# THEOREMS RESPECTING ALGEBRAIC ELIMINATION

## By


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*Theorem respecting Algebraic Elimination, connected with the Question of the Possibility of resolving in finite Terms the general Equation of the Fifth Degree.* Extracted by Permission, from a Communication recently made to the Royal Irish Academy. *By Professor Sir* William Rowan Hamilton, *Astronomer Royal of Ireland**.

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*Theorem I.* If $x$ be eliminated between two equations, of the following forms, namely, 1st, an equation of the fifth degree, of the form

$$0 = x^5 + \mathrm{D}x + \mathrm{E}, \tag{1.}$$

in which the roots are supposed to be all unequal, and the coefficients D and E to be, both of them, different from 0, and, 2nd, an equation of the form

$$y = \mathrm{Q}x + f(x), \tag{2.}$$

in which $f(x)$ denotes any rational function of $x$, whether integral or fractional,

$$f(x) = \frac{\mathrm{M}'x^{\mu'} + \mathrm{M}''x^{\mu''} + \&\mathrm{c.}}{\mathrm{K}'x^{\kappa'} + \mathrm{K}''x^{\kappa''} + \&\mathrm{c.}}; \tag{3.}$$

and if, in the result of this elimination, which will always be an equation of the fifth degree in $y$, of the form

$$0 = y^5 + \mathrm{A}'y^4 + \mathrm{B}'y^3 + \mathrm{C}'y^2 + \mathrm{D}'y + \mathrm{E}', \tag{4.}$$

we suppose that the coefficients are such as to satisfy, *independently of* Q, the second as well as the first of the two conditions

$$\mathrm{A}' = 0, \quad \mathrm{C}' = 0, \tag{5.}$$

in virtue of the values of the constants

$$\mathrm{M}', \mathrm{M}'', \ldots, \mu', \mu'', \ldots, \mathrm{K}', \mathrm{K}'', \ldots, \kappa', \kappa'', \ldots \tag{6.}$$

in the rational function $f(x)$; I say that then those constants (6.) must be such as to admit of our reducing that rational function to the form

$$f(x) = qx + (x^5 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x), \tag{7.}$$

$q$ being some new constant, and $\phi(x)$ being some new rational function of $x$, which does not contain the polynome $x^5 + \mathrm{D}x + \mathrm{E}$ as a divisor.

* Communicated by the Author.

*Demonstration.*—Let $x_1\,x_2\,x_3\,x_4\,x_5$ denote the five roots of the equation (1.), which are supposed to be all unequal among themselves, and different from 0; and let us put for abridgement

$$\left.\begin{aligned} f(x_1) - \frac{x_1}{x_5} f(x_5) &= h_1, \\ f(x_2) - \frac{x_2}{x_5} f(x_5) &= h_2, \\ f(x_3) - \frac{x_3}{x_5} f(x_5) &= h_3, \\ f(x_4) - \frac{x_4}{x_5} f(x_5) &= h_4, \\ \frac{f(x_5)}{x_5} = q, \quad \mathrm{Q} + q &= \mathrm{Q}'. \end{aligned}\right\} \tag{8.}$$

We shall then have

$$\left.\begin{array}{lll} f(x_1) = h_1 + qx_1, & f(x_2) = h_2 + qx_2, & \\ f(x_3) = h_3 + qx_3, & f(x_4) = h_4 + qx_4, & f(x_5) = qx_5, \end{array}\right\} \tag{9.}$$

and the result (4.) of the elimination of $x$ between the equations (1.) and (2.), may be expressed as follows:

$$0 = (y - \mathrm{Q}'x_1 - h_1)(y - \mathrm{Q}'x_2 - h_2)(y - \mathrm{Q}'x_3 - h_3)(y - \mathrm{Q}'x_4 - h_4)(y - \mathrm{Q}'x_5). \tag{10.}$$

