ON CONJUGATE FUNCTIONS, OR ALGEBRAIC COUPLES

By

William Rowan Hamilton

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On Conjugate Functions, or Algebraic Couples, as tending to illustrate generally the Doctrine of Imaginary Quantities, and as confirming the Results of Mr Graves respecting the Existence of Two independent Integers in the complete expression of an Imaginary Logarithm. By W. R. Hamilton, *M.R.I.A., Astronomer Royal for Ireland.*

[*Report of the Fourth Meeting of the British Association for the Advancement of Science; held at Edinburgh in 1834.* (John Murray, London, 1835), pp. 519–523.]

Admitting, at first, the usual things about imaginaries, let

$$u + v\sqrt{-1} = \phi.(x + y\sqrt{-1}), \tag{a.}$$

in which x, y are one pair of real quantities, and u, v are another pair, depending on the former, and therefore capable of being thus denoted, $u_{x,y}$, $v_{x,y}$. It is easy to prove that these two functions $u_{x,y}$, $v_{x,y}$, must satisfy the two following equations between their partial differential coefficients of the first order:

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{du}{dy} = -\frac{dv}{dx}. \tag{b.}$$

Professor Hamilton calls these the *Equations of Conjugation*, between the functions u, v, because they are the necessary and sufficient conditions in order that the imaginary expression $u+v\sqrt{-1}$ should be a function of $x+y\sqrt{-1}$. And he thinks that without any introduction of imaginary symbols, the two real relations (b.), between two real functions, might have been suggested by analogies of algebra, as constituting between those two functions a connexion useful to study, and as leading to the same results which are usually obtained by imaginaries. Dismissing, therefore, for the present, the conception and language of imaginaries, Mr Hamilton proposes to consider a few properties of such *Conjugate Functions*, or *Algebraic Couples*; defining two functions to be *conjugate* when they satisfy the two equations of conjugation, and calling, under the same circumstances, the pair or couple (u, v) *a function of the pair* (x, y).

An easy extension of this view leads to the consideration of relations between several pairs, and generally to reasonings and operations upon pairs analogous to reasonings and operations on single quantities. For all such reasonings it is necessary to establish definitions: the following definitions of sum and product of pairs appear to Mr Hamilton natural:

$$(x, y) + (a, b) = (x + a, y + b), \tag{c.}$$

$$(x, y) \times (a, b) = (xa - yb, xb + ya), \tag{d.}$$

and conduct to meanings of all integer powers and other rational functions of pairs, enabling us to generalize any ordinary algebraic equation from single quantities to pairs, and so to interpret the research of all its roots, without introducing imaginaries.

Without stopping to justify these definitions of sum and product which will probably be admitted without difficulty, Mr Hamilton inquires what analogous meaning should be attached to an exponential pair, or to the notation $(a,b)^{(x,y)}$; or, finally, what form ought to be assigned to the conjugate functions $u_{x,y}$, $v_{x,y}$, in the exponential equation

$$(a,b)^{(x,y)} = (u_{x,y}, v_{x,y}). \tag{e.}$$

In the theory of quantities, the most fundamental properties of the exponential function $a^x = \phi(x)$ are these:

$$\phi(x)\phi(\xi) = \phi(x+\xi), \quad \text{and} \quad \phi(1) = a; \tag{f.}$$

Mr Hamilton thinks it right, therefore, in the theory of pairs, to establish by definition the analogous properties,

$$(a,b)^{(x,y)}(a,b)^{(\xi,\eta)} = (a,b)^{(x+\xi,y+\eta)}, \tag{g.}$$

and

$$(a,b)^{(1,0)} = (a,b). \tag{h.}$$

Combining these properties with the equation (e.) and with the definition (d.) of product, and defining an equation between pairs to involve two equations between quantities, Mr Hamilton obtains the following pair of ordinary differential equations, or equations in differences, to be combined with the two equations of conjugation:

$$\left.\begin{aligned} u_{x,y}v_{\xi,\eta} - v_{x,y}u_{\xi,\eta} &= u_{x+\xi,y+\eta}, \\ u_{x,y}v_{\xi,\eta} + v_{x,y}u_{\xi,\eta} &= v_{x+\xi,y+\eta}, \end{aligned}\right\} \tag{i.}$$

and also the following pair of conditions,

$$u_{1,0} = a, \quad v_{1,0} = b. \tag{k.}$$

Solving the pair of equations (i.), he finds

$$\left.\begin{aligned} u_{x,y} &= f(\alpha' y + \beta' x).\cos(\alpha y + \beta x), \\ v_{x,y} &= f(\alpha' y + \beta' x).\sin(\alpha y + \beta x), \end{aligned}\right\} \tag{l.}$$

