ON ANHARMONIC CO-ORDINATES

By

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Sir William Rowan Hamilton.

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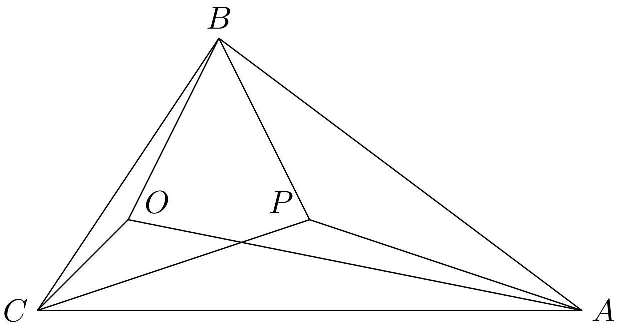
[April 9th, 1860]

1. Let A B C be any given triangle; and let O, P be any two points in its plane, whereof O shall be supposed to be given or constant, but P variable. Then, by a well-known theorem, respecting the six segments into which the sides are cut by right lines drawn from the vertices of a triangle to any common point the three following *anharmonics of pencils* have a product equal to positive unity:—

$$(\text{A . PCOB}) \,.\, (\text{B . PAOC}) \,.\, (\text{C . PBOA}) = +1.$$



It is, therefore, allowed to establish the following system of three equations, of which any one is a consequence of the other two:—

$$\frac{y}{z} = (\text{A . PCOB}); \quad \frac{z}{x} = (\text{B . PAOC}); \quad \frac{x}{y} = (\text{C . PBOA});$$

and, when this is done, I call the three quantities x, y, z, or any quantities proportional to them, the *Anharmonic Co-ordinates of the Point* P, with respect to the *given triangle* ABC, and to the *given point* O. And I *denote* that point P by the *symbol*,

$$\text{P} = (x, y, z); \quad \text{or,} \quad \text{P} = (tx, ty, tz); \quad \&\text{c.}$$

2. When the variable point P takes the given position O, the three anharmonics of pencils above mentioned become each equal to unity; so that we may write then,

$$x = y = z = 1.$$

The given point O is therefore denoted by the symbol,

$$\text{O} = (1, 1, 1);$$

on which account I call it the *Unit-Point.*

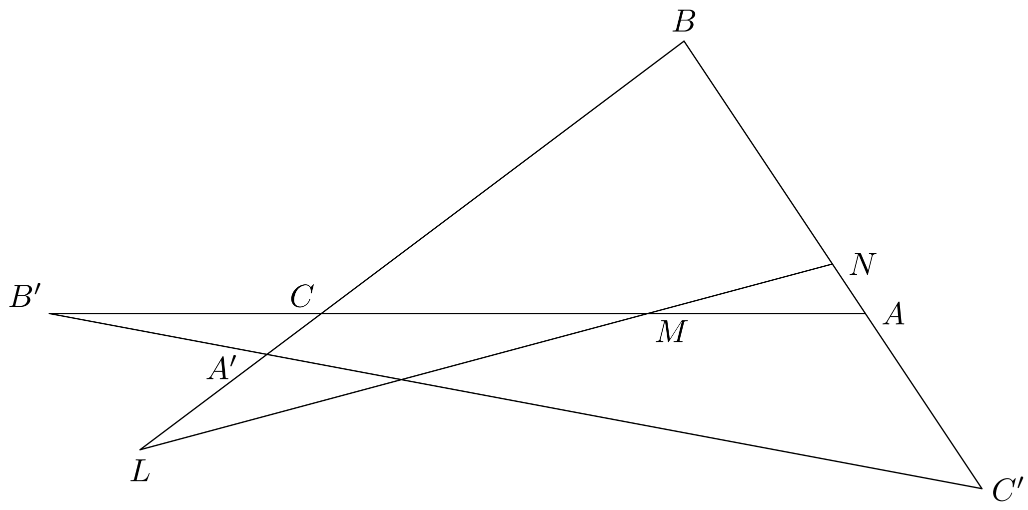
3. When the variable point P comes to coincide with the given point A, so as to be at the vertex of the first pencil, but on the second ray of the second pencil, and on the fourth ray of the third, without being at the vertex of either of the two latter pencils, then the first anharmonic becomes indeterminate, but the second is equal to zero, and the third is infinite. We are, therefore, to consider y and z, but not x, as vanishing for this position of P; and consequently may write,

$$\text{A} = (1, 0, 0).$$

In like manner,

$$\text{B} = (0, 1, 0), \quad \text{and} \quad \text{C} = (0, 0, 1);$$

and on account of these simple representations of its three corners, I call the given triangle ABC the *Unit-Triangle.*



