, as was obtained in the last article by an entirely different method. The general formula (60) may also, in virtue of the equations (13), (59), (62), (63), (70), (116), and (4), be written thus:

$$18\sqrt{\mathrm{H}_4} = -5^9(\theta-\theta^2)\varpi\mathrm{L}; \tag{186}$$

which agrees, by (94), with the general result (145), and in which we have now

$$\varpi = 1 \,.\, 2 \,.\, 3 \,.\, 1 \,.\, 2 \,.\, 1 = 12; \tag{187}$$

while L may be calculated by the definitions (62) and (63), which give, at present, by the values (184) for α, ϵ, η, ι,

$$\mu = \tfrac{5}{6}(1-\zeta), \quad \nu = -2(2+\zeta), \tag{188}$$

and

$$\mathrm{L} = -\tfrac{15}{8}(23+3\zeta): \tag{189}$$

and thus we arrive again at the same value of $\sqrt{\mathrm{H}_4}$ as before. The same value of L may be obtained in other ways, by other formulæ of this paper; for example, by those of the 24th and 25th articles, which give, in the present question,

$$\mathrm{L}' = -2^3 3^1 5^2 23; \quad \mathrm{L}'' = +2^3 3^2 5^3. \tag{190}$$

We may also decompose L into three factors M, which are here:

$$\mathrm{M}_1 = -\tfrac{1}{2}(3+4\zeta); \quad \mathrm{M}_2 = \tfrac{1}{2}(3-\zeta); \quad \mathrm{M}_3 = \tfrac{5}{2}; \tag{191}$$

and which conduct still to the same result.

38. An equation of the fifth degree, which, like that here assumed as an example, has all its roots unequal, may have those roots arranged in 120 different ways; and any one of these arrangements may be taken as the basis of a verification such as that contained in the last five articles. But we have seen that no such change of arrangement will affect the value of either H_1 or H_2; and with respect to H_4 which has been more particularly under our consideration in this paper, it is not difficult to perceive that an interchange of any two of the four last roots (x_2, x_3, x_4, x_5, or x'', x''', x^{IV}, x^V), of the proposed equation of the fifth degree, will merely change the sign of the square-root, $\sqrt{\mathrm{H}_4}$, in the foregoing formulæ, without making any change in the value of H_4 itself, which has been shown to depend on the first root (x_1 or x') alone. It will, however, be instructive to exemplify this last-mentioned dependence, by applying the foregoing general processes to the case of the equation of the fifth degree (147),

the two first roots being made to change places with each other, in such a manner that the order shall now be chosen as follows:

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 0, \quad x_4 = -1, \quad x_5 = -2, \tag{192}$$

or (since the sum of all five vanishes),

$$x' = 1, \quad x'' = 2, \quad x''' = 0, \quad x^{IV} = -1, \quad x^{V} = -2. \tag{193}$$

We find, for this new case, by calculations of the same sort as in recent articles of this paper, the following new system of equations for the values of the six functions V:

$$\left.\begin{array}{lll} \tfrac{2}{125}\mathrm{V}_{345} = 12 + 4\mathrm{D}; & \tfrac{2}{125}\mathrm{V}_{453} = -9 - 4\mathrm{D}; & \tfrac{2}{125}\mathrm{V}_{534} = -3; \\ \tfrac{2}{125}\mathrm{V}_{435} = 12 - 4\mathrm{D}; & \tfrac{2}{125}\mathrm{V}_{543} = -9 + 4\mathrm{D}; & \tfrac{2}{125}\mathrm{V}_{354} = -3; \end{array}\right\} \tag{194}$$

in which, D has again the meaning assigned by (30): and, consequently,

$$\left.\begin{array}{r} \frac{2\theta^2}{125}(\mathrm{V}_{345} + \theta^2\mathrm{V}_{453} + \theta\mathrm{V}_{534}) = 3(5\theta^2 - 2\theta) - 4\zeta; \\ \tfrac{2}{125}(\mathrm{V}_{354} + \theta^2\mathrm{V}_{543} + \theta\mathrm{V}_{435}) = 3(5\theta - 2\theta^2) - 4\zeta; \end{array}\right\} \tag{195}$$

$$\left.\begin{array}{r} 2^4 3^3 5^{-9}\sqrt{\mathrm{H}_4} = \{5(5\theta^2 - 2\theta) - 4\zeta\}^3 - \{3(5\theta - 2\theta^2) - 4\zeta\}^3; \\ \sqrt{\mathrm{H}_4} = 2^{-3}5^9 7(\theta - \theta^2)(55 - 6\zeta); \end{array}\right\} \tag{196}$$

and

$$\mathrm{H}_4 = -2^{-6}3^1 5^{19} 7^2 (497 - 132\zeta): \tag{197}$$

results which differ from those obtained with the former arrangement of the five roots of the proposed equation (147), but of which the agreement with the general formulæ of the present paper may be evinced by processes similar to those of the last article.