Comparing (10.) with (4.), and observing that the form of the equation (1.) gives the relations

$$0 = x_1 + x_2 + x_3 + x_4 + x_5, \tag{11.}$$

$$0 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2, \tag{12.}$$

$$\begin{aligned} 0 = {} & x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 \\ & + x_1x_3x_4 + x_2x_4x_5 + x_3x_5x_1 + x_4x_1x_2 + x_5x_2x_3, \end{aligned} \tag{13.}$$

we easily find these expressions for $\mathrm{A}'$ and $\mathrm{C}'$, namely,

$$\mathrm{A}' = -(h_1 + h_2 + h_3 + h_4), \tag{14.}$$

and

$$\begin{aligned} \mathrm{C}' = {} & -\mathrm{Q}'^2(h_1x_1^2 + h_2x_2^2 + h_3x_3^2 + h_4x_4^2) \\ & + \mathrm{Q}' \left\{ \begin{array}{l} \;\; h_1h_2(x_1 + x_2) + h_1h_3(x_1 + x_3) + h_1h_4(x_1 + x_4) \\ + h_2h_3(x_2 + x_3) + h_2h_4(x_2 + x_4) + h_3h_4(x_3 + x_4) \end{array} \right\} \\ & - (h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4). \end{aligned} \tag{15.}$$

If, then, the coefficient $\mathrm{C}'$, as well as $\mathrm{A}'$, is to vanish independently of Q, and consequently of $\mathrm{Q}'$, we must have the four following equations:

$$0 = h_1 + h_2 + h_3 + h_4; \tag{16.}$$

$$0 = h_1 x_1^2 + h_2 x_2^2 + h_3 x_3^2 + h_4 x_4^2; \tag{17.}$$

$$\begin{aligned} 0 = {} & h_1 h_2 (x_1 + x_2) + h_1 h_3 (x_1 + x_3) + h_1 h_4 (x_1 + x_4) \\ & + h_2 h_3 (x_2 + x_3) + h_2 h_4 (x_2 + x_4) + h_3 h_4 (x_3 + x_4); \end{aligned} \tag{18.}$$

$$0 = h_1 h_2 h_3 + h_1 h_2 h_4 + h_1 h_3 h_4 + h_2 h_3 h_4; \tag{19.}$$

which give, by elimination of $h_4$,

$$0 = h_1 (x_1^2 - x_4^2) + h_2 (x_2^2 - x_4^2) + h_3 (x_3^2 - x_4^2), \tag{20.}$$
$$0 = h_1^2 x_1 + h_2^2 x_2 + h_3^2 x_3 + (h_1 + h_2 + h_3)^2 x_4, \tag{21.}$$
$$0 = (h_2 + h_3)(h_3 + h_1)(h_1 + h_2). \tag{22.}$$

Of the three factors of the last of these equations, it is manifestly indifferent *which* we employ; since the conclusions which can be drawn from the consideration of any one of these three factors can also be drawn from the consideration of either of the other two, by merely interchanging two of the three roots $x_1\, x_2\, x_3$, without altering the other of those three roots, or the two remaining roots $x_4\, x_5$ of the equation (1.). We shall therefore take the first of the three factors of (22.), namely, the equation

$$0 = h_2 + h_3; \tag{23.}$$

which reduces the two equations (20.) and (21.) to the two following, obtained by elimination of $h_3$,

$$0 = h_1 (x_1^2 - x_4^2) + h_2 (x_2^2 - x_3^2), \tag{24.}$$
$$0 = h_1^2 (x_1 + x_4) + h_2^2 (x_2 + x_3). \tag{25.}$$

These two last equations give, by elimination of $h_2$,

$$0 = h_1^2 (x_1 + x_4)\{(x_1 + x_4)(x_1 - x_4)^2 + (x_2 + x_3)(x_2 - x_3)^2\}; \tag{26.}$$

in which we cannot suppose the factor $x_1 + x_4$ to vanish, because the relations

$$0 = x_1^5 + \mathrm{D} x_1 + \mathrm{E}, \quad 0 = x_4^5 + \mathrm{D} x_4 + \mathrm{E}, \tag{27.}$$

give

$$\left.\begin{aligned} \mathrm{D} &= -(x_1^4 + x_1^3 x_4 + x_1^2 x_4^2 + x_1 x_4^3 + x_4^4), \\ \mathrm{E} &= (x_1 + x_4)(x_1^2 + x_4^2) x_1 x_4, \end{aligned}\right\} \tag{28.}$$