4. Again, let the sides of this given triangle ABC be cut by a given transversal A′B′C′, and by a variable transversal LMN. Then, by another very well-known theorem respecting segments, we shall have the relation,

$$(\text{LBA}'\text{C}) \,.\, (\text{MCB}'\text{A}) \,.\, (\text{NAC}'\text{B}) = +1;$$

it is therefore permitted to establish the three equations,

$$\frac{m}{n} = (\text{LBA}'\text{C}), \quad \frac{n}{l} = (\text{MCB}'\text{A}), \quad \frac{l}{m} = (\text{NAC}'\text{B});$$

where l, m, n, or any quantities proportional to them, are what I call the *Anharmonic Co-ordinates of the Line* LMN, with respect to the given triangle ABC, and the given transversal A′B′C′. And I *denote* the *line* LMN by the *symbol*,

$$\overline{\text{LMN}} = [l, m, n].$$

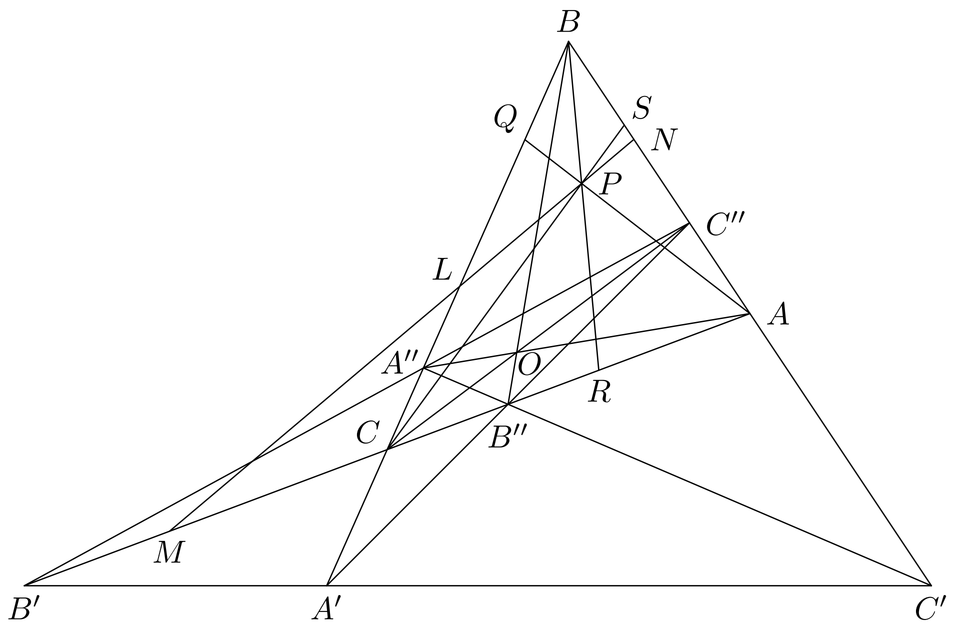
For example, if this variable line come to coincide with the given line A′B′C′, then

$$l = m = n;$$

so that this given line may be thus denoted,

$$\overline{\text{A}'\text{B}'\text{C}'} = [1, 1, 1];$$

on which account I call the *given transversal* A′B′C′ the *Unit-Line* of the Figure. The *sides*, BC, &c., of the given triangle ABC, take on this plan the symbols $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$.



5. Suppose now that the *unit-point* and the *unit-line* are related to each other, as being (in a known sense) *pole* and *polar*, with respect to the given or *unit-triangle*; or, in other words, let the lines OA, OB, OC be supposed to meet the sides BC, CA, AB of that given triangle, in points A″, B″, C″, which are, with respect to those sides, the harmonic conjugates of the points A′, B′, C′, in which the same sides are cut by the given transversal A′B′C′. Also, let the variable *point* P be situated upon the variable *line* LMN; and let Q, R, S be the intersection of AP, BP, CP with BC, CA, AB. Then, because

$$(\text{BA}'\text{CA}'') = (\text{CB}'\text{AB}'') = (\text{AC}'\text{BC}'') = -1,$$

we have

$$\begin{cases} -\dfrac{m}{n} = (\text{LBA}''\text{C}), & -\dfrac{n}{l} = (\text{MCB}''\text{A}), & -\dfrac{l}{m} = (\text{NAC}''\text{B}), \\ -\dfrac{n}{m} = (\text{LCA}''\text{B}), & -\dfrac{l}{n} = (\text{MAB}''\text{C}), & -\dfrac{m}{l} = (\text{NBC}''\text{A}); \end{cases}$$