39. As a last example, if the arrangement of the same five roots be

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = -1, \quad x_5 = -2, \tag{198}$$

we then find easily that all the six quantities V vanish, and, therefore, that we have, with this arrangement,

$$\sqrt{\mathrm{H}_4} = 0, \quad \mathrm{H}_4 = 0. \tag{199}$$

All these results respecting the numerical values of H_4, for different arrangements of the roots of the proposed equation (147), are included in the common expression:

$$\mathrm{H}_4 = -2^{-4}3^3 5^{18}\left(\frac{5(72x' + 5x'^3) - 2\zeta(38x' - 17x'^3)}{5x'^4 - 15x'^2 + 4}\right)^2; \tag{200}$$

which results from the formula (85), combined with (79) and (86) (87) (88): and thus we have a new confirmation of the correctness of the foregoing calculations.

40. It is then proved, in several different ways, that the quantity H_4, in the formulæ which have been marked in this paper (a), (b), (c), (d), and which have been proposed by Professor BADANO for the solution of the general equation of the fifth degree, is not a symmetric function of the five roots of that equation. And since it has been shown that the expression of this quantity H_4 contains in general the imaginary radical ζ or $\sqrt{-15}$, which changes sign in passing to the expression of the analogous quantity H_6, we see that these two quantities, H_4 and H_6, are not generally equal to each other, as Professor BADANO, in a supplement to his essay, appears to think they must be. They are, on the contrary, found to be in general the two unequal roots of a quadratic equation, namely,

$$\mathrm{H}_4^2 + \mathrm{Q}\mathrm{H}_4 + \mathrm{R}^2 = 0, \tag{201}$$

in which

$$\mathrm{Q} = -(\mathrm{H}_4 + \mathrm{H}_6) = 2^{-13}3^{-3}5^{15}\varpi^2(5\mathrm{L}'^2 - 3\mathrm{L}''^2), \tag{202}$$

and

$$\mathrm{R} = \sqrt{\mathrm{H}_4} \,.\, \sqrt{\mathrm{H}_6} = -2^{-14}3^{-3}5^{15}\varpi^2(5\mathrm{L}'^2 - 3\mathrm{L}''^2), \tag{203}$$

ϖ, L', and L'', having the signification already assigned; and the values of the coefficients Q and R depend essentially, in general, on the choice of the root x', although they can always be expressed as rational functions of that root.

41. It does not appear to be necessary to write here the analogous calculations, which show that the two remaining quantities H_3 and H_5, which enter into the same formulæ (a), (b), (c), (d), are not, in general, symmetric functions of the five roots of the proposed equation of the fifth degree, nor equal to each other, but roots of a quadratic equation, of the same kind with that considered in the last article. But it may be remarked, in illustration of this general result, that for the particular equation of the fifth degree which has been marked (147) we find, with the arrangement (148) of the five roots, the values:

$$\mathrm{H}_3 = 2^{-3}3^{-2}5^9(1809 - 914\zeta), \quad \mathrm{H}_5 = 2^{-3}3^{-2}5^9(1809 + 914\zeta); \tag{204}$$

with the arrangement (192),

$$\mathrm{H}_3 = 2^{-2}3^{-2}5^9(1269 + 781\zeta), \quad \mathrm{H}_5 = 2^{-2}3^{-2}5^9(1269 - 781\zeta); \tag{205}$$

and, with the arrangement (198),

$$\mathrm{H}_3 = 0, \quad \mathrm{H}_5 = 0. \tag{206}$$

The general decomposition of these quantities H_3 and H_5, into factors of the fifth dimension, referred to in a former article, results easily from the equations of definition (42) and (43), which give:

$$\left.\begin{aligned} 2\mathrm{H}_3 &= (h + h')(h + \theta h')(h + \theta^2 h'); \\ 2\mathrm{H}_5 &= (i + i')(i + \theta i')(i + \theta^2 i'). \end{aligned}\right\} \tag{207}$$

And the same equations, when combined with (40) and (38), show that the combinations

$$\mathrm{H}_3^2 - \mathrm{H}_4 = h^3 h'^3, \quad \mathrm{H}_5^2 - \mathrm{H}_6 = i^3 i'^3, \tag{208}$$

are exact cubes of rational functions of the five roots of the equation of the fifth degree, which functions are each of the tenth dimension relatively to those five roots, and are symmetric relatively to four of them; while each of these functions hh' and ii', decomposes itself into two factors, which are also rational functions of the five roots, and are no higher than the fifth dimension.

42. In the foregoing articles, we have considered only those six quantities H which were connected with the composition of the six functions V, determined by the definition (33). But if we establish the expressions,

$$\left.\begin{array}{l} \mathrm{T}_{2cde} = \mathrm{V}_{cde} + \mathrm{V}'_{cde} + \mathrm{V}''_{cde} + \mathrm{V}'''_{cde}, \\ \mathrm{T}_{c2ed} = \mathrm{V}_{cde} + \qquad - \qquad - \\ \mathrm{T}_{de2c} = \mathrm{V}_{cde} - \qquad + \qquad - \\ \mathrm{T}_{edc2} = \mathrm{V}_{cde} - \qquad - \qquad + \end{array}\right\} \tag{209}$$