α β α' β' being any four constants, independent of x and y, and the function f being such that

$$f(r) = 1 + \frac{r}{1} + \frac{r^2}{1.2} + \frac{r^3}{1.2.3} + \&\text{c.}; \tag{m.}$$

and having established the following, among many other general properties of conjugate functions, that if two such functions be put under the forms

$$\left.\begin{aligned} u_{x,y} &= f(\rho_{x,y}).\cos\theta_{x,y} \\ v_{x,y} &= f(\rho_{x,y}).\sin\theta_{x,y} \end{aligned}\right\} \tag{n.}$$

f still retaining its late meaning, the functions $\rho_{x,y}$ $\theta_{x,y}$ are also conjugate, he concludes that the 4 constants of (l.) are connected by these two relations

$$\beta' = +\alpha, \quad \alpha' = -\beta, \tag{o.}$$

so that the general expressions for two conjugate exponential functions are:

$$\left.\begin{aligned} u_{x,y} &= f(\alpha x - \beta y).\cos(\alpha y + \beta x), \\ v_{x,y} &= f(\alpha x - \beta y).\sin(\alpha y + \beta x); \end{aligned}\right\} \tag{p.}$$

and it only remains to introduce the constants of the *base-pair* (a, b), by the conditions (k.). These conditions give

$$a = f(\alpha).\cos\beta, \quad b = f(\alpha).\sin\beta, \tag{q.}$$

and therefore, finally,

$$\left.\begin{aligned} \alpha &= \int_1^{\sqrt{a^2+b^2}} \frac{dr}{r}, \\ \beta &= \beta_0 + 2i\pi, \end{aligned}\right\} \tag{r.}$$

i being an arbitrary integer, and β_0 being a quantity which may be assumed as $> -\pi$, but not $> \pi$, and may then be determined by the equations

$$\cos\beta_0 = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\beta_0 = \frac{b}{\sqrt{a^2+b^2}}. \tag{s.}$$

The form of the direct exponential pair $(a, b)^{(x,y)}$, (or of the direct conjugate exponential functions u, v,) is now entirely determined; but the process has introduced *one* arbitrary integer i.

Another arbitrary integer is introduced by reversing the process, and seeking the *inverse exponential* or *logarithmic pair*,

$$(x, y) = \operatorname*{Log}_{(a,b)}(u, v). \tag{t.}$$

Professor Hamilton finds for this inverse problem the formulæ

$$x = \frac{\alpha\rho + \beta\theta}{\alpha^2 + \beta^2}, \quad x = \frac{\alpha\theta - \beta\rho}{\alpha^2 + \beta^2}; \tag{u.}$$

in which α β are the constants deduced as before by (r.) from the *base-pair* (a, b), and involving the integer i in the expression of β; while ρ and θ are deduced from u and v, with a new arbitrary integer k in θ, by expressions analogous to (r.), namely,

$$\left.\begin{aligned} \rho &= \int_1^{\sqrt{u^2+v^2}} \frac{dr}{r}, \\ \theta &= \theta_0 + 2k\pi, \end{aligned}\right\} \tag{v.}$$

in which θ_0 is supposed $> -\pi$ but not $> \pi$, and

$$\cos\theta_0 = \frac{u}{\sqrt{u^2+v^2}}, \quad \sin\theta_0 = \frac{v}{\sqrt{u^2+v^2}}. \tag{w.}$$

By the definition of quotient, which the definition (d.) of product suggests, the formulæ (u.) may be briefly comprised in the following expression of a logarithmic pair:

$$(x, y) = \frac{(\rho, \theta)}{(\alpha, \beta)}; \tag{x.}$$

and, reciprocally, the direct exponential pair (u, v), as already determined, may be concisely expressed by this other form of the same equation,

$$(\rho, \theta) = (x, y)(\alpha, \beta), \tag{y.}$$

if we still suppose

$$\left.\begin{aligned} (u, v) &= (f(\rho).\cos\theta, f(\rho).\sin\theta), \\ (a, b) &= (f(\alpha).\cos\beta, f(\alpha).\sin\beta). \end{aligned}\right\} \tag{z.}$$

Thus all the foregoing results respecting exponential and logarithmic pairs may be comprised in the equations (y.) and (z.).

When translated into the language of imaginaries, they agree with the results respecting imaginary exponential functions, direct and inverse, which were published by Mr Graves in the *Philosophical Transactions* for 1829, and it was in meditating on those results of Mr Graves that Mr Hamilton was led, several years ago, to this theory of conjugate functions,* as tending to illustrate and confirm them. For example, Mr Graves had found, for the logarithm of unity to the Napierian base, the expression

$$\operatorname*{Log}_{e}.1 = \frac{2k\pi\sqrt{-1}}{1 + 2i\pi\sqrt{-1}},$$

which is more general than the usual expression. This result of Mr. Graves appeared erroneous to the author of the excellent Report on Algebra, which was lately printed for the Association; but it is confirmed by Mr Hamilton's theory, which conducts to it under the form of a relation between real pairs, namely,

$$\operatorname*{Log}_{(e,0)}(1, 0) = \frac{(0, 2k\pi)}{(1, 2i\pi)}$$

and the connexion of this result with that Report was thought to justify an greater fulness in the present communication† that would have been proper otherwise on a question so abstract and mathematical.

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* An Essay on this theory of Conjugate Functions was presented some years ago by Professor Hamilton to the Royal Irish Academy, and will be published in one of the next forthcoming volumes of its *Transactions*.

† Since this communication was prepared, Professor Hamilton has learned that Professor Ohm of Berlin has been conducted by a different method to results respecting Imaginary Logarithms, which agree with those of Mr Graves: as do also the results obtained in other ways, by Mons. Vincent and by Mr Warren. The partial differential equations (b.) have been noticed and employed, for a different purpose, by Mr Murphy of Cambridge.