as well as

$$\begin{cases} \dfrac{y}{z} = (\text{QCA}''\text{B}), & \dfrac{z}{x} = (\text{RAB}''\text{C}), & \dfrac{x}{y} = (\text{SBC}''\text{A}), \\ \dfrac{z}{y} = (\text{QBA}''\text{C}), & \dfrac{x}{z} = (\text{RCB}''\text{A}), & \dfrac{y}{x} = (\text{SAC}''\text{B}); \end{cases}$$

and therefore,

$$\frac{-lx}{nz} = (\text{MARC}); \quad \frac{-my}{nz} = (\text{LBQC}).$$

But, by the pencil through P,

$$(\text{MARC}) = (\text{LQBC});$$

and by the *definition* of the symbol (ABCD), for any four collinear points,

$$(\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}},$$

which is here throughout adopted, we have the *identity*,

$$(\text{ABCD}) + (\text{ACBD}) = 1;$$

therefore

$$(\text{MARC}) + (\text{LBQC}) = 1,$$

or,

$$lx + my + nz = 0.$$

6. We arrive then at the following *Theorem*, which is of fundamental importance in the present system of Anharmonic Co-ordinates:—

"If the unit-point O be the pole of the unit-line A′B′C′, with respect to the unit-triangle ABC, and if a variable point P, or (x, y, z), be situated anywhere on a variable right line LMN, or $[l, m, n]$, then the sum of the products of the corresponding co-ordinates of point and line is zero."

7. It may already be considered as an evident consequence of this Theorem, that any *homogeneous equation* of the p^{th} *dimension*,

$$f_p(x, y, z) = 0,$$

represents a *curve of the* p^{th} *order*, considered as the *locus* of the variable *point* P; and that any homogeneous equation of the q^{th} dimension, of the form

$$F_q(l, m, n) = 0,$$

may in like manner be considered as the *tangential equation* of a *curve of the* q^{th} *class*, which is the *envelope* of the variable *line* LMN. But any examples of such applications must be reserved for a future communication. Meantime, I may just mention that I have been, for some time back, in possession of an analogous method for treating Points, Lines, Planes, Curves, and Surfaces in Space, by a system of Anharmonic Co-ordinates.

8. As regards the *advantages* of the Method which has been thus briefly sketched, the *first* may be said to be its geometrical *interpretability*, in a manner *unaffected by perspective.* The *relations*, whether between *variables* or between *constants*, which enter into the formula of this method, are *all projective*; and because they *all* depend upon, and are referred to, *anharmonic functions*, of groups or of pencils.

9. In the *second* place, we may remark that the great principle of *geometrical Duality* is recognized from the very outset. Confining ourselves, for the moment (as in the foregoing articles), to figures in a *given plane*, we have seen that the *anharmonic co-ordinates* of a *point*, and those of a *right line*, are deduced by processes absolutely *similar*, the one from a system of *four given points*, and the other from a system of *four given right lines.* And the *fundamental equation* ($lx + my + nz = 0$) which has been found to *connect* these *two systems* of co-ordinates, is evidently one of the most perfect *symmetry*, as regards *points* and *lines.* An analogous symmetry will show itself afterwards, in relation to points and planes.

10. The *third advantage* of the anharmonic method may be stated to consist in its possessing an *increased number of disposable constants.* Thus, within the plane, *trilinear* co-ordinates give us *only six* such constants, corresponding to the *three* disposable *positions* of the *sides* of that assumed *triangle*, to the perpendicular distances from which the co-ordinates are supposed to be proportional; but *anharmonics*, by admitting an *arbitrary unit-point*, enable us to treat *two other constants* as disposable, the *number* of such constants being thus raised from *six* to *eight.* Again, in *space*, whereas *quadriplanar* co-ordinates, considered as the *ratios of the distances* from *four assumed planes*, allow of only *twelve* disposable constants, corresponding to the possible selection of the *four planes of reference*, *anharmonic co-ordinates*, on the contrary, which admit either *five planes* or *five points* as *data*, and which might, therefore, be called *quinquiplanar* or *quinquipunctual*, permit us to dispose of no fewer than *fifteen constants as arbitrary*, in the general treatment of *surfaces.*

[May 28th, 1860]

11. To myself, it naturally appears as a *fourth advantage* of the anharmonic method, that it is found to harmonize well with the method of *quaternions*, and was in fact *suggested* thereby; though not without suggestions from other methods previously known.