which include the definition (33), and give,

$$\left.\begin{array}{l} \mathrm{V}'_{cde} = \tfrac{1}{4}(\mathrm{T}_{2cde} + \mathrm{T}_{c2ed} - \mathrm{T}_{de2c} - \mathrm{T}_{edc2}), \\ \mathrm{V}''_{cde} = \tfrac{1}{4}(\mathrm{T}_{2cde} - \qquad + \qquad - \qquad), \\ \mathrm{V}'''_{cde} = \tfrac{1}{4}(\mathrm{T}_{2cde} - \qquad - \qquad + \qquad), \end{array}\right\} \tag{210}$$

we are conducted to expressions for the squares of the three functions V', V'', V''', which are entirely analogous to those marked (a) and (b), and have accordingly been assigned under such forms by Professor BADANO, involving eighteen new quantities $\mathrm{H}_7, \ldots \mathrm{H}_{24}$; which quantities, however, are not found to be symmetric functions of the five roots of the equation of the fifth degree, though they are symmetric relative to four of them.

43. In making the investigations which conduct to this result, it is convenient to establish the following definitions, analogous to, and in combination with, that marked (111):

$$\left.\begin{array}{l} 4\mathrm{Y}'_{cde} = \mathrm{X}_{2cde} + \mathrm{X}_{c2ed} - \mathrm{X}_{de2c} - \mathrm{X}_{edc2}, \\ 4\mathrm{Y}''_{cde} = \mathrm{X}_{2cde} - \qquad + \qquad - \qquad , \\ 4\mathrm{Y}'''_{cde} = \mathrm{X}_{2cde} - \qquad - \qquad + \qquad ; \end{array}\right\} \tag{211}$$

for thus we obtain,

$$\left.\begin{array}{l} \mathrm{X}_{2cde} = \mathrm{Y}_{cde} + \mathrm{Y}'_{cde} + \mathrm{Y}''_{cde} + \mathrm{Y}'''_{cde}, \\ \mathrm{X}_{c2ed} = \mathrm{Y}_{cde} + \qquad - \qquad - \qquad , \\ \mathrm{X}_{de2c} = \mathrm{Y}_{cde} - \qquad + \qquad - \qquad , \\ \mathrm{X}_{edc2} = \mathrm{Y}_{cde} - \qquad - \qquad + \qquad ; \end{array}\right\} \tag{212}$$

$$\left.\begin{aligned}
\mathrm{V}'_{cde} &= (\omega^4 - \omega)\mathrm{Y}'_{cde} + (\omega^3 - \omega^2)\mathrm{Y}''_{dce},\\
\mathrm{V}''_{cde} &= (\omega^4 - \omega)\mathrm{Y}''_{cde} - (\omega^3 - \omega^2)\mathrm{Y}'_{dce},\\
\mathrm{V}''_{cde} &= (\omega^4 + \omega - 2)\mathrm{Y}'''_{cde} - (\omega^3 + \omega^2 - 2)\mathrm{Y}'''_{dce}.
\end{aligned}\right\} \tag{213}$$

Introducing also the following notations, analogous to (112),

$$\left.\begin{aligned}
\mathrm{Y}'_{345} &= \mathrm{Y}_5^{\backprime\prime} + \mathrm{Y}_5^{\backprime\prime\prime}, & \mathrm{Y}''_{435} &= \mathrm{Y}_5^{\backprime\prime} - \mathrm{Y}_5^{\backprime\prime\prime},\\
\mathrm{Y}'_{453} &= \mathrm{Y}_3^{\backprime\prime} + \mathrm{Y}_3^{\backprime\prime\prime}, & \mathrm{Y}''_{543} &= \mathrm{Y}_3^{\backprime\prime} - \mathrm{Y}_3^{\backprime\prime\prime},\\
\mathrm{Y}'_{534} &= \mathrm{Y}_4^{\backprime\prime} + \mathrm{Y}_4^{\backprime\prime\prime}, & \mathrm{Y}''_{354} &= \mathrm{Y}_4^{\backprime\prime} - \mathrm{Y}_4^{\backprime\prime\prime};
\end{aligned}\right\} \tag{214}$$

$$\left.\begin{aligned}
\mathrm{Y}''_{345} &= \mathrm{Y}_5^{\backprime\backprime\prime} + \mathrm{Y}_5^{\backprime\backprime\prime\prime}, & \mathrm{Y}'_{435} &= \mathrm{Y}_5^{\backprime\backprime\prime} - \mathrm{Y}_5^{\backprime\backprime\prime\prime},\\
\mathrm{Y}''_{453} &= \mathrm{Y}_3^{\backprime\backprime\prime} + \mathrm{Y}_3^{\backprime\backprime\prime\prime}, & \mathrm{Y}'_{543} &= \mathrm{Y}_3^{\backprime\backprime\prime} - \mathrm{Y}_3^{\backprime\backprime\prime\prime},\\
\mathrm{Y}''_{534} &= \mathrm{Y}_4^{\backprime\backprime\prime} + \mathrm{Y}_4^{\backprime\backprime\prime\prime}, & \mathrm{Y}'_{354} &= \mathrm{Y}_4^{\backprime\backprime\prime} - \mathrm{Y}_4^{\backprime\backprime\prime\prime};
\end{aligned}\right\} \tag{215}$$