and we have supposed that E does not vanish; and since, for a similar reason, we cannot suppose that $x_2 + x_3$ vanishes, we see that we must conclude

$$h_1 = 0, \quad h_2 = 0, \quad h_3 = 0, \quad h_4 = 0, \tag{29.}$$

unless we can suppose that the third factor of (26.) vanishes, that is, unless

$$(x_1 + x_4)(x_1 - x_4)^2 + (x_2 + x_3)(x_2 - x_3)^2 = 0. \tag{30.}$$

Let us then examine into the meaning of this last condition, and the circumstances under which it can be satisfied.

If we put, for abridgement,

$$x_2 + x_3 = -\alpha, \quad x_2 x_3 = \beta, \tag{31.}$$

the condition (30.) will become

$$0 = x_4^3 - x_4^2 x_1 - x_4 x_1^2 + x_1^3 - \alpha^3 + 4\alpha\beta; \tag{32.}$$

and we shall have, in virtue of the relations (11.) (12.) (13.), two other equations between $x_4$, $x_1$, $\alpha$, $\beta$, namely,

$$0 = x_4^2 + x_4(x_1 - \alpha) + x_1^2 - x_1\alpha + \alpha^2 - \beta, \tag{33.}$$

and

$$0 = x_1^3 - x_1^2\alpha + x_1(\alpha^2 - \beta) - \alpha^3 + 2\alpha\beta; \tag{34.}$$

between which three equations, (32.) (33.) (34.), we shall now proceed to eliminate $x_4$ and $x_1$. For this purpose we may begin by multiplying (33.) by $x_1$, and adding the product to (32.); a process which gives, by (34.),

$$0 = x_4^3 - x_4 x_1 \alpha + x_1^3 + 2\alpha\beta, \tag{35.}$$

a relation more simple than (32.). In the next place we may observe that, in general, the result of elimination of any variable $x$ between any two equations of the forms

$$\left.\begin{array}{l} 0 = p' + q'x + r'x^2 + s'x^3, \\ 0 = p'' + q''x + r''x^2, \end{array}\right\} \tag{36.}$$

is

$$\begin{aligned} 0 = p'^2 r''^3 - p'q'q''r''^2 - 2p'r'p''r''^2 + p'r'q''^2 r'' + 3p's'p''q''r'' - p's'q''^3 + q'^2 p'' r''^2 \\ -q'r'p''q''r'' - 2q's'p''^2 r'' + q's'p''q''^2 + r'^2 p''^2 r'' - r's'p''^2 q'' + s'^2 p''^3. \end{aligned} \tag{37.}$$

Applying this general formula to the elimination of $x_4$ between the equations (35.) and (33.), and making, for that purpose,

$$\left.\begin{array}{l} p' = x_1^3 + 2\alpha\beta, \quad q' = -x_1\alpha, \quad r' = 0, \quad s' = 1, \\ p'' = x_1^2 - x_1\alpha + \alpha^2 - \beta, \quad q'' = x_1 - \alpha, \quad r'' = 1, \end{array}\right\} \tag{38.}$$

we find, after some easy reductions,

$$0 = 4x_1^6 - 4x_1^5\alpha + x_1^4(8\alpha^2 - 6\beta) + x_1^3(-8\alpha^3 + 14\alpha\beta) + x_1^2(6\alpha^4 - 12\alpha^2\beta + 3\beta^2)$$
$$+x_1(-2\alpha^5 + 7\alpha^3\beta - 7\alpha\beta^2) + \alpha^6 - 7\alpha^4\beta + 13\alpha^2\beta^2 - \beta^3; \tag{39.}$$

which is easily reduced by (34.) to the form

$$0 = x_1^2(2\alpha^4 - 2\alpha^2\beta + \beta^2) + x_1(2\alpha^5 - 7\alpha^3\beta + \alpha\beta^2) + \alpha^6 - 3\alpha^4\beta + 5\alpha^2\beta^2 - \beta^3. \tag{40.}$$