12. Thus, if α, β, γ denote the given vectors, OA, OB, OC, from a given origin O, while a, b, c are three given and constant scalars, but t, u, v are three variable scalars, subject to the condition that their sum is zero,

$$t + u + v = 0;$$

then the equation,

$$\mathrm{OP} = \rho = \frac{t^r a\alpha + u^r b\beta + v^r c\gamma}{t^r\alpha + u^r\beta + v^r c},$$

in which ρ is any positive and whole exponent, expresses general that the *locus* of the point P is a *curve of the* r^{th} *order*, in the given plane of ABC; which curve has the property, that it is met in r coincident points, by any one of the three sides of the given triangle ABC. But the coefficients t^r u^r v^r are examples here of what have been above called anharmonic co-ordinates.

[June 25th, 1860]

13. Proceeding to *space*, let A, B, C, D be the four corners of a given triangular pyramid, and let E be any fifth given point, which is not on any one of the four faces of that pyramid. Let P be any sixth point of space; and let $x\,y\,z\,w$ be four positive or negative numbers, such that

$$\frac{x}{w} = (\text{BC . AEDP}), \quad \frac{y}{w} = (\text{CA . BEDP}), \quad \frac{z}{w} = (\text{AB . CEDP});$$

the right-hand member of these equations representing *anharmonics of pencils of planes*, in a way which is easily understood, with the help of the definition (5) of the symbol (ABCD). Then I call x, y, z, w (or any numbers proportional to them), the *Anharmonic Co-ordinates* of the *Point* P, with respect to what may be said to be the *Unit-Pyramid*, ABCD, because its corners may (on the present plan) be thus denoted,

$$\text{A} = (1, 0, 0, 0); \quad \text{B} = (0, 1, 0, 0); \quad \text{C} = (0, 0, 1, 0); \quad \text{D} = (0, 0, 0, 1);$$

and with respect to that fifth given point E, which may be called the *Unit-Point*, because its symbol, in the present system, may be thus written:—

$$\text{E} = (1, 1, 1, 1).$$

And I denote the general or variable point by the symbol,

$$\text{P} = (x, y, z, w).$$

14. When we have thus five given points, A ... E, of which no four are situated in any common plane, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect each other, deriving in this way as system of ten lines, ten planes, and ten points, whereof the latter may be thus denoted:

$$\begin{aligned}
\text{A}' &= \text{BC . ADE} = (0, 1, 1, 0), \quad \text{B}' = \&\text{c.}, \quad \text{C}' = \&\text{c.};\\
\text{A}_1 &= \text{AE . BCD} = (0, 1, 1, 1), \quad \text{B}_1 = \&\text{c.}, \quad \text{C}_1 = \&\text{c.};\\
\text{A}_2 &= \text{AD . BCE} = (1, 0, 0, 1), \quad \text{B}_2 = \&\text{c.}, \quad \text{C}_2 = \&\text{c.};\\
\text{D}_1 &= \text{DE . ABC} = (1, 1, 1, 0);
\end{aligned}$$

and the harmonic conjugates of these last points, with respect to the ten given lines on which they are situated, may on the same plan be represented by the following symbols:—

$$\begin{aligned}
\text{A}'' &= (0, 1, -1, 0), \quad \text{B}'' = \&\text{c.}, \quad \text{C}'' = \&\text{c.};\\
\text{A}'_1 &= (2, 1, 1, 1), \quad \text{B}'_1 = \&\text{c.}, \quad \text{C}'_1 = \&\text{c.};\\
\text{A}'_2 &= (1, 0, 0, -1), \quad \text{B}'_2 = \&\text{c.}, \quad \text{C}'_2 = \&\text{c.};\\
\text{D}_1 &= (1, 1, 1, 2);
\end{aligned}$$

so that

$$(\text{BA}'\text{CA}'') = \ \dots = (\text{EA}_1\text{AA}'_1) = \ \dots = (\text{DA}_2\text{AA}'_2) = \ \dots = (\text{ED}_1\text{DD}'_1) = -1.$$