and

$$\left.\begin{aligned}
\mathrm{Y}'''_{345} &= \mathrm{Y}_5^{\backprime\backprime\backprime\prime} + \mathrm{Y}_5^{\backprime\backprime\backprime\prime\prime}, & \mathrm{Y}'''_{435} &= \mathrm{Y}_5^{\backprime\backprime\backprime\prime} - \mathrm{Y}_5^{\backprime\backprime\backprime\prime\prime},\\
\mathrm{Y}'''_{453} &= \mathrm{Y}_3^{\backprime\backprime\backprime\prime} + \mathrm{Y}_3^{\backprime\backprime\backprime\prime\prime}, & \mathrm{Y}'''_{543} &= \mathrm{Y}_3^{\backprime\backprime\backprime\prime} - \mathrm{Y}_3^{\backprime\backprime\backprime\prime\prime},\\
\mathrm{Y}'''_{534} &= \mathrm{Y}_4^{\backprime\backprime\backprime\prime} + \mathrm{Y}_4^{\backprime\backprime\backprime\prime\prime}, & \mathrm{Y}'''_{354} &= \mathrm{Y}_4^{\backprime\backprime\backprime\prime} - \mathrm{Y}_4^{\backprime\backprime\backprime\prime\prime};
\end{aligned}\right\} \tag{216}$$

we find, by (30), results analogous to (113) and (114), namely,

$$\left.\begin{aligned}
\mathrm{V}'_{345} &= \mathrm{BY}_5^{\backprime\prime} + \mathrm{CY}_5^{\backprime\prime\prime}, & \mathrm{V}'_{435} &= \mathrm{BY}_5^{\backprime\backprime\prime} - \mathrm{CY}_5^{\backprime\backprime\prime\prime},\\
\mathrm{V}'_{453} &= \mathrm{BY}_3^{\backprime\prime} + \mathrm{CY}_3^{\backprime\prime\prime}, & \mathrm{V}'_{543} &= \mathrm{BY}_3^{\backprime\backprime\prime} - \mathrm{CY}_3^{\backprime\backprime\prime\prime},\\
\mathrm{V}'_{534} &= \mathrm{BY}_4^{\backprime\prime} + \mathrm{CY}_4^{\backprime\prime\prime}, & \mathrm{V}'_{354} &= \mathrm{BY}_4^{\backprime\backprime\prime} - \mathrm{CY}_4^{\backprime\backprime\prime\prime};
\end{aligned}\right\} \tag{217}$$

$$\left.\begin{aligned}
\mathrm{V}''_{345} &= \mathrm{CY}_5^{\backprime\backprime\prime} + \mathrm{BY}_5^{\backprime\backprime\prime\prime}, & \mathrm{V}''_{435} &= \mathrm{CY}_5^{\backprime\prime} - \mathrm{BY}_5^{\backprime\prime\prime},\\
\mathrm{V}''_{453} &= \mathrm{CY}_3^{\backprime\backprime\prime} + \mathrm{BY}_3^{\backprime\backprime\prime\prime}, & \mathrm{V}''_{543} &= \mathrm{CY}_3^{\backprime\prime} - \mathrm{BY}_3^{\backprime\prime\prime},\\
\mathrm{V}''_{534} &= \mathrm{CY}_4^{\backprime\backprime\prime} + \mathrm{BY}_4^{\backprime\backprime\prime\prime}, & \mathrm{V}''_{354} &= \mathrm{CY}_4^{\backprime\prime} - \mathrm{BY}_4^{\backprime\prime\prime};
\end{aligned}\right\} \tag{218}$$

and

$$\left.\begin{aligned}
\mathrm{V}'''_{345} &= \mathrm{DY}_5^{\backprime\backprime\backprime\prime} - 5\mathrm{Y}_5^{\backprime\backprime\backprime\prime\prime}, & \mathrm{V}'''_{435} &= \mathrm{DY}_5^{\backprime\backprime\backprime\prime} + 5\mathrm{Y}_5^{\backprime\backprime\backprime\prime\prime},\\
\mathrm{V}'''_{453} &= \mathrm{DY}_3^{\backprime\backprime\backprime\prime} - 5\mathrm{Y}_3^{\backprime\backprime\backprime\prime\prime}, & \mathrm{V}'''_{543} &= \mathrm{DY}_3^{\backprime\backprime\backprime\prime} + 5\mathrm{Y}_3^{\backprime\backprime\backprime\prime\prime},\\
\mathrm{V}'''_{534} &= \mathrm{DY}_4^{\backprime\backprime\backprime\prime} - 5\mathrm{Y}_4^{\backprime\backprime\backprime\prime\prime}, & \mathrm{V}'''_{354} &= \mathrm{DY}_4^{\backprime\backprime\backprime\prime} + 5\mathrm{Y}_4^{\backprime\backprime\backprime\prime\prime}.
\end{aligned}\right\} \tag{219}$$

And squaring the eighteen expressions (217) (218) (219), we obtain others, for the eighteen functions V'^2, V''^2, V'''^2, which depend indeed on eighteen others of the forms Y, determined by the definitions (211) (214) (215) (216), but which are free, by (54) and (55), from the imaginary fifth root of unity, ω, except so far as that root enters by means of the combination D, of which the square is $= 5$.