Again, applying the same general formula (37.) to the elimination of $x_1$ between the equations (34.) and (40.), by making now

$$\left.\begin{array}{l} p' = -\alpha^3 + 2\alpha\beta, \quad q' = \alpha^2 - \beta, \quad r' = -\alpha, \quad s' = 1, \\ p'' = \alpha^6 - 3\alpha^4\beta + 5\alpha^2\beta^2 - \beta^3, \quad q'' = 2\alpha^5 - 7\alpha^3\beta + \alpha\beta^2, \\ r'' = 2\alpha^4 - 2\alpha^2\beta + \beta^2, \end{array}\right\} \tag{41.}$$

we find after reductions,

$$0 = 25\alpha^{18} - 250\alpha^{16}\beta + 975\alpha^{14}\beta^2 - 1850\alpha^{12}\beta^3 + 1725\alpha^{10}\beta^4 - 700\alpha^8\beta^5 + 100\alpha^6\beta^6, \tag{42.}$$

that is,

$$0 = 25\alpha^6(\alpha^2 - 2\beta)^2(\alpha^4 - 3\alpha^2\beta + \beta^2)^2. \tag{43.}$$

But this condition cannot be satisfied, consistently with the supposition which we have already made that neither D nor E vanishes; because, by expressions similar to (28.), we have

$$\mathrm{D} = -(\alpha^4 - 3\alpha^2\beta + \beta^2), \quad \mathrm{E} = -\alpha\beta(\alpha^2 - 2\beta). \tag{44.}$$

We must therefore reject the supposition (30.), and adopt the only other alternative, namely, (29.); and hence we have, by (9.),

$$f(x_1) = qx_1, \quad f(x_2) = qx_2, \quad f(x_3) = qx_3, \quad f(x_4) = qx_4, \quad f(x_5) = qx_5. \tag{45.}$$

In this manner we find, that, under the circumstances supposed in the enunciation of the theorem, the function

$$f(x) - qx$$

vanishes, for every value of $x$ which makes the polynome $x^5 + \mathrm{D}x + \mathrm{E}$ vanish; and since these values have been supposed unequal, we must have, therefore,

$$f(x) - qx = (x^5 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x), \tag{46.}$$

the function $(\phi x)$ being rational, like $f(x)$, and not containing $x^5 + \mathrm{D}x + \mathrm{E}$ as a divisor; which was the thing to be proved.

*Corollary.* It is evident that, under the circumstances above supposed, the coefficients $\mathrm{B}'$, $\mathrm{D}'$, $\mathrm{E}'$ of (4.) will be expressed as follows:

$$\mathrm{B}' = 0, \quad \mathrm{D}' = \mathrm{Q}'^4\mathrm{D}, \quad \mathrm{E}' = \mathrm{Q}'^5\mathrm{E}; \tag{47.}$$

that is, the equation of the fifth degree in $y$ will be of the form

$$0 = y^5 + \mathrm{Q}'^4\mathrm{D}y + \mathrm{Q}'^5\mathrm{E}. \tag{48.}$$

At the same time the relation between $y$ and $x$ will reduce itself, by (2.) and (7.), to the form

$$y = \mathrm{Q}'x + (x^5 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x), \tag{49.}$$

$\mathrm{Q}'$ still denoting $\mathrm{Q} + q$. If then, we were to establish this additional supposition

$$\mathrm{D}' = \tfrac{1}{5}\mathrm{B}'^2, \tag{50.}$$

in order to complete the reduction of (4.) to De Moivre's solvible form, we should have

$$\mathrm{Q}'^4 = 0, \tag{51.}$$

that is,

$$\mathrm{Q}' = 0; \tag{52.}$$

the equation of the fifth degree in $y$ would become

$$y^5 = 0, \tag{53.}$$

and the relation between $y$ and $x$ would become

$$y = (x^5 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x); \tag{54.}$$

and thus, although the equation in $y$ would indeed by easily solvible, yet it would entirely fail to give the least assistance towards resolving the proposed equation of the fifth degree in $x$.

Observatory, Dublin, May 13, 1836.