15. Let any plane Π intersect the three given lines DA, DB, DC, in points Q, R, S; and let $l\,m\,n\,r$ be any positive or negative numbers, such that

$$\frac{l}{r} = (\mathrm{DA}_2'\mathrm{AQ}), \quad \frac{m}{r} = (\mathrm{DB}_2'\mathrm{BR}), \quad \frac{n}{r} = (\mathrm{DC}_2'\mathrm{CS});$$

then I call l, m, n, r, or any numbers proportional to them, the *Anharmonic Co-ordinates* of the *Plane* Π; which *plane* I also denote by the *Symbol*,

$$\Pi = [l, m, n, r].$$

In particular the four faces of the unit pyramid come thus to be denoted by the symbols,

$$\mathrm{BCD} = [1, 0, 0, 0], \quad \mathrm{CAD} = [0, 1, 0, 0], \quad \mathrm{ABD} = [0, 0, 1, 0], \quad \mathrm{ABC} = [0, 0, 0, 1];$$

and the six planes through its edges, and through the unit point, are denoted thus:—

$$\mathrm{BCE} = [1, 0, 0, -1]; \quad \mathrm{CAE} = [0, 1, 0, -1]; \quad \mathrm{ABE} = [0, 0, 1, -1];$$

$$\mathrm{ADE} = [0, 1, -1, 0]; \quad \mathrm{BDE} = [-1, 0, 1, 0]; \quad \mathrm{CDE} = [1, -1, 0, 0];$$

in connexion with which last planes it may be remarked that we have, generally, as a consequence of the foregoing definitions, the formulae,

$$\frac{n}{m} = (\mathrm{BA}''\mathrm{CL}), \quad \frac{l}{n} = (\mathrm{CB}''\mathrm{AM}), \quad \frac{m}{l} = (\mathrm{AC}''\mathrm{BN}),$$

if L, M, N be the points in which a variable plane Π intersects the sides BC, &c., of the given triangle ABC: as we have also, generally,

$$\frac{z}{y} = (\mathrm{AD}\,.\,\mathrm{CEBP}), \quad \frac{x}{z} = (\mathrm{BD}\,.\,\mathrm{AECP}), \quad \frac{y}{x} = (\mathrm{CD}\,.\,\mathrm{BEAP}).$$

16. If a *point*, $\mathrm{P} = (x, y, z, w)$, be situated *on a plane*, $\Pi = [l\,m\,n\,r]$, then I find that the following relation between their co-ordinates exists, which is entirely analogous to that already assigned (6) for the case of a *point* and *line* in a given plane, and is of fundamental importance in the application of the present *Anharmonic Method* to *space*:

$$lx + my + nz + rw = 0;$$

or in words, "*the sum of the products of corresponding co-ordinates*, of point and plane, *is zero*."

For example, all planes through the unit point $(1, 1, 1, 1)$ are subject to the condition,

$$l + m + n + r = 0,$$

as may be seen for the six planes (15) already drawn through that point E; and the six points $\mathrm{A}''\,\mathrm{B}''\,\mathrm{C}''\,\mathrm{A}_2'\mathrm{B}_2'\mathrm{C}_2'$ (14), in which the six edges BC, CA, AB, DA, DB, DC, of the given or unit pyramid ABCD, intersect the six corresponding edges of the inscribed and homologous pyramid $\mathrm{A}_1\mathrm{B}_1\mathrm{C}_1\mathrm{D}_1$, with the unit point E for their centre of homology, are all ranged on one common plane of homology, of which the *equation* and the *symbol* may be thus written,

$$x + y + r + w = 0, \quad [\mathrm{E}] = [1, 1, 1, 1],$$

and which may be called (comp. 4) the *Unit-Plane*.

17. Any four collinear points, P_0, P_1, P_2, P_3, have their anharmonic symbols connected by two equations of the forms,

$$(\mathrm{P}_1) = t(\mathrm{P}_0) + u(\mathrm{P}_2), \quad (\mathrm{P}_3) = t'(\mathrm{P}_0) + u'(\mathrm{P}_2)$$

each including four ordinary linear equations between the co-ordinates of the four points, such as

$$x_1 = tx_0 + ux_2, \quad y_1 = ty_0 + uy_2, \quad \&c.;$$

and the anharmonic of their *group* is then found to be expressed by the formula,

$$(\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3) = \frac{ut'}{tu'}.$$

And similarly, if any four planes $\Pi_0 \ldots \Pi_3$ be collinear (that is, if they have any one right line common to them all), their symbols satisfy two linear equations of the corresponding forms,