44. If, now, we write like Professor BADANO (who uses, indeed, as has been stated already, a notation slightly different),

$$\left.\begin{aligned}
\mathrm{V}'''^2_{453} &= \mathrm{H}_{19} + \sqrt{\mathrm{H}_{20}} + \sqrt[3]{\mathrm{H}_{21} + \sqrt{\mathrm{H}_{22}}} + \sqrt[3]{\mathrm{H}_{23} - \sqrt{\mathrm{H}_{24}}};\\
\mathrm{V}'''^2_{534} &= \mathrm{H}_{19} + \sqrt{\mathrm{H}_{20}} + \theta\sqrt[3]{\mathrm{H}_{21} + \sqrt{\mathrm{H}_{22}}} + \theta^2\sqrt[3]{\mathrm{H}_{23} - \sqrt{\mathrm{H}_{24}}};\\
\mathrm{V}'''^2_{345} &= \mathrm{H}_{19} + \sqrt{\mathrm{H}_{20}} + \theta^2\sqrt[3]{\mathrm{H}_{21} + \sqrt{\mathrm{H}_{22}}} + \theta\sqrt[3]{\mathrm{H}_{23} - \sqrt{\mathrm{H}_{24}}};
\end{aligned}\right\} \tag{a$'''$}$$

and

$$\left.\begin{aligned}
\mathrm{V}_{543}^{\prime\prime\prime 2} &= \mathrm{H}_{19} - \sqrt{\mathrm{H}_{20}} + \sqrt[3]{\mathrm{H}_{21} - \sqrt{\mathrm{H}_{22}}} + \sqrt[3]{\mathrm{H}_{23} + \sqrt{\mathrm{H}_{24}}};\\
\mathrm{V}_{435}^{\prime\prime\prime 2} &= \mathrm{H}_{19} - \sqrt{\mathrm{H}_{20}} + \theta\sqrt[3]{\mathrm{H}_{21} - \sqrt{\mathrm{H}_{22}}} + \theta^2\sqrt[3]{\mathrm{H}_{23} + \sqrt{\mathrm{H}_{24}}};\\
\mathrm{V}_{354}^{\prime\prime\prime 2} &= \mathrm{H}_{19} - \sqrt{\mathrm{H}_{20}} + \theta^2\sqrt[3]{\mathrm{H}_{21} - \sqrt{\mathrm{H}_{22}}} + \theta\sqrt[3]{\mathrm{H}_{23} + \sqrt{\mathrm{H}_{24}}};
\end{aligned}\right\} \tag{b$'''$}$$

together with twelve other expressions similar to these, and to those already marked (a) and (b), but involving the functions V' and V''; we shall have, as the same author has remarked, a system of converse formulæ, analogous to (c) and (d), for the determination of the values of the eighteen quantities $\mathrm{H}_7, \ldots \mathrm{H}_{24}$. Among these, we shall content ourselves with here examining one of the most simple, namely the following:

$$\mathrm{H}_{19} = \tfrac{1}{6}(\mathrm{V}_{345}^{\prime\prime\prime 2} + \mathrm{V}_{453}^{\prime\prime\prime 2} + \mathrm{V}_{534}^{\prime\prime\prime 2} + \mathrm{V}_{354}^{\prime\prime\prime 2} + \mathrm{V}_{543}^{\prime\prime\prime 2} + \mathrm{V}_{435}^{\prime\prime\prime 2}); \tag{220}$$

for the purpose of showing, by an example, that this quantity is not independent of the arrangement of the five roots of the original equation of the fifth degree.

45. Resuming with this view, the equation (147), and the arrangement of the roots (148), we find the following system of the twenty-four values of the function X_{bcde}:

$$\left.\begin{array}{llll}
\mathrm{X}_{2345} = -500; & \mathrm{X}_{3254} = -90; & \mathrm{X}_{4523} = 240; & \mathrm{X}_{5432} = 500;\\
\mathrm{X}_{2453} = 1165; & \mathrm{X}_{4235} = -935; & \mathrm{X}_{5324} = -515; & \mathrm{X}_{3542} = -1165;\\
\mathrm{X}_{2534} = 90; & \mathrm{X}_{5243} = 935; & \mathrm{X}_{3425} = 515; & \mathrm{X}_{4352} = -240;
\end{array}\right\} \tag{221}$$

$$\left.\begin{array}{llll}
\mathrm{X}_{2354} = -620; & \mathrm{X}_{3245} = -295; & \mathrm{X}_{5423} = 145; & \mathrm{X}_{4532} = 70;\\
\mathrm{X}_{2543} = 620; & \mathrm{X}_{5234} = -720; & \mathrm{X}_{4325} = 720; & \mathrm{X}_{3452} = -70;\\
\mathrm{X}_{2435} = -145; & \mathrm{X}_{4253} = 375; & \mathrm{X}_{3524} = -375; & \mathrm{X}_{5342} = 295;
\end{array}\right\} \tag{222}$$