*Second Theorem of Algebraic Elimination, connected with the Question of the Possibility of resolving, in finite Terms, Equations of the Fifth Degree. By Professor Sir* William Rowan Hamilton, *Astronomer Royal of Ireland.*

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*Theorem II.* If $x$ be eliminated between a proposed equation of the fifth degree,

$$x^5 + \mathrm{A}x^4 + \mathrm{B}x^3 + \mathrm{C}x^2 + \mathrm{D}x + \mathrm{E} = 0, \tag{55.}$$

and an assumed equation, of the form

$$y = \mathrm{Q}x + f(x), \tag{2.}$$

in which $f(x)$ denotes any rational function of $x$,

$$f(x) = \frac{\mathrm{M}'x^{\mu'} + \mathrm{M}''x^{\mu''} + \dots}{\mathrm{K}'x^{\kappa'} + \mathrm{K}''x^{\kappa''} + \dots}; \tag{3.}$$

and if the constants of this function be such as to reduce the result of the elimination to the form

$$y^5 + \mathrm{B}'y^3 + \mathrm{D}'y + \mathrm{E}' = 0, \tag{56.}$$

*independently of* Q: then not only must we have

$$\mathrm{A} = 0, \quad \mathrm{C} = 0, \tag{57.}$$

so that the proposed equation of the fifth degree must be of the form

$$x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E} = 0, \tag{58.}$$

but also the function $f(x)$ must be of the form

$$f(x) = qx + (x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x), \tag{59.}$$

$q$ being some constant multiplier, and $\phi(x)$ some rational function of $x$, which does not contain the polynome $x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E}$ as one of the factors of its denominator; unless we have either, first,

$$\mathrm{E} = 0; \tag{60.}$$

or else, secondly,

$$5\mathrm{D} = \mathrm{B}^2, \tag{61.}$$

or, as the third and only remaining case of exception,

$$5^5\mathrm{E}^4 + 2^2\mathrm{B}\mathrm{E}^2(2^2 5^3\mathrm{D}^2 - 3^2 5^2\mathrm{B}^2\mathrm{D} + 3^3\mathrm{B}^4) + 2^4\mathrm{D}^3(2^4\mathrm{D}^2 - 2^3\mathrm{B}^2\mathrm{D} + \mathrm{B}^4) = 0. \tag{62.}$$

*Demonstration.*—If we denote by $x_1\, x_2\, x_3\, x_4\, x_5$ the five roots of the proposed equation of the fifth degree, and put, as is permitted,

$$\left.\begin{aligned} &f(x_1) = h_1 + qx_1, \quad f(x_2) = h_2 + qx_2, \quad f(x_3) = h_3 + qx_3,\\ &f(x_4) = h_4 + qx_4, \quad f(x_5) = qx_5, \end{aligned}\right\} \tag{9.}$$

and

$$\mathrm{Q} + q = \mathrm{Q}', \tag{8.}$$

the result of the elimination of $x$ between the two equations (55.) and (2.) may be denoted thus,

$$(y - \mathrm{Q}'x_1 - h_1)(y - \mathrm{Q}'x_2 - h_2)(y - \mathrm{Q}'x_3 - h_3)(y - \mathrm{Q}'x_4 - h_4)(y - \mathrm{Q}'x_5) = 0; \tag{10.}$$

and if this result is to be of the form (56.), independently of Q, and therefore also of $\mathrm{Q}'$, we must have the six following relations:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \tag{11.}$$

$$\begin{aligned} &x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2\\ &+ x_1x_3x_4 + x_2x_4x_5 + x_3x_5x_1 + x_4x_1x_2 + x_5x_2x_3 = 0, \end{aligned} \tag{13.}$$

$$h_1 + h_2 + h_3 + h_4 = 0, \tag{16.}$$

$$h_1(x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5) + h_2(\&\mathrm{c.}) + h_3(\&\mathrm{c.}) + h_4(\&\mathrm{c.}) = 0, \tag{63.}$$

$$h_1h_2(x_3+x_4+x_5) + h_1h_3(\&\mathrm{c.}) + h_1h_4(\&\mathrm{c.}) + h_2h_3(\&\mathrm{c.}) + h_2h_4(\&\mathrm{c.}) + h_3h_4(\&\mathrm{c.}) = 0, \tag{64.}$$