$$[\Pi_1] = t[\Pi_0] + u[\Pi_2], \quad [\Pi_3] = t'[\Pi_0] + u'[\Pi_2];$$

and the anharmonic of the *pencil* is,

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = \frac{ut'}{tu'}.$$

18. If $\phi(x\,y\,z\,w)$ be any *rational fraction*, the numerator and denominator of which are any two given homogeneous and linear functions of the co-ordinates of a *variable point*; and if we determine a *line* Λ, and *three planes* Π_0, Π_1, Π_2 through that line, by the four *local* equations,

$$\phi = \frac{0}{0}, \quad \phi = \infty, \quad \phi = 1, \quad \phi = 0;$$

then I find that the function ϕ may be expressed as the anharmonic of a *pencil of planes*, as follows:—

$$\phi(x\,y\,z\,w) = (\Pi_0\,\Pi_1\,\Pi_2\,\Pi);$$

where Π is the variable plane $\Lambda\mathrm{P}$, which passes through the fixed line Λ, and through the variable point $\mathrm{P} = (x\,y\,z\,w)$.

19. And in like manner, as the *geometrical dual* (9) of this last theorem (18), if $\Phi(l\,m\,n\,r)$ be any rational fraction, of which the numerator and denominator are any two given functions, homogeneous and linear, of the co-ordinates of a *variable plane*; and if we determine a *line* Λ, and *three points* P_0, P_1, P_2, on that line, by the four *tangential* equations,

$$\Phi = \frac{0}{0}, \quad \Phi = \infty, \quad \Phi = 1, \quad \Phi = 0;$$

I find that the proposed function Φ may then be thus expressed as the anharmonic of a *group of points*.

$$\Phi(l\,m\,n\,r) = (\mathrm{P}_0\,\mathrm{P}_1\,\mathrm{P}_2\,\mathrm{P});$$

P here denoting the variable point $\Lambda\cdot\Pi$, in which the fixed line Λ intersects the variable plane $\Pi = [l\,m\,n\,r]$.

20. All problems respecting *intersections of lines with planes*, &c., are resolved, with the help of the Fundamental Theorem (16) respecting the relation which exists between the anharmonic co-ordinates of point and plane, as easily by the present method, as by the known method of *quadriplanar* co-ordinates (10); and indeed, by the very *same mechanism*, of which it is therefore unnecessary here to speak.

But it may be proper to say a few words respecting the application of the anharmonic method to *Surfaces* (7); although here again the known mechanism of *calculation* may in great part be preserved unchanged, and only the *interpretations* need be *new*.

21. In general, it is easy to see (comp. 7) that, in the present method, as in older ones, the *order* of a curved surface is denoted by the *degree* of its *local equation*, $f(x\,y\,z\,w) = 0$; and that the *class* of the same surface is expressed, in like manner, by the degree of its *tangential equation*, $F(l\,m\,n\,r) = 0$: because the *former* degree (or dimension) determines the *number of points* (distinct or coincident, and real or imaginary), in which the surface, considered as a locus, is *intersected* by an arbitrary right *line*; while the *latter* degree determines the *number of planes* which can be drawn *through* an arbitrary right line, so as to touch the same surface, considered as an *envelope*. It may be added, that I find the *partial derivatives of each* of these two functions, f and F, to be proportional to the *co-ordinates* which enter as variables into the *other*; thus we may write

$$[D_x f, D_y f, D_z f, D_w f],$$

as the symbol (15) of the *tangent plane* to the *locus* f, at the point $(x\,y\,z\,w)$; and

$$(D_l F, D_m F, D_n F, D_r F),$$

as a symbol for the *point of contact* of the *envelope* F, with the plane $[l\,m\,n\,r]$: whence it is easy to conceive how problems respecting the *polar reciprocals* of *surfaces* are to be treated.

22. As a very simple *example*, the surface of the *second order* which passes through the *nine points*, above called A B C D E A$'$ A$_2$ C$'$C$_2$, is easily found to have for its *local* equation $0 = f = xz - yw$; whence the co-ordinates of its tangent plane are, $l = D_x f = z$, $m = D_y f = -w$, $n = D_z f = x$, $r = D_w f = -y$, and its *tangential* equation is, therefore, $0 = F = ln - mr$, so that it is also a surface of the *second class.* In fact it is the *hyperboloid* on which the gauche quadrilateral ABCD is superscribed, and which passes also through the point E; and the known *double generation*, and *anharmonic properties*, of this surface, may easily be deduced from either of the foregoing forms of its anharmonic equation, whereof the first may (by 13, 15) be expressed as an equality between the anharmonic functions of two *pencils of planes*, in either of the two following ways:— (BC . AEDP) = (DA . BECP); (AB . CEDP) = (CD . BEAP).