which give, by (211),

$$\left.\begin{array}{lll}
4\mathrm{Y}_{345}^{\prime\prime\prime} = -150; & 4\mathrm{Y}_{453}^{\prime\prime\prime} = 1450; & 4\mathrm{Y}_{534}^{\prime\prime\prime} = -1600;\\
4\mathrm{Y}_{435}^{\prime\prime\prime} = 150; & 4\mathrm{Y}_{543}^{\prime\prime\prime} = 550; & 4\mathrm{Y}_{354}^{\prime\prime\prime} = -400;
\end{array}\right\} \tag{223}$$

and, therefore, by (216),

$$\left.\begin{array}{lll}
8\mathrm{Y}_{5}^{\backprime\backprime\backprime\prime} = 0; & 8\mathrm{Y}_{3}^{\backprime\backprime\backprime\prime} = 2000; & 8\mathrm{Y}_{4}^{\backprime\backprime\backprime\prime} = -2000;\\
8\mathrm{Y}_{5}^{\backprime\backprime\backprime\prime\prime} = -300; & 8\mathrm{Y}_{3}^{\backprime\backprime\backprime\prime\prime} = 900; & 8\mathrm{Y}_{4}^{\backprime\backprime\backprime\prime\prime} = -1200;
\end{array}\right\} \tag{224}$$

whence, by (219),

$$\left.\begin{array}{lll}
\frac{2}{125}\mathrm{V}_{345}^{\prime\prime\prime} = 3; & \frac{2}{125}\mathrm{V}_{453}^{\prime\prime\prime} = -9 + 4\mathrm{D}; & \frac{2}{125}\mathrm{V}_{534}^{\prime\prime\prime} = 12 - 4\mathrm{D};\\
\frac{2}{125}\mathrm{V}_{435}^{\prime\prime\prime} = -3; & \frac{2}{125}\mathrm{V}_{543}^{\prime\prime\prime} = 9 + 4\mathrm{D}; & \frac{2}{125}\mathrm{V}_{354}^{\prime\prime\prime} = -12 - 4\mathrm{D};
\end{array}\right\} \tag{225}$$

and the squares of these six second members are

$$9, \quad 161 \mp 72\mathrm{D}, \quad 224 \mp 96\mathrm{D}, \tag{226}$$

so that we have, by (220), with this arrangement of the five roots of the equation (147),

$$\text{H}_{19} = 2^{-1}3^{-1}5^{6}197. \tag{227}$$

But with the arrangement (192), we find, by similar calculations,

$$\left.\begin{array}{lll} \frac{2}{125}\text{V}'''_{345} = 6 + 4\text{D}; & \frac{2}{125}\text{V}'''_{453} = -9 - 4\text{D}; & \frac{2}{125}\text{V}'''_{534} = -3; \\ \frac{2}{125}\text{V}'''_{435} = -6 + 4\text{D}; & \frac{2}{125}\text{V}'''_{543} = 9 - 4\text{D}; & \frac{2}{125}\text{V}'''_{354} = +3; \end{array}\right\} \tag{228}$$

of which the squares are

$$116 \pm 48\text{D}, \quad 161 \pm 72\text{D}, \quad 9; \tag{229}$$

and we have now

$$\text{H}_{19} = 2^{-1}3^{-1}5^{6}11^{1}13, \tag{230}$$

a value different from that marked (227). And, finally, with the arrangement of the roots (198), we find instead of the quantities (225) or (228), the following:

$$\mp 18 - 8\text{D}, \quad \pm 6, \quad 0, \tag{231}$$

of which the squares are

$$644 \pm 288\text{D}, \quad 36, \quad 0, \tag{232}$$

and give still another value for the quantity H now under consideration, namely,

$$\text{H}_{19} = 2^{1}3^{-1}5^{7}17. \tag{233}$$

46. The twelve other expressions which have been referred to, as being analogous to (a) and (b), are of the forms:

$$\text{V}'^{2}_{345} = \text{H}_7 + \sqrt{\text{H}_8} + \sqrt[3]{\text{H}_9 + \sqrt{\text{H}_{10}}} + \sqrt[3]{\text{H}_{11} - \sqrt{\text{H}_{12}}}; \tag{a$'$}$$
$$\text{V}'^{2}_{354} = \text{H}_7 - \sqrt{\text{H}_8} + \sqrt[3]{\text{H}_9 - \sqrt{\text{H}_{10}}} + \sqrt[3]{\text{H}_{11} + \sqrt{\text{H}_{12}}}; \tag{b$'$}$$
$$\text{V}''^{2}_{534} = \text{H}_{13} + \sqrt{\text{H}_{14}} + \sqrt[3]{\text{H}_{15} + \sqrt{\text{H}_{16}}} + \sqrt[3]{\text{H}_{17} - \sqrt{\text{H}_{18}}}; \tag{a$''$}$$
$$\text{V}''^{2}_{435} = \text{H}_{13} - \sqrt{\text{H}_{14}} + \sqrt[3]{\text{H}_{15} - \sqrt{\text{H}_{16}}} + \sqrt[3]{\text{H}_{17} + \sqrt{\text{H}_{18}}}; \tag{b$''$}$$

and they give, as the simplest of the expressions deduced from them, the two following, which are analogous to that marked (220):

$$\text{H}_7 = \tfrac{1}{6}(\text{V}'^{2}_{345} + \text{V}'^{2}_{453} + \text{V}'^{2}_{534} + \text{V}'^{2}_{354} + \text{V}'^{2}_{543} + \text{V}'^{2}_{435}); \tag{234}$$
$$\text{H}_{13} = \tfrac{1}{6}(\text{V}''^{2}_{345} + \text{V}''^{2}_{453} + \text{V}''^{2}_{534} + \text{V}''^{2}_{354} + \text{V}''^{2}_{543} + \text{V}''^{2}_{435}). \tag{235}$$