$$h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4 = 0; \tag{19.}$$

of which the two first give

$$\mathrm{A} = 0, \quad \mathrm{C} = 0, \tag{57.}$$

and the last three may, by attending to the first and third, and by eliminating $h_4$, be written thus:

$$h_1(x_1^2 - x_4^2) + h_2(x_2^2 - x_4^2) + h_3(x_3^2 - x_4^2) = 0, \tag{20.}$$

$$h_1^2x_1 + h_2^2x_2 + h_3^2x_3 + (h_1 + h_2 + h_3)^2x_4 = 0, \tag{21.}$$

$$(h_2 + h_3)(h_3 + h_1)(h_1 + h_2) = 0. \tag{22.}$$

Selecting, as we are at liberty to do, the first of the three factors of (22.), namely,

$$h_2 + h_3 = 0, \tag{23.}$$

and eliminating $h_3$ by this, we reduce the two conditions (20.) and (21.) to the two following:

$$h_1(x_1^2 - x_4^2) + h_2(x_2^2 - x_3^2) = 0, \tag{24.}$$
$$h_1^2(x_1 + x_4) + h_2^2(x_2 + x_3) = 0, \tag{25.}$$

which give, by elimination of $h_2$,

$$h_1^2(x_1 + x_4)\{(x_1 + x_4)(x_1 - x_4)^2 + (x_2 + x_3)(x_2 - x_3)^2\} = 0. \tag{26.}$$

And from these equations (of which some occurred in the investigation of the former theorem, but are now for greater clearness repeated,) we see that we must have

$$h_1 = 0, \quad h_2 = 0, \quad h_3 = 0, \quad h_4 = 0, \tag{29.}$$

and therefore, by (9.),

$$f(x_1) = qx_1, \quad f(x_2) = qx_2, \quad f(x_3) = qx_3, \quad f(x_4) = qx_4, \quad f(x_5) = qx_5, \tag{45.}$$

unless we have either

$$x_1 + x_4 = 0, \tag{65.}$$

or else

$$(x_1 + x_4)(x_1 - x_4)^2 + (x_2 + x_3)(x_2 - x_3)^2 = 0, \tag{30.}$$

or at least some one of those other relations into which (65.) and (30.) may be changed, by changing the arrangement of the roots of the proposed equation of the fifth degree.

The alternative (65.), combined with (57.), gives evidently

$$\mathrm{E} = 0; \tag{60.}$$

but the meaning of the alternative (30.) is a little less easy to examine, now that we do not suppose the coefficient B to vanish, as we did in the investigation of the former theorem. However, the following process is tolerably simple. We may conceive that $x_1\, x_2\, x_3\, x_4$ are roots of a certain biquadratic equation,

$$x^4 + ax^3 + bx^2 + cx + d = 0, \tag{66.}$$

and may express, by means of its coefficients $a\, b\, c\, d$, the symmetric functions of $x_1\, x_2\, x_3\, x_4$ which enter into the development of the product formed by multiplying together the condition (30.), and these two other similar conclusions,

$$(x_1 + x_3)(x_1 - x_3)^2 + (x_2 + x_4)(x_2 - x_4)^2 = 0, \tag{67.}$$
$$(x_1 + x_2)(x_1 - x_2)^2 + (x_3 + x_4)(x_3 - x_4)^2 = 0. \tag{68.}$$

If we put, for abridgement,

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = \mathrm{f}, \tag{69.}$$

$$-x_1x_2(x_1+x_2) - x_3x_4(x_3+x_4) = \mathrm{g}, \tag{70.}$$

$$-x_1x_3(x_1+x_3) - x_2x_4(x_2+x_4) - x_1x_4(x_1+x_4) - x_2x_3(x_2+x_3) = \mathrm{h}, \tag{71.}$$

$$\{x_1x_3(x_1+x_3) + x_2x_4(x_2+x_4)\}\{x_1x_4(x_1+x_4) + x_2x_3(x_2+x_3)\} = \mathrm{i}, \tag{72.}$$