For the case of the equation (147), and the arrangement of roots (148), we find the numerical values:

$$\left.\begin{array}{lll} \frac{2}{5}\text{V}'_{345} = -126\text{B} - 7\text{C}; & \frac{2}{5}\text{V}'_{453} = 202\text{B} - 11\text{C}; & \frac{2}{5}\text{V}'_{534} = 25\text{B} + 50\text{C}; \\ \frac{2}{5}\text{V}''_{435} = -126\text{C} + 7\text{B}; & \frac{2}{5}\text{V}''_{543} = 202\text{C} + 11\text{B}; & \frac{2}{5}\text{V}''_{354} = 25\text{C} - 50\text{B}; \end{array}\right\} \tag{236}$$

$$\left.\begin{array}{lll}\frac{2}{5}\mathrm{V}'_{435} = -18\mathrm{B} + 47\mathrm{C}; & \frac{2}{5}\mathrm{V}'_{543} = 100\mathrm{B} - 175\mathrm{C}; & \frac{2}{5}\mathrm{V}'_{354} = -61\mathrm{B} - 52\mathrm{C}; \\ \frac{2}{5}\mathrm{V}''_{345} = -18\mathrm{C} - 47\mathrm{B}; & \frac{2}{5}\mathrm{V}''_{453} = 100\mathrm{C} + 175\mathrm{B}; & \frac{2}{5}\mathrm{V}''_{534} = -61\mathrm{C} + 52\mathrm{B};\end{array}\right\} \quad (237)$$

which may be obtained, either by the method of article 43., combined with the values (221) (222) of the twenty-four functions X; or by the formulæ (210), combined with the following table:

$$\left.\begin{array}{ll}\frac{2}{5}\mathrm{T}_{2345} = -175\mathrm{B} - 25\mathrm{C}; & \frac{2}{5}\mathrm{T}_{2435} = -150 - 11\mathrm{B} - 77\mathrm{C}; \\ \frac{2}{5}\mathrm{T}_{2453} = +377\mathrm{B} + 89\mathrm{C}; & \frac{2}{5}\mathrm{T}_{2543} = 450 + 111\mathrm{B} + 27\mathrm{C} + 200\mathrm{D}; \\ \frac{2}{5}\mathrm{T}_{2534} = 150 + 77\mathrm{B} - 11\mathrm{C}; & \frac{2}{5}\mathrm{T}_{2354} = -450 - 111\mathrm{B} - 27\mathrm{C} - 200\mathrm{D};\end{array}\right\} \quad (238)$$

and with the condition, that, if we write for abridgment,

$$\mathrm{T}_{bcde} = \mathrm{T}^{(0)}_{bcde} + \mathrm{BT}'_{bcde} + \mathrm{CT}''_{bcde} + \mathrm{DT}'''_{bcde}, \quad (239)$$

we have in general the relations,

$$\left.\begin{array}{l}\mathrm{T}_{edcb} = \mathrm{T}^{(0)}_{bcde} - \mathrm{BT}'_{bcde} - \mathrm{CT}''_{bcde} + \mathrm{DT}'''_{bcde}; \\ \mathrm{T}_{cebd} = \mathrm{T}^{(0)}_{bcde} + \mathrm{CT}'_{bcde} - \mathrm{BT}''_{bcde} - \mathrm{DT}'''_{bcde}.\end{array}\right\} \quad (240)$$

And hence, for the same equation of the fifth degree, and the same arrangement of the roots, we find, by (54) and (55):

$$\left.\begin{array}{l}\mathrm{H}_7 = -2^{-2}3^{-1}5^4(10975 + 706\mathrm{D}); \\ \mathrm{H}_{13} = -2^{-2}3^{-1}5^4(10975 - 706\mathrm{D}).\end{array}\right\} \quad (241)$$

But, for the same equation (147), with the arrangement of the roots (192), we find, by similar calculations, the values:

$$\left.\begin{array}{l}\mathrm{H}_7 = -2^{-2}3^{-1}5^4(10975 - 1472\mathrm{D}); \\ \mathrm{H}_{13} = -2^{-2}3^{-1}5^4(10975 + 1472\mathrm{D});\end{array}\right\} \quad (242)$$

and with the arrangement (198),

$$\left.\begin{array}{l}\mathrm{H}_7 = -2^{-2}3^{-1}5^4(10975 + 3832\mathrm{D}); \\ \mathrm{H}_{13} = -2^{-2}3^{-1}5^4(10975 - 3832\mathrm{D}).\end{array}\right\} \quad (243)$$