the condition (68.) will become

$$\mathrm{f} + \mathrm{g} = 0, \tag{73.}$$

and the product of the two other conditions (67.) and (68.) will become

$$\mathrm{f}^2 + \mathrm{hf} + \mathrm{i} = 0, \tag{74.}$$

so that the product of all the three conditions becomes

$$\mathrm{f}^3 + (\mathrm{g}+\mathrm{h})\mathrm{f}^2 + (\mathrm{gh}+\mathrm{i})\mathrm{f} + \mathrm{gi} = 0; \tag{75.}$$

and the symmetric functions f, g + h, gh + i, gi, may be expressed as follows:

$$\mathrm{f} = -a^3 + 3ab - 3c, \tag{76.}$$

$$\mathrm{g} + \mathrm{h} = ab - 3c, \tag{77.}$$

$$\mathrm{gh} + \mathrm{i} = a^3c - 4a^2d - 2abc + 3c^2, \tag{78.}$$

$$\mathrm{gi} = a^5d - 4a^3bd + 4a^2cd + abc^2 - c^3. \tag{79.}$$

Again, the proposed equation of the fifth degree,

$$x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E} = 0, \tag{58.}$$

must be exactly divisible by the quadratic equation (66.), because all the roots of the latter are also roots of the former; and therefore we must have

$$\mathrm{B} = b - a^2, \quad \mathrm{D} = d - a^2b, \quad \mathrm{E} = -ad, \tag{80.}$$

and

$$c = ab. \tag{81.}$$

This relation $c = ab$ reduces the expression (76.) … (79.) to the following,

$$\left.\begin{aligned} \mathrm{f} &= -a^3, \\ \mathrm{g}+\mathrm{h} &= -2ab, \\ \mathrm{gh}+\mathrm{i} &= a^4b + a^2b^2 - 4a^2d, \\ \mathrm{gi} &= a^5d; \end{aligned}\right\} \tag{82.}$$

and thereby reduces the condition (75.), that is, the product of the three conditions (66.) (67.) (68.), to the form

$$-a^9 - 3a^7b - a^5b^2 + 5a^5d = 0, \tag{83.}$$

which gives either

$$a = 0, \tag{84.}$$

or else

$$a^4 + 3a^2b + b^2 = 5d, \tag{85.}$$

and therefore, by (80.), either

$$\mathrm{E} = 0, \tag{60.}$$

or else

$$5\mathrm{D} = \mathrm{B}^2. \tag{61.}$$

Thus,when we set aside these two particular cases, we see by (45.), that under the circumstances supposed in the enunciation of the theorem, the function $f(x) - qx$ vanishes, for every value of $x$ which makes the polynome $x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E}$ vanish; and that therefore if we set aside the third and only remaining case of exception, namely, the case in which the proposed equation of the fifth degree has two equal roots, and in which consequently the condition (62.) is satisfied, the function $f(x)$ must be of the form (59.); which was the thing to be proved.

*Corollary.*—Setting aside the three excepted cases (60.) (61.) (62.), the coefficients of the equation (50.) of the fifth degree in $y$ will be expressed as follows,

$$\mathrm{B}' = \mathrm{Q}'^2\mathrm{B}, \quad \mathrm{D}' = \mathrm{Q}'^4\mathrm{D}, \quad \mathrm{E}' = \mathrm{Q}'^5\mathrm{E}; \tag{86.}$$

and if we attempt to reduce it to De Moivre's solvible form, by making

$$\mathrm{D}' = \tfrac{1}{5}\mathrm{B}'^2, \tag{50.}$$

we find

$$\mathrm{Q}'^4 = 0, \tag{51.}$$

that is,

$$\mathrm{Q}' = 0, \tag{52.}$$

so that the relation between $y$ and $x$ reduces itself to the form

$$y = (x^5 + \mathrm{B}x^3 + \mathrm{D}x + \mathrm{E}) \,.\, \phi(x), \tag{87.}$$

which can give no assistance towards resolving the proposed equation (58.) of the fifth degree in $x$.

Observatory, Dublin, June 11, 1836.