We see, therefore, that in this example, the difference of the two quantities H_7 and H_{13} is neither equal to zero, nor independent of the arrangement of the five roots of the equation of the fifth degree. However, it may be noticed that in the same example, the sum of the same two quantities H_7 and H_{13} has not been altered by altering the arrangement of the roots; and in fact, by the method of the 43rd article, we find the formula:

$$\left.\begin{array}{rl}-\frac{48}{5}(\mathrm{H}_7 + \mathrm{H}_{13}) = & (\mathrm{X}_{2345} - \mathrm{X}_{5432})^2 + (\mathrm{X}_{2453} - \mathrm{X}_{3542})^2 + (\mathrm{X}_{2534} - \mathrm{X}_{4352})^2 \\ & + (\mathrm{X}_{3254} - \mathrm{X}_{4523})^2 + (\mathrm{X}_{4235} - \mathrm{X}_{5324})^2 + (\mathrm{X}_{5243} - \mathrm{X}_{3425})^2 \\ & + (\mathrm{X}_{2354} - \mathrm{X}_{4532})^2 + (\mathrm{X}_{2543} - \mathrm{X}_{3452})^2 + (\mathrm{X}_{2435} - \mathrm{X}_{5342})^2 \\ & + (\mathrm{X}_{3245} - \mathrm{X}_{5423})^2 + (\mathrm{X}_{5234} - \mathrm{X}_{4325})^2 + (\mathrm{X}_{4253} - \mathrm{X}_{3524})^2;\end{array}\right\} \quad (244)$$

of which the second member is in general a symmetric function of the five roots, and gives, in the case of the equation (147), by (221) and (222), the following numerical value, agreeing with recent results,

$$\mathrm{H}_7 + \mathrm{H}_{13} = -2^{-1}3^{-1}5^6 439. \quad (245)$$

47. It seems useless to add to the length of this communication, by entering into any additional details of calculation: since the foregoing investigations will probably be thought to have sufficiently established the inadequacy of Professor BADANO's method* for the general solution of equations of the fifth degree, notwithstanding the elegance of those systems of radicals which have been proposed by that author for the expression of the twenty-four values of LAGRANGE's function t^5. Indeed, it is not pretended that a full account has been given, in the present paper, of the reasons which Professor BADANO has assigned for believing that the twenty-four quantities which have been called H are all symmetric† functions of the five roots of the equation of the fifth degree; and that those quantities are connected by certain relations among themselves, which would, if valid, conduct to the following expression for resolving an equation of that degree, analogous to the known radical expressions for the solution of less elevated equations:

$$\begin{aligned} t^5 = \mathrm{K}_1 + \sqrt{\mathrm{K}_2} + \sqrt[3]{\mathrm{K}_3 + \sqrt{\mathrm{K}_4}} + \sqrt[3]{\mathrm{K}_3 - \sqrt{\mathrm{K}_4}} \\ + \sqrt{\{\mathrm{K}_5 + \sqrt{\mathrm{K}_6} + \sqrt[3]{\mathrm{K}_7 + \sqrt{\mathrm{K}_8}} + \sqrt[3]{\mathrm{K}_7 - \sqrt{\mathrm{K}_8}}\}} \\ + \sqrt{\{\mathrm{K}_5 + \sqrt{\mathrm{K}_6} + \theta \sqrt[3]{\mathrm{K}_7 + \sqrt{\mathrm{K}_8}} + \theta^2 \sqrt[3]{\mathrm{K}_7 - \sqrt{\mathrm{K}_8}}\}} \\ + \sqrt{\{\mathrm{K}_5 + \sqrt{\mathrm{K}_6} + \theta^2 \sqrt[3]{\mathrm{K}_7 + \sqrt{\mathrm{K}_8}} + \theta \sqrt[3]{\mathrm{K}_7 - \sqrt{\mathrm{K}_8}}\}}. \end{aligned}$$

But it has been shown, in the foregoing articles, that at least some of the relations here referred to, between the twenty-four quantities H, do not in general exist; since we have not, for example, the relation of equality between H_4 and H_6, which would be required, in order to justify the substitution of a single symbol K_4 for these two quantities. It has also been shown that each of these two unequal quantities, H_4 and H_6, in general changes its value, when the arrangement of the five roots of the original equation is changed in a suitable manner: and that H_7, H_{13}, H_{19}, are also unequal, and change their values, at least in the example above chosen. And thus it appears, to the writer of the present paper, that the investigations now submitted to the Academy, by establishing (as in his opinion they do) the failure of this new and elegant attempt of an ingenious Italian analyst, have thrown some additional light on the impossibility (though otherwise proved before) of resolving the general equation of the fifth degree by any finite combination of radicals and rational functions.

* Professor BADANO's rule is, to substitute, in each H, for each power of x', the fifth part of the sum of the corresponding powers of the five roots, $x', \dots x^V$; and he proposes to extend the same method to equations of all higher degrees.

† "Dunque le H sono quantità costanti sotto la sostituzione di qualcunque radice dell'equazione." To show that the constancy, thus asserted, does not exist, has been the chief object proposed in the present paper; to which the writer has had opportunities of making some additions, since it was first communicated to the Academy.